McKay Centralizer Algebras

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Abstract

For a finite subgroup G of the special unitary group SU_2, we study the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$ of G acting on the k-fold tensor product of its defining representation $\mathsf{V} = \mathbb{C}^2$. These subgroups are in bijection with the simply-laced affine Dynkin diagrams. The McKay correspondence relates the representation theory of these groups to the associated Dynkin diagram, and we use this connection to show that the structure and representation theory of $\mathsf{Z}_k(\mathsf{G})$ as a semisimple algebra is controlled by the combinatorics of the corresponding Dynkin diagram.

Introduction

In 1980, John McKay [Mc] made the remarkable discovery that there is a natural one-to-one correspondence between the finite subgroups of the special unitary group SU_2 and the simply-laced affine Dynkin diagrams, which can be described as follows. Let $\mathsf{V} = \mathbb{C}^2$ be the defining representation of SU_2 and let G be a finite subgroup of SU_2 with irreducible modules $\mathsf{G}^\lambda, \lambda \in \Lambda(\mathsf{G})$. The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (also known as the McKay graph or McKay quiver) has vertices indexed by the $\lambda \in \Lambda(\mathsf{G})$ and $a_{\lambda,\mu}$ edges from λ to μ if G^μ occurs in $\mathsf{G}^\lambda \otimes \mathsf{V}$ with multiplicity $a_{\lambda,\mu}$. Almost a century earlier, Felix Klein had determined that a finite subgroup of SU_2 must be one of the following: (a) a cyclic group $\mathbf{C}_n$ of order n, (b) a binary dihedral group $\mathbf{D}_n$ of order $4n$, or (c) one of the 3 exceptional groups: the binary tetrahedral group $\mathbf{T}$ of order 24, the binary octahedral group $\mathbf{O}$ of order 48, or the binary icosahedral group $\mathbf{I}$ of order 120. McKay's observation was that the representation graph of $\mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$ corresponds exactly to the Dynkin diagram $\hat{\mathsf{A}}_{\mathsf{n}-1}$, $\hat{\mathsf{D}}_{\mathsf{n}+2}$, $\hat{\mathsf{E}}_6$, $\hat{\mathsf{E}}_7$, $\hat{\mathsf{E}}_8$, respectively (see Section 4.1 below).

In this paper, we examine the McKay correspondence from the point of view of Schur-Weyl duality. Since the McKay graph provides a way to encode the rules for tensoring by V, it is natural to consider the k-fold tensor product module $\mathsf{V}^{\otimes k}$ and to study the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$ of endomorphisms that commute with the action of G on $\mathsf{V}^{\otimes k}$. The algebra $\mathsf{Z}_k(\mathsf{G})$ provides essential information about the structure of $\mathsf{V}^{\otimes k}$ as a G-module, as the projection maps from $\mathsf{V}^{\otimes k}$ onto its irreducible G-summands are idempotents in $\mathsf{Z}_k(\mathsf{G})$, and the multiplicity of G^λ in $\mathsf{V}^{\otimes k}$ is the dimension of the $\mathsf{Z}_k(\mathsf{G})$-irreducible module corresponding to λ. The problem of studying centralizer algebras of tensor powers of the natural G-module $\mathsf{V} = \mathbb{C}^2$ for $\mathsf{G} \subseteq \mathsf{SU}_2$ via the McKay correspondence is discussed in [GHJ, 4.7.d] in the general framework of derived towers, subfactors, and von Neumann algebras, an approach not adopted here. Our aim is to develop the structure and representation theory of the algebras $\mathsf{Z}_k(\mathsf{G})$ and to show how they are controlled by the combinatorics of the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (the Dynkin diagram) via double-centralizer theory (Schur-Weyl duality). In particular,

- the irreducible $\mathsf{Z}_k(\mathsf{G})$-modules are indexed by the vertices of $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ which correspond to the irreducible modules G^λ that occur in $\mathsf{V}^{\otimes k}$;
- the dimensions of these modules enumerate walks on $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ of k steps;
- the dimension of $\mathsf{Z}_k(\mathsf{G})$ equals the number of walks of $2k$ steps on $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ starting and ending at the node 0, which corresponds to the trivial G-module and is the affine node of the Dynkin diagram;
- the Bratteli diagram of $\mathsf{Z}_k(\mathsf{G})$ (see Section 4.2) is constructed recursively from $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$; and
- when k is less than or equal to the diameter of the graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$, the algebra $\mathsf{Z}_k(\mathsf{G})$ has generators labeled by nodes of $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$, and the relations they satisfy are determined by the edge structure of $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$.

