Focusing Strategies for Synthetic Connectives

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conclusions

<1>

<1>

focusing

<1>

focusing = polarization

<1>

focusing = polarization
+ synthetic rules
+ strategy

<2>

<2>

ordinary focusing (Andreoli'92)	$\longleftrightarrow$	**maximal multi-focusing (CMS'08)**

<2>

ordinary focusing (Andreoli'92)	$^{-} \longleftrightarrow ^{+}$	**maximal multi-focusing (CMS'08)**

<3>

<3>

yes we can

<3>

yes we can

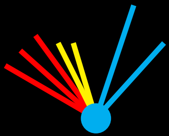
have canonicity
in the sequent calculus

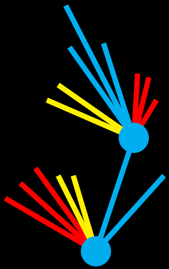
focusing = polarization

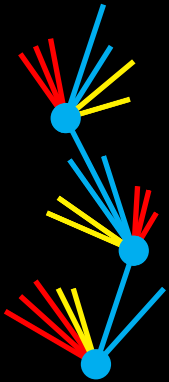
\+ synthetic rules

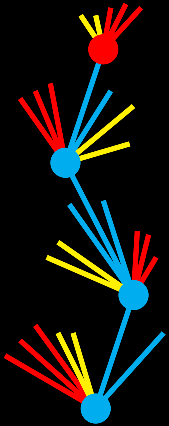
\+ strategy

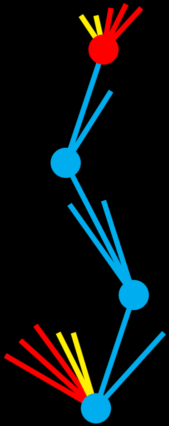
<disjunctive>

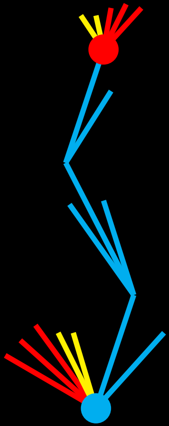


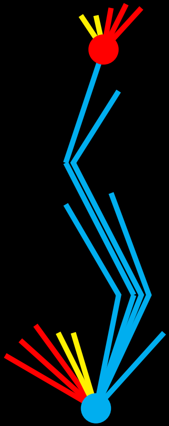












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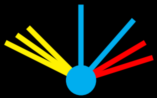
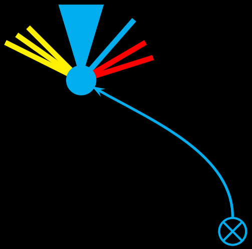
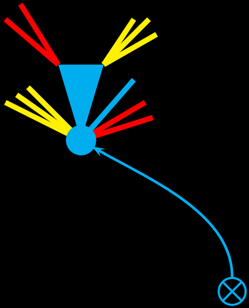
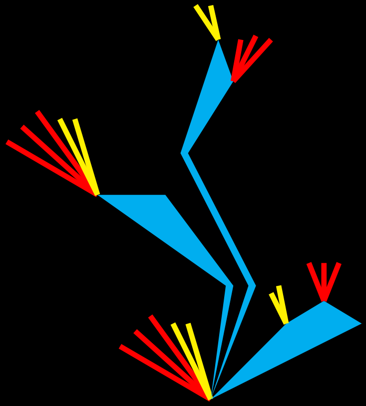








```
    </multiplicative>
</disjunctive>
```

focusing = polarization

+ synthetic rules

+ strategy

$$P ::= \;|\; \otimes \;|\;\;|\; \oplus \;|\;\;|$$

$$P, Q \quad ::= \quad \big| \; P \otimes Q \; \big| \quad \big| \; P \oplus Q \; \big| \quad \big|$$

$$P, Q \quad ::= \quad \mid P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid$$

$$P, Q \quad ::= \quad | \; P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \; |$$

product **sum**

$$P, Q \quad ::= \quad \mid \; P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid$$

$$N, M \quad ::= \quad \mid \; N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid$$

product	sum
$P \otimes Q \mid \mathbf{1}$ / $N \parr M \mid \bot$	$P \oplus Q \mid \mathbf{0}$ / $N \,\&\, M \mid \top$

$$P, Q \quad ::= p \mid P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid$$

$$N, M \quad ::= n \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid$$

product **sum**

$$P, Q \ ::= \ p \mid P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid N$$

$$N, M \ ::= \ n \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid P$$

product	sum
$P \otimes Q$ | $\mathbf{1}$	$P \oplus Q$ | $\mathbf{0}$
$N \parr M$ | $\bot$	$N \,\&\, M$ | $\top$

interface

$$P, Q \quad ::= \; p \mid P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid N$$

$$N, M \quad ::= \; n \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid P$$

product **sum**

<neutral expression>

$$E, F ::= \; e \;\Big|\; E \times F \;\Big|\; 1 \;\Big|\; E + F \;\Big|\; 0 \;\Big|\; \updownarrow E$$

