15-312 Lecture on Inductive Definitions and Proofs by Induction

A Cartoon View of Inductive Definitions

If we think about objects (e.g., natural numbers, terms, derivations themselves) as boxes, then we have an inductive definition when big boxes are defined in term of smaller boxes. There may be more than one way to define a box.

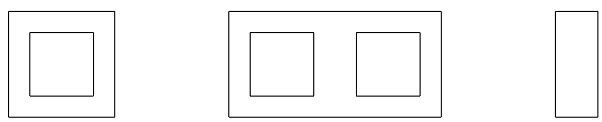


Figure 1: Inductive Definition

This should be finite however: if I keep on opening boxes, there should be a point where there is no box to open any more.

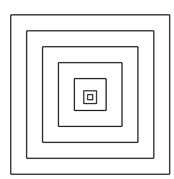


Figure 2: Not an Inductive Definition

A Cartoon View of Proofs by Rule Induction

Now you want to prove that a property P holds of of all your boxes. How do you do it?

The induction principle on boxes say that:

> ***If**, for each way to build a box,*
>
> - P(*"smaller box"*) *implies* P(*"bigger box"*)
>   [This is your inductive hypothesis]
>
> ***Then** P holds of all boxes.*

Note that if some particular way to build a box does not uses smaller boxes, then you just need to prove that P holds for it.

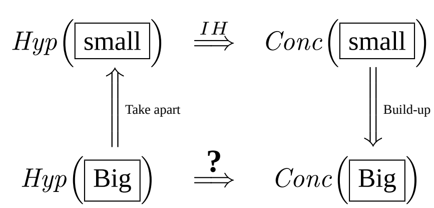
Let's apply it to the boxes in Figure 1: **If**

- Given $P\left(\boxed{\text{small}}\right)$, I can show $P\left(\boxed{\text{Big}}\right)$
- and given $P\left(\boxed{\text{small}}\right)$ and $P\left(\boxed{\text{small}}\right)$, I can show $P\left(\boxed{\text{Big}}\right)$
- and I can show $P\left(\boxed{\text{Big}}\right)$

Then P holds of all boxes one can build from Figure 1.

A Cartoon View of an Inductive Case

In practice, P is always an implication of from hypotheses (Hyp) to a conclusion ($Conc$), so that $P = Hyp \Rightarrow Conc$. How is an inductive case proved? Use the following diagram:



Example

Let's consider the following property of this deductive system for lists and reversal:

$$\frac{}{\mathsf{nil}\ \mathsf{list}}\ \mathbf{nil} \qquad\qquad \frac{n\ \mathsf{nat} \quad l\ \mathsf{list}}{\mathsf{cons}(n, l)\ \mathsf{list}}\ \mathbf{cons}$$

$$\frac{a \ \mathsf{list}}{\mathsf{rev}\ \mathsf{nil}\ a\ a}\ \mathbf{rev\_nil} \qquad\qquad \frac{\mathsf{rev}\ l\ \mathsf{cons}(n,a)\ x}{\mathsf{rev}\ \mathsf{cons}(n,l)\ a\ x}\ \mathbf{rev\_cons}$$

> *For all derivations* $\mathcal{L} :: l$ list *and* $\mathcal{A} :: a$ list, *there exists a derivation* $\mathcal{R} ::$ rev $l\ a\ x$.

Here our boxes are derivations of lists and reversals. Let's call this property P. Then

- Hyp = "Given derivations $\mathcal{L} :: l$ list and $\mathcal{A} :: a$ list", and
- $Conc$ = "There exists a derivation $\mathcal{R} ::$ rev $l\ a\ x$"

Let's use the above technique to prove this property.

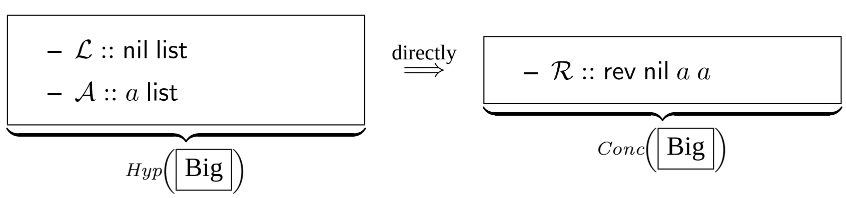
Proof: The proof proceeds by induction the given derivation of $\mathcal{L}$. Since there are two ways to construct a derivation, we need to examine two cases:

- $\mathcal{L} = \dfrac{}{\mathsf{nil}\ \mathsf{list}}\ \mathbf{nil}$, with $l = \mathsf{nil}$.