Since $\mathsf{G} \subseteq \mathsf{SU}_2$, the centralizer algebras satisfy the reverse inclusion $\mathsf{Z}_k(\mathsf{SU}_2) \subseteq \mathsf{Z}_k(\mathsf{G})$. It is well known that $\mathsf{Z}_k(\mathsf{SU}_2)$ is isomorphic to the Temperley-Lieb algebra $\mathsf{TL}_k(2)$. Thus, the centralizer algebras constructed here all contain a Temperley-Lieb subalgebra. The dimension of $\mathsf{TL}_k(2)$ is the Catalan number $\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}$, which counts walks of $2k$ steps that begin and end at 0 on the representation graph of SU_2, i.e. the Dynkin diagram $\mathsf{A}_{+\infty}$, (see (1.3)).

Our paper is organized as follows:

(1) In Section 1, we derive general properties of the centralizer algebras $\mathsf{Z}_k(\mathsf{G})$. Many of these results hold for subgroups G of SU_2 that are not necessarily finite. We study the tower $\mathsf{Z}_0(\mathsf{G}) \subseteq \mathsf{Z}_1(\mathsf{G}) \subseteq \mathsf{Z}_2(\mathsf{G}) \subseteq \cdots$ and show that $\mathsf{Z}_k(\mathsf{G})$ can be constructed from $\mathsf{Z}_{k-1}(\mathsf{G})$ by adjoining generators that are (essential) idempotents; usually there is just one except when we encounter a branch node in the graph. By using the Jones basic construction, we develop a procedure for constructing idempotent generators of $\mathsf{Z}_k(\mathsf{G})$ inspired by the Jones-Wenzl idempotent construction.

(2) Section 2 examines the special case that G is the cyclic subgroup $\mathbf{C}_n$. In Theorems 2.7 and 2.16 we present dimension formulas for $\mathsf{Z}_k(\mathbf{C}_n)$ and for its irreducible modules and explicitly exhibit a basis of matrix units for $\mathsf{Z}_k(\mathbf{C}_n)$. These matrix units can be viewed using diagrams that correspond to subsets of $\{1, 2, \ldots, 2k\}$ that satisfy a special $\mathsf{mod}\, n$ condition (see Remark 2.13). We also consider the case that G is the infinite cyclic group $\mathbf{C}_\infty$, which has as its representation graph the Dynkin diagram A_∞. Our results on $\mathbf{C}_\infty$, which are summarized in Theorem 2.21, show that $\mathsf{Z}_k(\mathbf{C}_\infty)$ can be regarded, in some sense, as the limiting case of $\mathsf{Z}_k(\mathbf{C}_n)$ as n grows large. The algebra $\mathsf{Z}_k(\mathbf{C}_\infty)$ is isomorphic to the planar rook algebra PR_k (see Remark 2.22).

(3) Section 3 is devoted to the case that G is the binary dihedral group $\mathbf{D}_n$. We compute $\mathsf{dim}\, \mathsf{Z}_k(\mathbf{D}_n)$ and the dimensions of the irreducible $\mathsf{Z}_k(\mathbf{D}_n)$-modules and construct a basis of matrix units for $\mathsf{Z}_k(\mathbf{D}_n)$ (see Theorems 3.13 and 3.29). These matrix units can be described diagrammatically using diagrams that correspond to set partitions of $\{1, 2, \ldots, 2k\}$ into at most 2 parts that satisfy a certain $\mathsf{mod}\, n$ condition. Theorem 3.42 treats the centralizer algebra $\mathsf{Z}_k(\mathbf{D}_\infty)$ of the infinite dihedral group $\mathbf{D}_\infty$, which has as its representation graph the Dynkin diagram D_∞ and can be viewed as the limiting case of the groups $\mathbf{D}_n$.

(4) In Section 4, we illustrate how the results of Section 2 can be used to compute $\dim \mathsf{Z}_k(\mathsf{G})$ for $\mathsf{G} = \mathbf{T}, \mathbf{O}, \mathbf{I}$ and the dimensions of the irreducible modules of these algebras. The case of $\mathbf{I}$ is noteworthy, as the expressions involve the Lucas numbers.