<polarized proposition>

$$\langle E, \pm \rangle$$

$$\langle E, \pm \rangle$$

n-exp

$$\langle E, \pm \rangle$$

n-exp **polarity**

$$[\![\langle E \times F, +\rangle]\!] = [\![\langle E, +\rangle]\!] \otimes [\![\langle F, +\rangle]\!]$$

$$[\![\langle E \times F, -\rangle]\!] = [\![\langle E, -\rangle]\!] \parr [\![\langle F, -\rangle]\!]$$

$$[\![\langle \updownarrow E, +\rangle]\!] = [\![\langle E, -\rangle]\!]$$

$$[\![\langle \updownarrow E, +\rangle]\!] = [\![\langle E, -\rangle]\!]$$

$$[\![\langle \updownarrow E, -\rangle]\!] = [\![\langle E, +\rangle]\!]$$

</polarized proposition>

delay

delay

↕↕

$$
\begin{aligned}
&[\![\langle (E \times (F + G)), o \rangle]\!] \\
&\qquad = [\![\langle (E \times \updownarrow \updownarrow (F + G)), o \rangle]\!] \\
&\qquad = [\![\langle (\updownarrow \updownarrow E \times (F + \updownarrow \updownarrow G)), o \rangle]\!]
\end{aligned}
$$

$$
\begin{aligned}
&[\![\langle (E \times (F + G)), o \rangle]\!] \\
&\qquad = [\![\langle (E \times \updownarrow\updownarrow (F + G)), o \rangle]\!] \\
&\qquad = [\![\langle (\updownarrow\updownarrow E \times (F + \updownarrow\updownarrow G)), o \rangle]\!] \\
&\qquad = \ldots
\end{aligned}
$$

$$[\![\langle \updownarrow \updownarrow E, o\rangle]\!] = [\![\langle E, o\rangle]\!]$$

delays are *undetectable*

delays are *undetectable*

as *connectives*

connective = static

connective = static

polarization = dynamic

focusing = polarization

+ synthetic rules

+ strategy

dogma

dogma

rule = proof constructor

dogma

rule = proof constructor
= syntax

dogma

rule = proof constructor
= syntax
= contract

dogma

rule = proof constructor
= syntax
= contract
(bet. prover & checker)

dogma

rule = proof constructor
= syntax
= contract
(bet. prover & checker)
= static

n-exps have normal forms

$$E \times (F + G) \approx (E \times F) + (E \times G)$$

$$E \times (F + G) \approx (E \times F) + (E \times G)$$

“as good as”

<pattern>

$$U ::= e \quad | \quad \updownarrow E$$

$$
\begin{array}{lcl}
U & ::= & e \quad | \quad \updownarrow E \\
\pi & ::= & \prod_{j \in J} U_j
\end{array}
$$

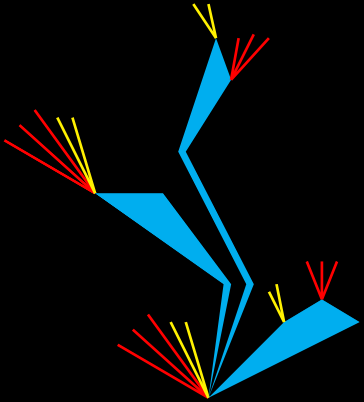
$$
\begin{aligned}
U &::= e \quad | \quad \updownarrow E \\
\pi &::= \prod_{j \in J} U_j \\
\sigma &::= \sum_{i \in I} \pi_i
\end{aligned}
$$

$$E \approx \sum_{i \in I} \prod_{j \in J_i} U_{ij}$$

</pattern>

<synthetic rule>

<synthetic rule>
    <positive>



$$E \approx (U_1 + (U_2 \times U_3) + (U_4 \times U_5 \times U_6))$$

$$E \approx (U_1 + \underline{(U_2 \times U_3)} + (U_4 \times U_5 \times U_6))$$

$$\dfrac{\dfrac{\vdash \Gamma, \langle U_1, + \rangle \qquad \vdash \Delta, \langle U_2, + \rangle}{\vdash \Gamma, \Delta, \langle U_2 \times U_3, + \rangle} \times}{\vdash \Gamma, \Delta, \langle E, + \rangle} +$$