  This is one of the cases where a big box is not built out of smaller boxes. So we need to build the derivation in the conclusions from scratch. Since we have derivation $\mathcal{A}$, we can do by applying rule **rev_nil**:

  $$\mathcal{R} = \frac{\begin{matrix}\mathcal{A}\\ a\ \mathsf{list}\end{matrix}}{\mathsf{rev}\ \mathsf{nil}\ a\ a}\ \mathbf{rev\_nil}$$

  Fitting all this in the diagram above, we have:



- $\mathcal{L} = \dfrac{\begin{matrix}\mathcal{D} \\ n\ \mathsf{nat}\end{matrix} \quad \begin{matrix}\mathcal{L}' \\ l'\ \mathsf{list}\end{matrix}}{\mathsf{cons}(n,l')\ \mathsf{list}}\ \mathbf{cons}$, with $l = \mathsf{cons}(n, l')$.

  This is an inductive case: the big box ($\mathcal{L}$) is built out of a smaller box ($\mathcal{L}'$). Here, we can appeal to the induction hypothesis. A first attempt at an IH is as follows:

  > *Given a derivation* $\mathcal{L}' :: l'$ list *and* $\mathcal{A} :: a$ list, *there exists a derivation* $\mathcal{R}' ::$ rev $l'\ a\ x$.

This is not very helpful however because a derivation of $\mathsf{rev}\ l'\ a\ x$ is of no use to obtain the desired derivation of $\mathsf{rev}\ \mathsf{cons}(n, l')\ a\ x$. The only way to get $\mathsf{cons}(n, l')$ as the first argument of rev is to apply rule **rev_cons**, but this requires $\mathsf{cons}(n, a)$ in the second argument of the conclusion of the induction hypothesis. The induction hypothesis we really want is as follows:

> *Given a derivation* $\mathcal{L}' :: l'\ \mathsf{list}$ *and* $\mathcal{A}' :: \mathsf{cons}(n, a)\ \mathsf{list}$, *there exists a derivation* $\mathcal{R}' :: \mathsf{rev}\ l'\ a\ x$.[1]

Since we already have $\mathcal{L}'$, to use it we need to build the derivation $\mathcal{A}'$, but this is easily done by applying rule **cons** to $\mathcal{A}$ and $\mathcal{D}$:

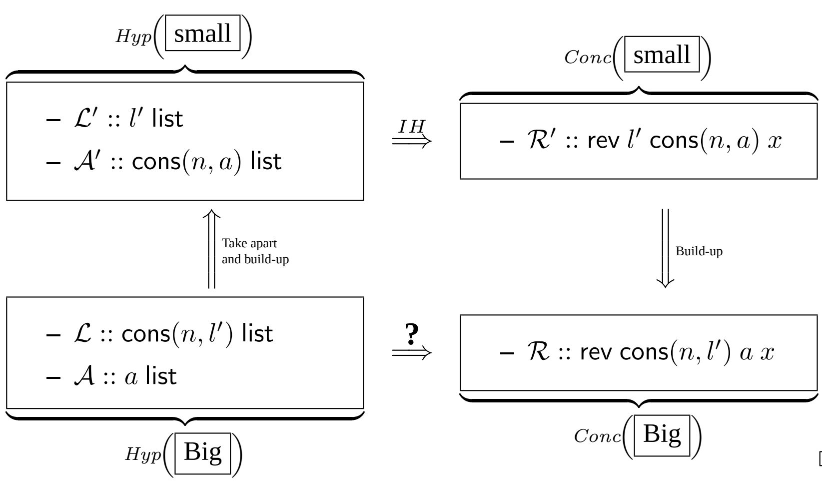
$$\mathcal{A}' = \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{A} \\ n\ \mathsf{nat} & a\ \mathsf{list}\end{array}}{\mathsf{cons}(n, a)\ \mathsf{list}}\ \mathbf{cons}$$

Now, we can apply rule **rev_cons** to $\mathcal{R}'$ to obtain the desired derivation $\mathcal{R}$ of $\mathsf{rev}\ \mathsf{cons}(n, l')\ a\ x$

$$\mathcal{R} = \dfrac{\begin{array}{c}\mathcal{R}' \\ \mathsf{rev}\ l'\ \mathsf{cons}(n, a)\ x\end{array}}{\mathsf{rev}\ \mathsf{cons}(n, l')\ a\ x}\ \mathbf{rev\_cons}$$

This concludes this branch of the proof, and therefore the whole proof since there are no other cases to consider.

Let's put this derivation too in the diagram in the previous section:



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[1] To be fully precise, the induction hypothesis is a generalization of both statements:

> *Given a derivation* $\mathcal{L}' :: l'\ \mathsf{list}$ *and* $\mathcal{A} :: A\ \mathsf{list}$ *for any* A, *there exists a derivation* $\mathcal{R}' :: \mathsf{rev}\ l'\ A\ x$.

Our trial and error process has consisted in finding the right intantiation for A.