Decidable and Undecidable Fragments of Halpern and Shoham's Interval Temporal Logic: Towards a Complete Classification

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[a] Guido Sciavicco was co-financed by the Spanish projects TIN 2006-15460-C04-01 and PET 2006_0406.

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- We focus on binary relations (i.e., unary modal operators).

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- 2005,2007: Bresolin, Goranko, Montanari and Sciavicco present the first decidable fragment (PNL), generating a natural question about whether is it possible to establish a complete classification of all fragments;

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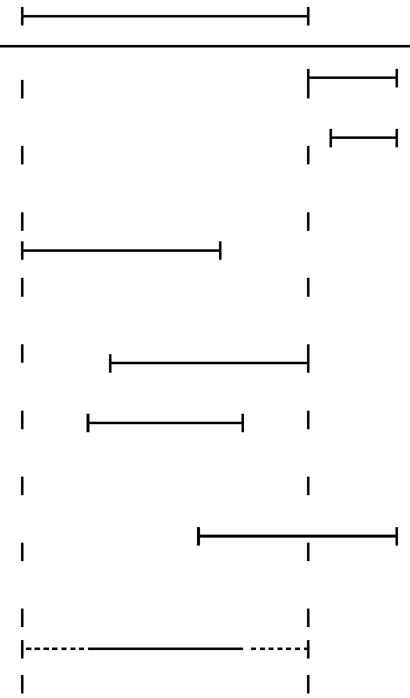
- Now: we present a partial classification of the over 5000 different fragments, narrowing down the 'unknown' territory.

Relations and Semantics

Op.	Semantics
$\langle A \rangle$	$\mathbf{M}, [a,b] \Vdash \langle A \rangle \phi \Leftrightarrow \exists c (b < c . \mathbf{M}, [b,c] \Vdash \phi)$
$\langle L \rangle$	$\mathbf{M}, [a,b] \Vdash \langle L \rangle \phi \Leftrightarrow \exists c, d (b < c < d . \mathbf{M}, [c,d] \Vdash \phi)$
$\langle B \rangle$	$\mathbf{M}, [a,b] \Vdash \langle B \rangle \phi \Leftrightarrow \exists c (a \leq c < b . \mathbf{M}, [a,c] \Vdash \phi)$
$\langle E \rangle$	$\mathbf{M}, [a,b] \Vdash \langle E \rangle \phi \Leftrightarrow \exists c (a < c \leq b . \mathbf{M}, [c,b] \Vdash \phi)$
$\langle D \rangle$	$\mathbf{M}, [a,b] \Vdash \langle D \rangle \phi \Leftrightarrow \exists c, d (a < c \leq d < b . \mathbf{M}, [c,d] \Vdash \phi)$
$\langle O \rangle$	$\mathbf{M}, [a,b] \Vdash \langle O \rangle \phi \Leftrightarrow \exists c, d (a < c \leq b < d . \mathbf{M}, [c,d] \Vdash \phi)$
$\langle D \rangle_{\sqsubset}$	$\mathbf{M}, [a,b] \Vdash \langle D \rangle_{\sqsubset} \phi \Leftrightarrow \exists c, d (a \leq c \leq d \leq b . \mathbf{M}, [c,d] \Vdash \phi \wedge [c,d] \neq [a,b])$



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- . . . but we have possibility of narrowing this number by using the inter-definability of operators, such as in the cases of $p = \langle A \rangle \langle A \rangle p$, or $\langle D \rangle p = \langle B \rangle \langle E \rangle p$.