$$E \approx (U_1 + (U_2 \times U_3) + \underline{(U_4 \times U_5 \times U_6)})$$

$$\dfrac{\dfrac{\vdash \Gamma, \langle U_4, +\rangle \qquad \vdash \Delta, \langle U_5, +\rangle \qquad \vdash \Omega, \langle U_6, +\rangle}{\vdash \Gamma, \Delta, \Omega, \langle U_4 \times U_5 \times U_6, +\rangle} \times}{\vdash \Gamma, \Delta, \Omega, \langle E, +\rangle} +$$

$$E \approx \sum_{i \in I} \prod_{j \in 1..n_i} U_{ij}$$

$$\frac{u \in I \qquad \vdash \Gamma_i, \langle U_{u1}, + \rangle \quad \cdots \quad \vdash \Gamma_{n_u}, \langle U_{un_u}, + \rangle}{\vdash \Gamma_1, \ldots, \Gamma_{n_u}, \langle E, + \rangle} \, \mathbf{P}$$

</positive>

</positive>
<reaction>

$$\vdash \Gamma_1, \langle U_1, + \rangle \qquad \cdots \qquad \vdash \Gamma_n, \langle U_n, + \rangle$$

$$(\Gamma_1, \ldots, \Gamma_n) \bigotimes (\prod_{i \in 1..n} U_i) =$$

$$\vdash \Gamma_1, \langle U_1, + \rangle \qquad \cdots \qquad \vdash \Gamma_n, \langle U_n, + \rangle$$

$$E \approx \textstyle\sum_{i \in I} \pi_i$$

$$(\Gamma_1, \ldots, \Gamma_n) \bigotimes \textstyle\prod_{i \in 1..n} U_i) \triangleq \vdash \Gamma_1, \langle U_1, + \rangle \ \cdots \ \vdash \Gamma_1, \langle U_n, + \rangle$$

$$\frac{u \in I \qquad \Gamma \bigotimes \pi_u}{\vdash \Gamma, \langle E, + \rangle}\, \mathbf{P}$$

$$\frac{u \in I \qquad \Gamma \bigotimes \pi_u}{\vdash \Gamma, \langle E, + \rangle}\,\mathbf{P}$$

$$\frac{\exists u \in I.\ \Gamma \bigotimes \pi_u}{\vdash \Gamma, \langle E, +\rangle}\ \mathbf{P}$$

$$\frac{\exists u \in I.\ \Gamma \bigotimes \pi_u}{\vdash \Gamma, \langle E, + \rangle}\ \mathbf{P}$$

$$\frac{\exists u \in I.\ \Gamma \bigotimes \pi_u}{\vdash \Gamma, \langle E, o \rangle}\ \mathbf{R}(o)$$

$$\frac{\exists u \in I.\ \Gamma \bigcirc \pi_u}{\vdash \Gamma, \langle E, o \rangle}\ \mathbf{R}(o, \bigcirc)$$

$$\frac{\mathsf{Q} u \in I.\ \Gamma \bigcirc \pi_u}{\vdash \Gamma, \langle E, o \rangle}\ \mathbf{R}(o, \bigcirc, \mathsf{Q})$$

$$\frac{Q u \in I.\ \Gamma \bigcirc \pi_u}{\vdash \Gamma, \langle E, o \rangle}\ \mathbf{R}(o, \bigcirc, Q)$$

$$\mathbf{P} = \mathbf{R}(+, \bigotimes, \exists)$$

$\mathbf{R}(-, \parr, \forall)$**?**

$$(\Gamma_1, \ldots, \Gamma_n) \parr \prod_{i \in 1..n} U_i = \\ \vdash \Gamma_1, \ldots, \Gamma_n, \langle U_1, - \rangle, \ldots, \langle U_n, - \rangle$$

$$\frac{\forall u \in I.\ \Gamma \parr \pi_u}{\vdash \Gamma, \langle E, - \rangle}\ \mathbf{N} = \mathbf{R}(-, \parr, \forall)$$

$$\frac{\forall u \in I.\ \Gamma \parr \pi_u}{\vdash \Gamma, \langle E, - \rangle}\ \mathbf{N} = \mathbf{R}(-, \parr, \forall)$$

$$\frac{\exists u \in I.\ \Gamma \bigotimes \pi_u}{\vdash \Gamma, \langle E, + \rangle}\ \mathbf{P} = \mathbf{R}(+, \bigotimes, \exists)$$

