McKay Tantalizer Algebras

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Summary

For each finite subgroup G of the special unitary group SU_2 and each $k \in \mathbb{N}$, we construct an algebra C_k^G, the centralizer of G on the tensor space $V^{\otimes k}$. We find that these algebras have beautiful combinatorics vis-à-vis walks on their Dynkin diagrams.

1. Background

The special unitary group of order 2, SU_2, is the group of 2 × 2 unitary matrices with determinant 1,

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\}$$

SU_2 acts naturally on the vector space $V = \mathbb{C}^2$ by matrix multiplication. If $g \in SU_2$ and $v \in V$, then gv is the product of the 2 × 2 matrix g and the 2 × 1 column vector v.

Let $v_0 = \binom{1}{0}$ and $v_1 = \binom{0}{1}$ form the standard basis for V. For $k > 0$, we define $V^{\otimes k}$ to be the 2^k-dimensional complex vector space with basis vectors

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}, \qquad i_j \in \{0,1\}.$$

We extend the action of SU_2 on V to an action on $V^{\otimes k}$ by

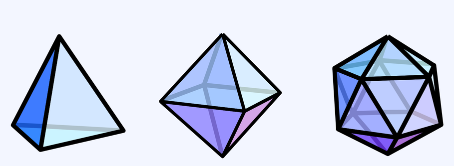
$$g(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = (gv_{i_1}) \otimes (gv_{i_2}) \otimes \cdots \otimes (gv_{i_k})$$

We now define the **tantalizer algebra** C_k^G (tensor power centralizer) to be the set of all endomorphisms that commute with G on $V^{\otimes k}$.

2. Finite subgroups of SU_2

There are infinitely many finite subgroups of SU_2, but they can be classified into five types:

- The cyclic groups
  There are infinitely many of these. A cyclic group can be generated by a single element g. Taking a high enough power of g gets us back to the identity—hence the term *cyclic*.
- The dicyclic groups
  There are infinitely many of these: one with 4 elements, one with 8, one with 12, and so on. The name comes from the fact that the binary dihedral group of $4m$ elements double-covers (whence *binary*) the symmetry group of a regular n-gon (whence *dihedral*).
- The binary tetrahedral group
- The binary octahedral group
- The binary icosahedral group



We call these three lonely groups the "exceptional cases" because they are so few. The binary tetrahedral group is so named because it double-covers the group of rotational symmetries of a regular tetrahedron, and likewise for the binary octahedral and binary icosahedral groups.

3. Research questions

Each tantalizer algebra C_k^G is a finite-dimensional vector space. Here are some questions we were interested in:

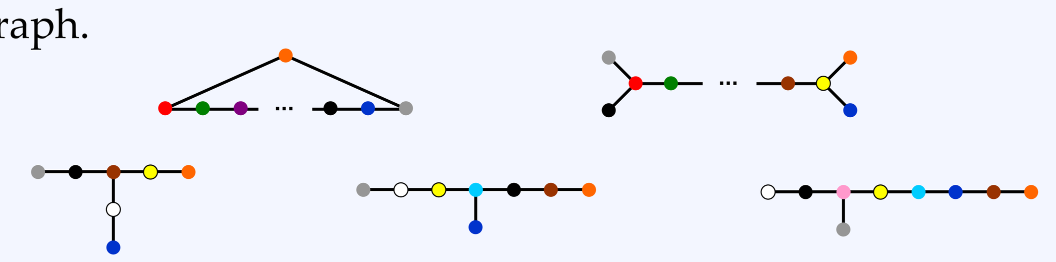
1. What is its dimension?
2. What is a basis for it?
3. If we look at the tantalizer algebras for different values of k, what patterns emerge?
4. If we look at the tantalizers for different groups of the same kind (e.g., different binary dihedral groups), what patterns emerge?
5. What patterns hold for all finite subgroups of SU_2 and for all values of k?
6. Is there a simple, unifying description of the tantalizer that will work for all finite subgroups of SU_2 and for all values of k?

How did we do?

We can answer question 1 for all the tantalizer algebras.
We can answer questions 1–4 for all but the exceptional cases.
Questions 5–6 are very hard and suggest directions for future research.

4. The McKay correspondence

The **McKay correspondence** is a celebrated proposition about the finite subgroups of SU_2. It associates to each subgroup $G \subseteq SU_2$ a unique affine, simply laced Dynkin graph.



Under the correspondence, vertices in the Dynkin graph correspond to irreducible representations of G, and an edge between vertices V_a and V_b indicates that V_a appears in $V \otimes V_b$.

We observed that the dimension of the tantalizer algebra C_k^G equals the number of paths of lengths $2k$ on the Dynkin graph of G that begin and end at the vertex corresponding to the trivial representation. It thus seems natural to seek a bijection

closed walks of length $2k$ at a vertex of the Dynkin graph of G
↔ elements of a basis of the tantalizer algebra of G

for each group G.