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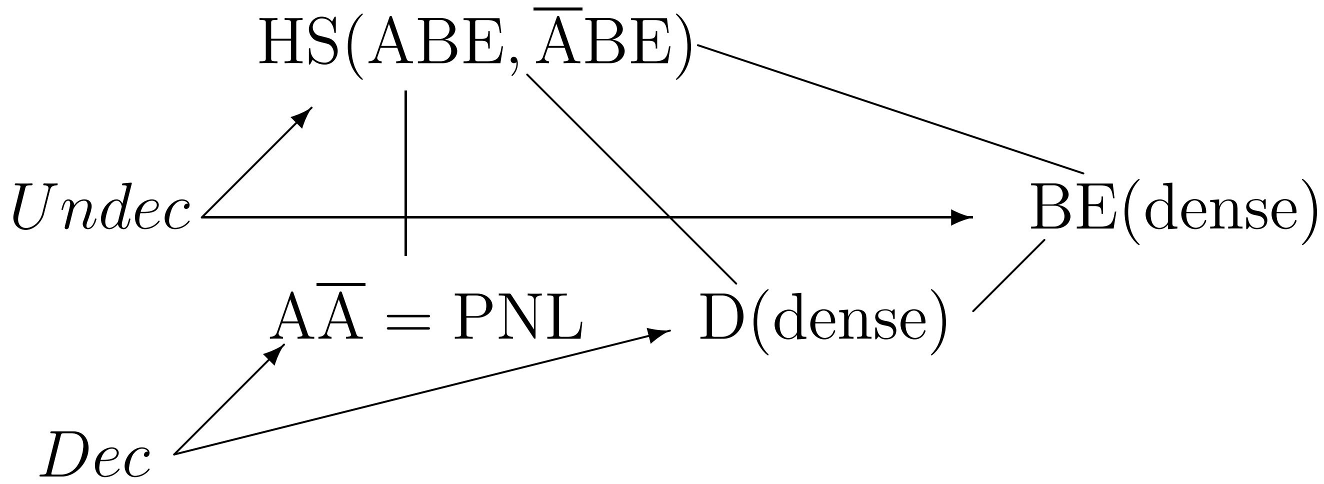
- Here we are particularly interested in undecidable fragments, so we aim to consider the smallest possible fragments;
- For the sake of simplicity, we now consider only the class of all linearly ordered sets, in the original, non-strict semantics, that is, including point-intervals.

An Overview

- A possible way to look at the variety of fragments to be classified is as follows:



Some New Undecidability Results

- We showed last year that are undecidable:

$$\mathsf{A\overline{A}B\overline{E}},\quad \mathsf{A\overline{A}E\overline{B}},\quad \mathsf{A\overline{A}D^*}$$

where $\mathsf{D}^* \in \{\mathsf{D}, \overline{\mathsf{D}}, \mathsf{D}_\sqsubset, \overline{\mathsf{D}}_\sqsubset\}$, and in this paper we add

$$\mathsf{AD^*E},\quad \mathsf{AD^*\overline{E}},\quad \mathsf{AD^*\overline{O}},\quad \mathsf{\overline{A}D^*B},\quad \mathsf{\overline{A}D^*\overline{B}},\quad \mathsf{\overline{A}D^*O}$$

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- The first and the second group differ for the technique that has been used to achieve the result.

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- There is, anyway, some interesting fragment for which we cannot even guess its decidability/undecidability, such as AB;
- Now, we give an idea of the techniques we used.

The Idea - 1

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- This problem can be shown to be undecidable by a simple application of the König's Lemma in the same way as it was used to show the undecidability of the $\mathbb{N} \times \mathbb{N}$ tiling problem from that of $\mathbb{Z} \times \mathbb{Z}$;

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- By such a reduction, we prove R.E.-hardness of the validity problem;

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- The propositional letters $t_{i,j}$ represent tiles:

$$
\begin{aligned}
B_1 = \quad & \neg \mathsf{u} \wedge \langle A \rangle \mathsf{u} \wedge [G](\mathsf{u} \rightarrow \\
& (\neg \pi \wedge \langle A \rangle \mathsf{u} \wedge \neg \langle D \rangle \mathsf{u} \wedge \neg \langle D \rangle \langle A \rangle \mathsf{u})), \\
B_2 = \quad & [G] \textstyle\bigwedge_{p \in \mathcal{AP}}((p \vee \langle A \rangle p) \rightarrow \langle A \rangle \mathsf{u}).
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The Idea - 3

- Tiles are placed over unit intervals, there are never two different tiles over the same unit, and the special symbol * distinguishes one level from the next one:

$$B_3 = \quad [G](\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge [G](* \rightarrow \neg\mathsf{tile}) \wedge [G]\neg(* \wedge \langle A \rangle *),$$
$$B_4 = \quad [G](\mathsf{tile} \leftrightarrow (\bigvee_{i=1}^{k} \mathtt{t_i} \wedge \bigwedge_{i,j=1,i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j}))).$$

