15-462 Computer Graphics I

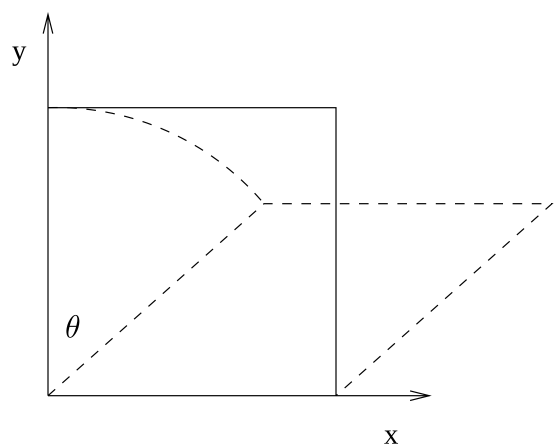
Midterm Examination

Sample Solution

March 5, 2002

1. Linear Transformations (40 pts)

Consider the following *skewing transformation.*



1. (20 pts) Show the 2-dimensional skewing transformation matrix for a given angle θ in homogeneous coordinates. This should be a 3×3 matrix. Explain your reasoning.

   We look at the action of the transformation on the basis vectors and the origin to determine the transformation matrix $\mathbf{S}$*.*

   - *The basis vector* $[1\ 0\ 0]^T$ *remains unchanged:*

   $$\mathbf{S}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

   - *The basis vector* $[0\ 1\ 0]^T$ *is rotated:*

   $$\mathbf{S}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\sin(\theta)\\\cos(\theta)\\0\end{bmatrix}$$

- *The origin remains unchanged:*

$$\mathbf{S}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore

$$\mathbf{S} = \begin{bmatrix} 1 & \sin(\theta) & 0 \\ 0 & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. (20 pts) Show how the skewing transformation can be represented as the composition of a scaling and a shearing transformation. Write out the auxiliary transformations explicitly as matrices and verify that the composition yields the skewing matrix from part 1.

First we scale along the y-direction with a factor of $\cos(\theta)$*:*

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we shear along the x-direction. The shear factor is $\tan(\theta) = \cot(90-\theta)$.

$$\mathbf{M}_2 = \begin{bmatrix} 1 & \tan(\theta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we verify

$$\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} 1 & \tan(\theta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \sin(\theta) & 0 \\ 0 & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{S}$$

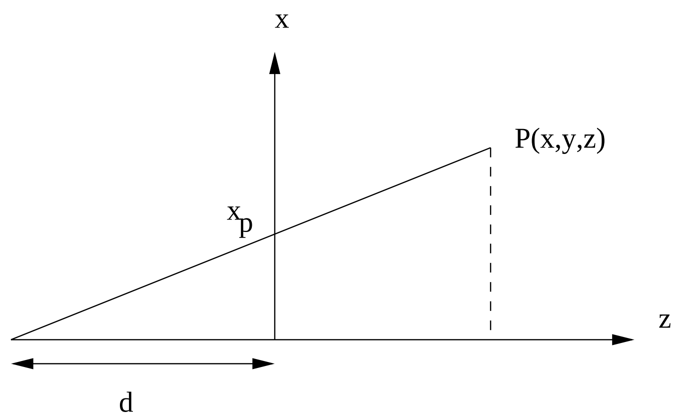
since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

2. Projections (30 pts)

In the textbook, the perspective projection matrix is given for the center of projection at the origin and the projection plane at $z = -d$ for a given distance d. In this problem we will develop a different perspective projection matrix the clarifies the relation between orthogonal and perspective projections. Your answers should be 4×4 matrices in homogeneous coordinates.

1. (20 pts) Give the perspective projection matrix with the center of projection at $x = 0, y = 0, z = d$ and the projection plane $z = 0$. Draw a picture to aid your reasoning.

   We draw the figure only for x (y is analogous).



From the picture we see that

$$\frac{x_p}{d} = \frac{x}{z+d}, \quad \frac{y_p}{d} = \frac{y}{z+d}, \quad z_p = 0.$$

This means

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{(z/d)+1} \\ \frac{y}{(z/d)+1} \\ 0 \\ 1 \end{bmatrix} = \frac{1}{(z/d)+1} \begin{bmatrix} x \\ y \\ 0 \\ (z/d)+1 \end{bmatrix}$$

Hence we obtain the projection matrix below and verify

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ (z/d)+1 \end{bmatrix} = w \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix}$$

for $w = (z/d) + 1$.

2. (10 pts) Give the orthogonal projection matrix onto the plane $z = 0$ and verify that we obtain the orthogonal projection matrix as the limit of the perspective projection matrix as d goes to infinity.

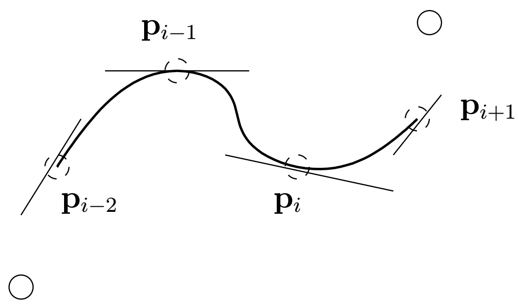
   The orthogonal projection satisfies $x_p = x$, $y_p = y$ and $z_p = 0$ and therefore has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   This is indeed the limit of the projection matrix from part (1) as d goes to infinity since $1/d$ goes to 0.

3. Splines (30 points)

In this problem with explore Catmull-Rom splines. In two dimension, they are guaranteed to interpolate the interior m points, given $m+2$ control points. Besides interpolation, we require that the tangent vector at each interior control point $\mathbf{p}_k$ is the average of the vectors from $\mathbf{p}_{k-1}$ to $\mathbf{p}_k$ and from $\mathbf{p}_k$ to $\mathbf{p}_{k+1}$.



1. (20 pts) Set up 4 equations that determine the Catmull-Rom geometry matrix, assuming we are trying to draw the segment from $\mathbf{p}_{i-1}$ to $\mathbf{p}_i$. For each, briefly note the geometric origin of the equation. You do not have to solve your equations.

   Recall that

   $$\begin{aligned} \mathbf{p}(u) &= \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 u^2 + \mathbf{c}_3 u^3, \\ \mathbf{p}'(u) &= \mathbf{c}_1 + 2\mathbf{c}_2 u + 3\mathbf{c}_3 u^2. \end{aligned}$$

   So we obtain

   $$\begin{array}{rclcll} \mathbf{p}(0) &=& \mathbf{p}_{i-1} &=& \mathbf{c}_0 & \textit{left end-point} \\ \mathbf{p}(1) &=& \mathbf{p}_i &=& \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 & \textit{right end-point} \\ \mathbf{p}'(0) &=& \frac{\mathbf{p}_i - \mathbf{p}_{i-2}}{2} &=& \mathbf{c}_1 & \textit{tangent at left end-point} \\ \mathbf{p}'(1) &=& \frac{\mathbf{p}_{i+1} - \mathbf{p}_{i-1}}{2} &=& \mathbf{c}_1 + 2\mathbf{c}_2 + 3\mathbf{c}_3 & \textit{tangent at right end-point} \end{array}$$

2. (5 pts) Explain to what extent Catmull-Rom splines allow local control.

   Moving a point will affect the tangent at the two adjacent points, and therefore the curve in two adjacent segments in both directions, but not beyond.

3. (5 pts) Do Catmull-Rom splines have the convex hull property?

   No. Even the given example violates the convex hull property at $\mathbf{p}_i$.