5. The cyclic groups

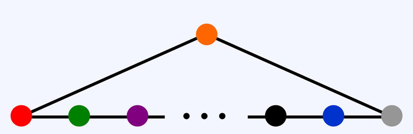
The cyclic group of order m, $\mathbf{C}_m$, is the subgroup of SU_2 generated by the single element

$$g_m = \begin{pmatrix} \omega_m & 0 \\ 0 & \omega_m^{-1} \end{pmatrix},$$

where $\omega_m = e^{2\pi i/m}$. The elements are

$$\mathbf{C}_m = \{1, g_m, g_m^2, \ldots, g_m^{m-1}\}.$$

Note that $g_m^m = 1$. Via the McKay correspondence, we associate to $\mathbf{C}_m$ the Dynkin graph

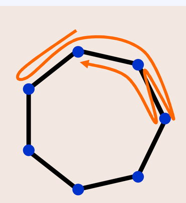


with m vertices.

We can completely describe the tantalizer algebra for all cyclic groups and for all values of k. We do so by exhibiting a bijection

closed walks of length $2k$ at a vertex of the Dynkin graph of $\mathbf{C}_m$
↔ elements of a basis of the tantalizer algebra of $\mathbf{C}_m$

Here, we illustrate it by an example.



Start with a walk of length $2k$ that starts and ends at a particular vertex of the Dynkin graph.
Here $k = 4$. This is the Dynkin graph of the cyclic group of order 9.

LRRRLRLL — Write it down as a sequence of lefts and rights.

RLLL / LRLL — Cut the walk in half and flip the first half.
Change lefts to rights and rights to lefts.

$v_1 \otimes v_0 \otimes v_0 \otimes v_0 \mapsto v_0 \otimes v_1 \otimes v_0 \otimes v_0$
everything else $\mapsto 0$

Now you can read the answer off as an endomorphism on $V^{\otimes k}$.
Change lefts to v_0 and rights to v_1.

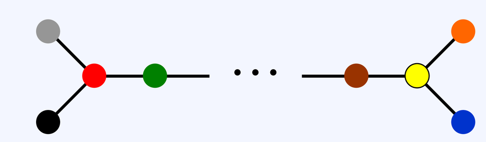
Example: Mapping a walk on a Dynkin graph to an endomorphism on $V^{\otimes k}$ that commutes with $\mathbf{C}_m$.

6. The dicyclic groups

The dicyclic group of order $4m$, $\mathbf{D}_m$, is the subgroup of SU_2 generated by the elements

$$g_m = \begin{pmatrix} \omega_{2m} & 0 \\ 0 & \omega_{2m}^{-1} \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

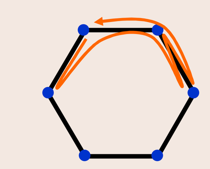
Via the McKay correspondence, we associate to $\mathbf{D}_m$ the Dynkin graph



with $m + 3$ vertices.

We can completely describe the tantalizer algebra for all dicyclic groups and for all values of k. This time, we do so by exhibiting a bijection

walks of length $2k$ at a vertex of the cyclic graph of $2m$ nodes
in which the first step is counterclockwise
↔ elements of a basis of the tantalizer algebra of $\mathbf{D}_m$



Start with a walk of length $2k$ between two adjacent vertices in the appropriate cyclic graph.
Here $k = 4$ and G is the binary dihedral group of 12 elements.

LRRRLRLL — Write it down as a sequence of lefts and rights.

RLLL / LRLL
LRRR / RLRR

Now cut the walk in half and flip each half individually;
i.e., we will get two pairs of half walks, one with the first half flipped and one with the second flipped.

$v_1 \otimes v_0 \otimes v_0 \otimes v_0 \mapsto v_0 \otimes v_1 \otimes v_0 \otimes v_0$
$v_0 \otimes v_1 \otimes v_1 \otimes v_1 \mapsto v_1 \otimes v_0 \otimes v_1 \otimes v_1$
everything else $\mapsto 0$

Then you can read the answer off as an endomorphism on $V^{\otimes k}$.

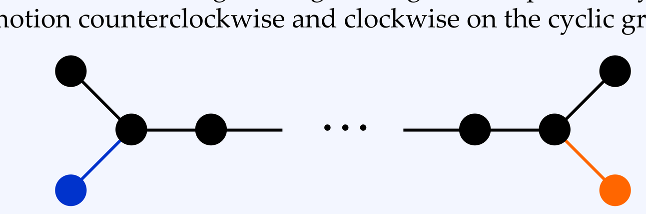
Example: Mapping a walk on a Dynkin graph to an endomorphism on $V^{\otimes k}$ that commutes with $\mathbf{D}_m$.

We also have a bijection

walks of length $2k$ at a vertex of the cyclic graph of $2m$ nodes
in which the first step is counterclockwise
↔ closed walks of length $2k$ at the trivial vertex of the Dynkin graph of $\mathbf{D}_m$

This bijection is described graphically below. Here $\kappa_0, \kappa_1, \ldots, \kappa_{2m-1}$ are the vertices of the cyclic graph.