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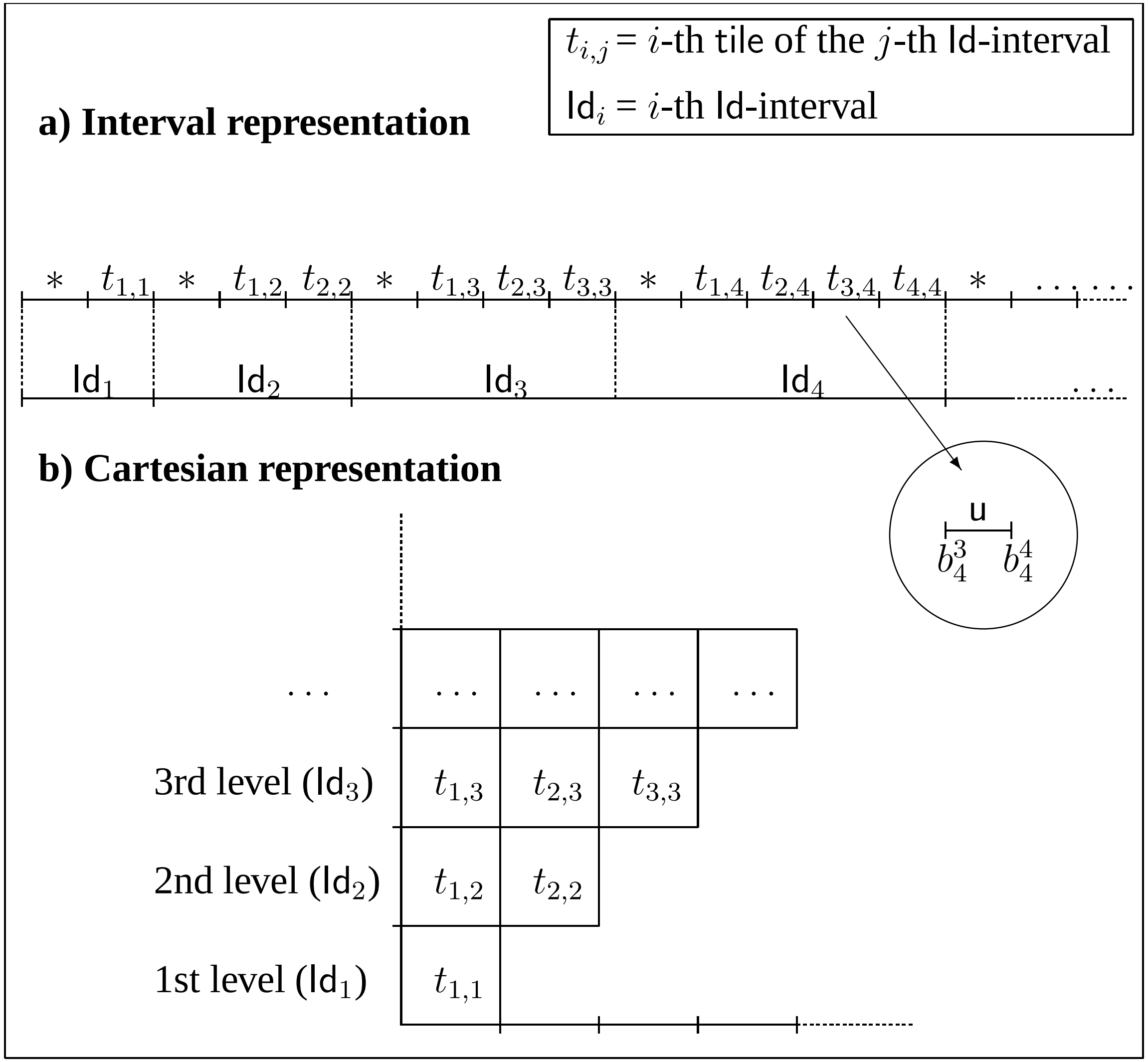
- Ids are collections of tiles separated by exactly one *:

$$B_5 = [G]((\mathsf{Id} \rightarrow (\neg\mathsf{u} \wedge \langle A\rangle\mathsf{Id} \wedge \neg\langle D\rangle\langle A\rangle\mathsf{Id})))\wedge$$
$$[G](\langle A\rangle\mathsf{Id} \leftrightarrow \langle A\rangle *),$$
$$B_6 = \langle A\rangle(* \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle *)),$$
$$B_7 = B_1 \wedge B_2 \wedge B_3 \wedge B_4 \wedge B_5 \wedge B_6.$$