N and P are exactly dual

```
</reaction>
```

```
    </reaction>
</synthetic rule>
```

```
    </reaction>
</synthetic rule>

<interface rule>
```

$$\frac{}{\vdash \langle e, + \rangle, \langle e, - \rangle}\ \mathbf{I}$$

$$\frac{\vdash \Gamma, \langle E, \mp \rangle}{\vdash \Gamma, \langle \updownarrow E, \pm \rangle}\ \mathbf{U}$$

</interface rule>

$$\frac{\vdash \Gamma, \langle E, +\rangle \qquad \vdash \Delta, \langle E, -\rangle}{\vdash \Gamma, \Delta}\ \mathbf{C}$$

$$\frac{\vdash \Gamma, \langle E, + \rangle \qquad \vdash \Delta, \langle E, - \rangle}{\vdash \Gamma, \Delta}\ \mathbf{C}$$

admissible

$$\frac{\vdash \Gamma, \langle E, + \rangle \qquad \vdash \Delta, \langle E, - \rangle}{\vdash \Gamma, \Delta}\ \mathbf{c}$$

admissible

eliminable

$$\frac{}{\vdash \langle E, +\rangle\, , \langle E, -\rangle}\ \mathbf{I^*}$$

$$\frac{}{\vdash \langle E, +\rangle, \langle E, -\rangle}\ \mathbf{I^*}$$

derivable

focusing = polarization

+ synthetic rules

+ strategy

phase alternation

phase alternation unnecessary

strategy = arrangement of P and N

Ordinary Focusing

(Andreoli'92)

`if` **goal has** $\langle E, - \rangle$
`then` **apply N**
`else` **apply P, I, or U**

focusing = polarization
+ synthetic rules
+ strategy

ordinary
focusing = polarization
+ synthetic rules
+ strategy

ordinary
focusing = polarization
+ synthetic rules
+ N strategy

ordinary
focusing = polarization
+ synthetic rules
+ N strategy

</1>

ordinary focusing = eager N

lazy N?

eager P?

Maximally Positive

```
if   goal has ⟨E, +⟩
     then apply P
     else apply N, I, or U
```

Maximally Positive

`if` **goal has** $\langle E, + \rangle$
`then` apply P if possible
`else` apply N, I, or U

if goal has $\langle E, - \rangle$
then apply N
else apply P, I, or U

$\updownarrow$

if goal has $\langle E, + \rangle$
then apply P if possible
else apply N, I, or U

<u>`if`</u> **goal has** $\langle E, - \rangle$

<u>`then`</u> apply N if possible

<u>`else`</u> apply P, I, or U

$\updownarrow$

<u>`if`</u> **goal has** $\langle E, + \rangle$

<u>`then`</u> apply P if possible

<u>`else`</u> apply N, I, or U

ordinary focusing $^{-}\longleftrightarrow^{+}$ **maximally positive**

ordinary focusing $^{-}\longleftrightarrow^{+}$ **maximally positive**

</2>

proof theory politics

“Bureaucracy of syntax

— Jean-Yves Girard
(proof nets)

“Bureaucracy of syntax

“[…] inessential […]

— Jean-Yves Girard
(proof nets)

“Bureaucracy of syntax

“[…] inessential […]

“[…] 75% of a cut-elimination proof is devoted to endless commutations of rules.”

— Jean-Yves Girard
(proof nets)

“It is embarrassing for proof theory that the natural question of *‘when are two proofs to be considered identical?’* lacks good answers [...]

— Alessio Guglielmi
(web-page)

“It is embarrassing for proof theory that the natural question of *‘when are two proofs to be considered identical?’* lacks good answers [...]

“I believe that *deductive nets* will be the right place where to work.”

— Alessio Guglielmi
(web-page)

“The focussing version […] is a good setting for studying *sequential models* […]

— Samson Abramsky
(concurrent game semantics)

“The focussing version […] is a good setting for studying *sequential models* […]

The *concurrent approach* […] is close to Geometry of Interaction and Proof nets.”

— Samson Abramsky
(concurrent game semantics)

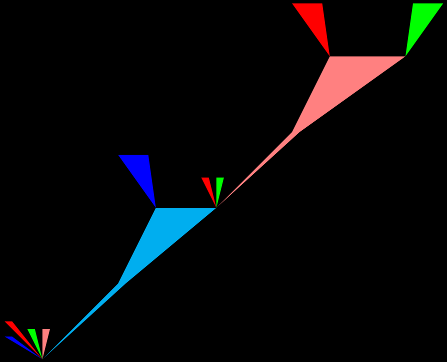
“Down with bureaucracy of syntax!”