"Main pathway"
Motion left and right along this segment maps directly to motion counterclockwise and clockwise on the cyclic graph.



Second branch
Walking onto and off of this branch in the Dynkin graph corresponds to walking between κ_m and κ_{m+2} (via κ_{m+1}) on the cyclic graph.

First branch
Walking onto and off of this branch in the Dynkin graph corresponds to walking from κ_0 to κ_1 to κ_2 on the cyclic graph.

Walking onto either branch also reverses the orientation of the main pathway.
If left meant counterclockwise before, it now means clockwise, etc.

Combining these results yields a bijection

closed walks of length $2k$ at a vertex of the Dynkin graph of $\mathbf{D}_m$
↔ elements of a basis of the tantalizer algebra of $\mathbf{D}_m$

7. The binary polyhedral groups

There are three groups that do not fit into the cyclic or dicyclic categories:

- **T**, the binary tetrahedral group (order 24);
- **O**, the binary octahedral group (order 48); and
- **I**, the binary icosahedral group (order 120).

We cannot yet describe the tantalizer algebras of these exceptional groups in great detail. However, we do know the dimensions of the algebras for certain, and we have conjectures about their structure.

The tools we use to prove the dimensions of the algebras are **Bratteli diagrams**, which are used to study chains of semisimple algebras.

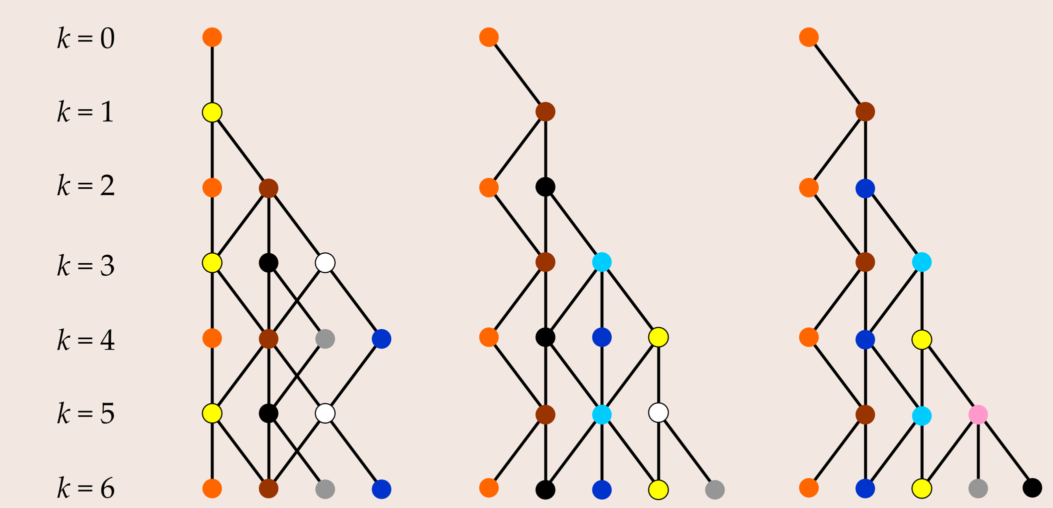


Figure: The Bratteli diagrams of **T**, **O**, and **I**. Bratteli diagrams are trees of infinite depth; only the first few levels are shown.

In a Bratteli diagram for a chain of algebras $A_0 \subseteq A_1 \subseteq \ldots$, the vertices at depth k are identified with the irreducible representations of A_k. The number of edges between a vertex v at depth k and a vertex w at depth $k+1$ is the number of times the representation v occurs in the restriction of the representation w to A_k.

Here, the algebra chain is $C_0^G \subseteq C_1^G \subseteq C_2^G \subseteq \ldots$.

This diagram is useful because it is known that if we label the root as 1 and all the other vertices with the sums of the labels of their parents (like Pascal's triangle), then dim A_k is the sum of the squares of the labels in row k. Using this theorem and some algebra, we proved that

$$\dim C_k^{\mathbf{T}} = \frac{4^{k-1} + 2}{3}$$

$$\dim C_k^{\mathbf{O}} = \frac{4^k + 6 \cdot 2^k + 8}{24}$$

$$\dim C_k^{\mathbf{I}} = \frac{4^{k-1} + 3L_{2k} + 5}{15} \qquad \text{where } L_k \text{ is the } k\text{th Lucas number.}$$

In addition, we conjecture that for $G = \mathbf{T}, \mathbf{O}, \mathbf{I}$, C_k^G is generated by the permutation S_k along with a single other element b_G. This element b_G only appears for sufficiently large k ($k \geq s$, where s is the position from the right where the Dynkin diagram branches).

Similarly, we conjecture that (for $k > s$) C_{k+1}^G is generated by C_k^G along with s_k, the transposition that switches k and $k + 1$.

This poster is available in electronic form at http://macalester.edu/~jbarnes/nola.pdf

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