The Idea -4


$t_{i,j}$ = i-th tile of the j-th ld-interval
ld_i = i-th ld-interval
a) Interval representation
$*$ $t_{1,1}$ $*$ $t_{1,2}$ $t_{2,2}$ $*$ $t_{1,3}$ $t_{2,3}$ $t_{3,3}$ $*$ $t_{1,4}$ $t_{2,4}$ $t_{3,4}$ $t_{4,4}$ $*$ . . . . . .
ld_1 ld_2 ld_3 ld_4 . . .
u
b_4^3 b_4^4
b) Cartesian representation
. . .
3rd level (ld_3) $t_{1,3}$ $t_{2,3}$ $t_{3,3}$
2nd level (ld_2) $t_{1,2}$ $t_{2,2}$
1st level (ld_1) $t_{1,1}$

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- We codify this relation by means of three propositional letters, namely bb (from the beginning point of a tile to the beginning point of the corresponding tile above), be, (beginning - ending), and eb (ending - beginning);

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- We codify this relation by means of three propositional letters, namely bb (from the beginning point of a tile to the beginning point of the corresponding tile above), be, (beginning - ending), and eb (ending - beginning);
- This helps us to formalize the intended properties of the "above connection" relation by means of a weak language.

The Idea - 6

$$
\begin{aligned}
B_8 &= [G]((\mathsf{bb} \vee \mathsf{be} \vee \mathsf{eb}) \leftrightarrow \mathsf{corr}),\\
B_9 &= [G]\neg(\mathsf{corr} \wedge \mathsf{ld}),\\
B_{10} &= [G]((\mathsf{corr} \rightarrow \neg\langle D\rangle\mathsf{ld}) \wedge (\mathsf{ld} \rightarrow \neg\langle D\rangle\mathsf{corr})),\\
B_{11} &= [G]((\mathsf{corr} \rightarrow \neg\langle A\rangle\mathsf{ld}) \wedge (\langle A\rangle(\mathsf{bb} \vee \mathsf{be}) \rightarrow \neg\langle A\rangle\mathsf{ld})),\\
B_{13} &= [G](\langle A\rangle\mathsf{tile} \leftrightarrow \langle A\rangle\mathsf{bb}),\\
B_{14} &= [A](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}) \leftrightarrow \langle E\rangle\mathsf{bb}),\\
B_{15} &= [G](\langle A\rangle\mathsf{tile} \leftrightarrow \langle A\rangle\mathsf{be}),\\
B_{16} &= [A]((\langle E\rangle\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}) \leftrightarrow \langle E\rangle\mathsf{be}),\\
B_{17} &= [G](\mathsf{u} \rightarrow (\mathsf{tile} \leftrightarrow \langle A\rangle\mathsf{eb})),\\
B_{18} &= [A](\langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}) \leftrightarrow \langle E\rangle\mathsf{eb}),
\end{aligned}
$$

The Idea - 6 (Cont'd)