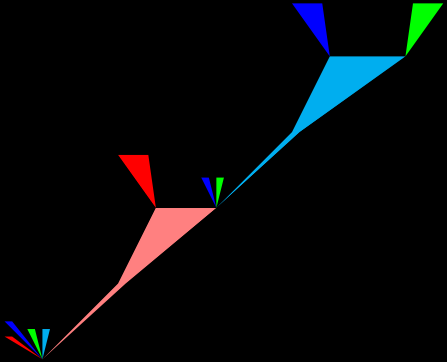
— Philip Wadler (patterns)

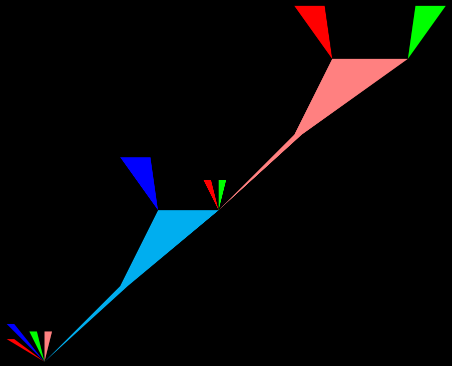
$$\dfrac{\dfrac{}{\vdash a^{\perp}, a}\,\mathbf{I} \qquad \dfrac{\dfrac{}{\vdash b^{\perp}, b}\,\mathbf{I} \qquad \dfrac{}{\vdash \mathbf{1}}\,\mathbf{P}}{\vdash b, \mathbf{1}, b^{\perp} \otimes \perp}\,\mathbf{P}}{\vdash a, b, \mathbf{1}, b^{\perp} \otimes \perp, a^{\perp} \otimes \perp}\,\mathbf{P}$$

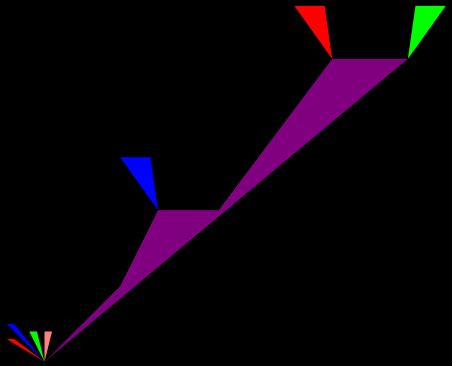
$$
\dfrac{\overset{\mathcal{D}}{\vdash a^{\perp}, \Gamma} \qquad \dfrac{\overset{\mathcal{E}}{\vdash b^{\perp}, \Delta} \qquad \overset{\mathcal{F}}{\vdash \Omega}}{\vdash \Delta, \Omega, b^{\perp} \otimes \perp}\,\mathbf{P}}{\vdash \Gamma, \Delta, \Omega, b^{\perp} \otimes \perp, a^{\perp} \otimes \perp}\,\mathbf{P}
$$

$$
\longrightarrow \qquad \dfrac{\overset{\mathcal{E}}{\vdash b^{\perp}, \Delta} \qquad \dfrac{\overset{\mathcal{D}}{\vdash a^{\perp}, \Gamma} \qquad \overset{\mathcal{F}}{\vdash \Omega}}{\vdash \Gamma, \Omega, a^{\perp} \otimes \perp}\,\mathbf{P}}{\vdash \Gamma, \Delta, \Omega, b^{\perp} \otimes \perp, a^{\perp} \otimes \perp}\,\mathbf{P}
$$









$$\dfrac{\dfrac{}{\vdash a^{\perp}, a}\,\mathbf{I} \qquad \dfrac{}{\vdash b^{\perp}, b}\,\mathbf{I} \qquad \dfrac{}{\vdash \mathbf{1}}\,\mathbf{P}}{\vdash a, b, \mathbf{1}, b^{\perp} \otimes \perp, a^{\perp} \otimes \perp}\,\mathbf{P} = \mathbf{P}, \mathbf{P}$$

maximal positive	**=**	**maximal multi-focus**

maximal positive	**=**	**maximal multi-focus**
	=	**canonical**

in bijection with proof-nets

in bijection with proof-nets

where known

yes

yes,
we can have
canonical sequent proofs

yes,
we can have
canonical sequent proofs

</3>

</talk>