15-312 Lecture on Judgments, Rules and Derivations

Domain of Discourse

- A formal *representation* of the informal entities we want to say something about
  - **Objects of discourse**
  - E.g., $\mathsf{s}(\mathsf{s}(\mathsf{s}\ \mathsf{z}))$ as the unary representation of 3
- For us, programming language notions
  - Expressions, Values, Types, Environments, ...
- Always given as finite structured constructions built incrementally from basic building blocks.
  - z, s z, $\mathsf{s}(\mathsf{s}\ \mathsf{z})$, ... are Ok.
  - 0, 1, 2, ... will not be Ok unless we provide a method for constructing all objects in the domain of discourse starting from a finite number of basic blocks.

Judgments

- Formal description of informal relationships/properties of objects of discourse
  - Written using a weird syntax, but just relations
  - Meant to formally mimick relations that hold in the informal world.
- 4 main distinctions

  Judgment forms (akin to relation domains — set of all tuples)

  - n nat, $e \Downarrow v$, $e : \tau$, ...
    - n, e, v, τ are **schematic variables** or *meta-variables*

  Judgment instances (akin to individual tuples)

  - z nat, $\mathsf{s}(\mathsf{s}\ \mathsf{z})$ nat, ♣ nat

  Judgment Extension **(set of all tuples in a specific relation)**

- Set of all instances that "hold"
- z nat, s z nat, s(s z) nat, ...
- but not ♣ nat, s ♣ nat, ...

Valid judgments (akin to individual tuples *in* a relation)

- Instances that "hold"
- z nat, s(s z) nat
- but not ♣ nat
- In general, infinitely many

We will often refer to *partial instances* such as s n nat, where basic building blocks and meta-variables are mixed.

We want to have a way to describe the valid instances of a judgment form (i.e., the judgment extension). The most naive way, listing them exaustively, is not very useful in general because many judgment forms have infinitely many instances. We want to give a finite, systematic, description of the valid judgments. To do so, we leverage the fact that the objects in the domain of discourse are described as constructions based on finite building blocks.

Inference Rules

Descriptions of when a schematic judgment follows from another judgment

$$\frac{a_1\ J \quad \ldots \quad a_n\ J}{a\ J}\,\mathbf{r}$$

$a_1\ J \quad \ldots \quad a_n\ J$ are **premises**; $a\ J$ is the *conclusion*, $\mathbf{r}$ is the **rule name** (sometimes omitted).

- Each judgment in a rule is a partial instance
  - May contain meta-variables
- **Axiom** if no premises ($n = 0$)
- A rule can be instantiated
- Examples

$$\frac{}{\mathsf{z}\ \mathsf{nat}}\,\mathbf{z\_nat} \qquad\qquad \frac{n\ \mathsf{nat}}{\mathsf{s}\ n\ \mathsf{nat}}\,\mathbf{s\_nat}$$

Note that these rules leverage the structure of expressions to describe *uniformly* which judgments hold.

Derivations

Justifications of judgment instances

- Denoted $\mathcal{D} :: J$

Defined inductively

- If
$$\frac{}{J}$$
is an instance of an axiom, then
$$\mathcal{D} = \frac{}{J}$$
is a derivation of J

- If
$$\frac{J_1 \quad \dots \quad J_n}{J}$$
is a rule instance and $\mathcal{D}_1 :: J_1, \dots, \mathcal{D}_n :: J_n$, then
$$\mathcal{D} = \frac{\mathcal{D}_1 \quad \dots \quad \mathcal{D}_n}{J}$$
is a derivation of J

Stacks rule instances into a tree.

Deductive Systems

Domain of discourse + set of rules = deductive system

Example:

Terms over an alphabet that includes z and s

+ $\frac{}{\mathsf{z}\ \mathsf{nat}}\ \mathbf{z\_nat}$ $\qquad$ $\frac{n\ \mathsf{nat}}{\mathsf{s}\ n\ \mathsf{nat}}\ \mathbf{s\_nat}$

= deductive system describing natural numbers in unary notation

Inductive Definitions

- In informal world, some relationships holds, some do not
  - 3 is a natural number; ♣ is not
- When formalizing informal world using judgment, we want:

  - Judgments that have a derivation $\leadsto$ relationships that hold
  - Judgments that do not have derivations $\leadsto$ relationships that do not hold
- Deductive system models *only* positive relationship
  - Does not say anything explicitly about relationships that do not hold
- Formally, a deductive system is assumed to be **closed** under inference rules in it
  - Defines all and only the judgments for which a *finite* derivation can be constructed from rules and axioms.
  - Judgment extension = smallest set of judgment instances closed under inference rules
- Deductive system constitutes an *inductive definition* of its judgments.
  - Judgment extension = strongest relation generated by inference rules