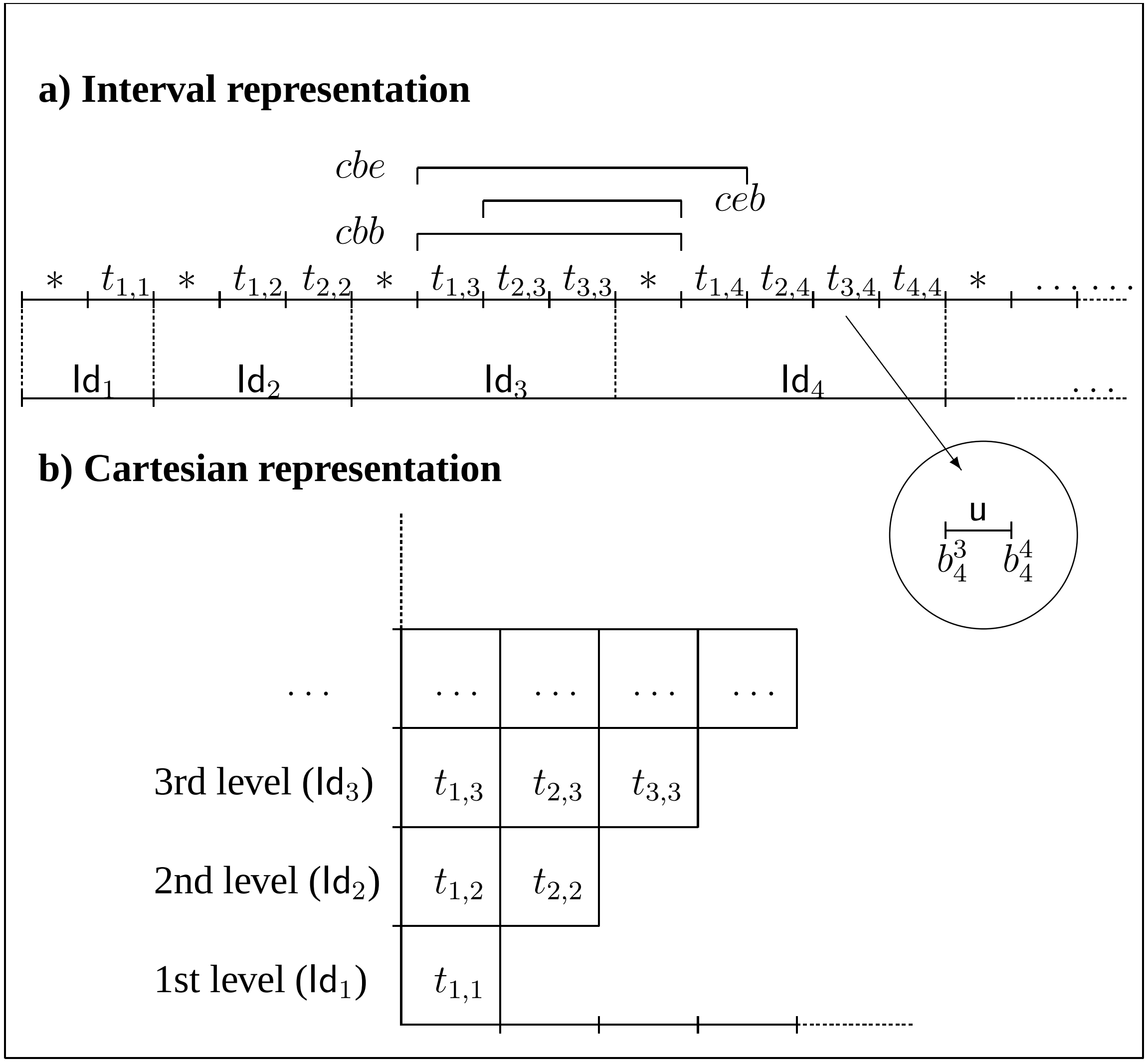
$$B_{20} = [G] \bigwedge_{c,c' \in \{\mathsf{bb},\mathsf{eb},\mathsf{be}\}, c \neq c'} \neg(c \wedge c'),$$

$$B_{21} = [G](\mathsf{bb} \to \neg\langle D\rangle\mathsf{bb} \wedge \neg\langle D\rangle\mathsf{eb} \wedge \neg\langle D\rangle\mathsf{be}),$$

$$B_{22} = [G](\mathsf{eb} \to \neg\langle D\rangle\mathsf{bb} \wedge \neg\langle D\rangle\mathsf{eb} \wedge \neg\langle D\rangle\mathsf{be}),$$

$$B_{23} = [G](\mathsf{be} \to \langle D\rangle\mathsf{eb} \wedge \neg\langle D\rangle\mathsf{bb} \wedge \neg\langle D\rangle\mathsf{be}),$$

The Idea -7


a) Interval representation
cbe
ceb
cbb
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- The "right correspondence" relation is simply the *meets* operator;
- The fundamental property is the commutativity of these two relations! So, we have that

$$[G]((\mathsf{tile} \wedge \langle A \rangle \mathsf{tile}) \longrightarrow \bigvee_{right(t_i)=left(t_j)}(\mathtt{t_i} \wedge \langle A \rangle \mathtt{t_j})),$$
$$[G](\langle A \rangle \mathsf{tile} \longrightarrow \bigvee_{up(t_i)=down(t_j)}(\langle A \rangle \mathtt{t_i} \wedge \langle A \rangle (\mathsf{bb} \wedge \langle A \rangle \mathtt{t_j}))).$$

encode exactly the Octant Tiling Problem.

More Considerations

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- We were recently able to narrow the dimension of the fragment, obtaining, for example, the undecidability of AD alone;
- This requires more than 50 formulas;
- Besides the results in themselves, we find this interesting as an expressivity exercise, which turns out to be useful when we apply interval logics to practical tasks.

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- Possibly, the main side-product of this classification will be the identification of more expressive decidable fragments, finally closing a 20-years-old open question;
- It is also worth noticing that the decidable fragments that have been found so far not only were not expected, but also the techniques used to show decidability are technically interesting.

A Partial Classification

K	$\overline{\mathrm{B}}\mathrm{E}$	$\overline{\mathrm{BE}}$	$\mathrm{A}\overline{\mathrm{A}}\mathrm{D}^*$	$\mathrm{AD}^*\mathrm{E}$	$\mathrm{AD}^*\overline{\mathrm{O}}$	$\overline{\mathrm{A}}\mathrm{D}^*\mathrm{B}$	$\overline{\mathrm{A}}\mathrm{D}^*\mathrm{O}$	$\mathrm{A}\overline{\mathrm{A}}$	D	$\mathrm{B}\overline{\mathrm{B}}$
Lin	Und	Und	Und	Und	Und	Und	Und	Dec	?	Dec
Den	Und	Und	Und	Und	Und	Und	Und	Dec	Den	Dec
Dis	Und	Und	Und	Und	Und	Und	Und	Dec	?	Dec

- Classes containing only infinite ascending/descending unbounded chains are omitted, as well as fragments obtained by the above ones by symmetry. It basically makes no difference to assume that point-intervals are included/excluded, and, when are included and the language does not allow to express a modal constant to capture them, we did not find differences when such a constant is included or not in the language explicitly.

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K	$\overline{\mathrm{B}}\mathrm{E}$	$\overline{\mathrm{BE}}$	$\mathrm{A}\overline{\mathrm{A}}\mathrm{D}^*$	$\mathrm{AD}^*\mathrm{E}$	$\mathrm{AD}^*\overline{\mathrm{O}}$	$\overline{\mathrm{A}}\mathrm{D}^*\mathrm{B}$	$\overline{\mathrm{A}}\mathrm{D}^*\mathrm{O}$	$\mathrm{A}\overline{\mathrm{A}}$	D	$\mathrm{B}\overline{\mathrm{B}}$
Lin	Und	Und	Und	Und	Und	Und	Und	Dec	?	Dec
Den	Und	Und	Und	Und	Und	Und	Und	Dec	Den	Dec
Dis	Und	Und	Und	Und	Und	Und	Und	Dec	?	Dec

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K	$\overline{\mathrm{B}}\mathrm{E}$	$\overline{\mathrm{BE}}$	$\mathrm{A}\overline{\mathrm{A}}\mathrm{D}^*$	$\mathrm{AD}^*\mathrm{E}$	$\mathrm{AD}^*\overline{\mathrm{O}}$	$\overline{\mathrm{A}}\mathrm{D}^*\mathrm{B}$	$\overline{\mathrm{A}}\mathrm{D}^*\mathrm{O}$	$\mathrm{A}\overline{\mathrm{A}}$	D	$\mathrm{B}\overline{\mathrm{B}}$
Lin	Und	Und	Und	Und	Und	Und	Und	Dec	?	Dec
Den	Und	Und	Und	Und	Und	Und	Und	Dec	Den	Dec
Dis	Und	Und	Und	Und	Und	Und	Und	Dec	?	Dec

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