The names for the exceptional subgroups $\mathbf{T}, \mathbf{O}, \mathbf{I}$ derive from the fact that they are 2-fold covers of classical polyhedral groups. Modulo the center $\mathcal{Z}(\mathsf{G}) = \{\mathbf{1}, -\mathbf{1}\}$, these groups have the following quotients: $\mathbf{T}/\{\mathbf{1}, -\mathbf{1}\} \cong \mathsf{A}_4$, the alternating group on 4 letters, which is the rotation group of the tetrahedron; $\mathbf{O}/\{\mathbf{1}, -\mathbf{1}\} \cong \mathsf{S}_4$, the symmetric group on 4 letters, which is the rotation group of the cube; and $\mathbf{I}/\{\mathbf{1}, -\mathbf{1}\} \cong \mathsf{A}_5$, the alternating group on 5 letters, which is the rotation group of the icosahedron. An exposition of this result based on an argument of Weyl can be found in [Si, Sec. 1.4]. Our sequel [BH] studies the exceptional centralizer algebras $\mathsf{Z}_k(\mathsf{G})$ for $\mathsf{G} = \mathbf{T}, \mathbf{O}, \mathbf{I}$, giving a basis for them and exhibiting remarkable connections between them, the Jones-Martin partition algebras, and partitions.

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1 McKay Centralizer Algebras

1.1 SU_2-modules

Consider the special unitary group SU_2 of 2×2 complex matrices defined by

$$\mathsf{SU}_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C}, \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}, \tag{1.1}$$

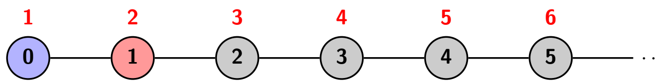
where $\bar{\alpha}$ denotes the complex conjugate of α. For each $r \geq 0$, SU_2 has an irreducible module $\mathsf{V}(r)$ of dimension $r+1$. The module $\mathsf{V} = \mathsf{V}(1) = \mathbb{C}^2$ corresponds to the natural two-dimensional representation on which SU_2 acts by matrix multiplication. Let $\mathsf{v}_{-1} = (1,0)^{\mathtt{t}}, \mathsf{v}_1 = (0,1)^{\mathtt{t}}$ (here $\mathtt{t}$ denotes transpose) be the standard basis for this action so that if $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ then $g\mathsf{v}_{-1} = \alpha\mathsf{v}_{-1} - \bar{\beta}\mathsf{v}_1$ and $g\mathsf{v}_1 = \beta\mathsf{v}_{-1} + \bar{\alpha}\mathsf{v}_1$.

Finite-dimensional modules for SU_2 are completely reducible and satisfy the Clebsch-Gordan formula,

$$\mathsf{V}(r) \otimes \mathsf{V} = \mathsf{V}(r-1) \oplus \mathsf{V}(r+1), \tag{1.2}$$

where $\mathsf{V}(-1) = 0$. The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{SU}_2)$ is the infinite graph with vertices labeled by $r = 0, 1, 2, \cdots$ and an edge connecting vertex r to vertex $r+1$ for each r (which can be thought of as the Dynkin diagram $\mathsf{A}_{+\infty}$). Vertex r corresponds to $\mathsf{V}(r)$, and the edges correspond to the tensor product rule (1.2). Above vertex r we place $\dim \mathsf{V}(r) = r+1$, as displayed below. The trivial module is indicated in blue and the defining module V in red.

SU_2 :



$(\mathsf{A}_{+\infty})$ (1.3)

1.2 Subgroups of SU_2 and their representation graphs

Let G be a subgroup of SU_2. Then G acts on the natural two-dimensional representation $\mathsf{V} = \mathbb{C}^2$ as 2×2 matrices with respect to the basis $\{\mathsf{v}_{-1}, \mathsf{v}_1\}$. We assume that the tensor powers $\mathsf{V}^{\otimes k}$ of V are completely reducible G-modules (which is always the case when G is finite), and let $\{\, \mathsf{G}^\lambda \mid \lambda \in \Lambda(\mathsf{G})\,\}$ denote a complete set of pairwise non-isomorphic irreducible finite-dimensional G-modules occurring in some $\mathsf{V}^{\otimes k}$ for $k = 0, 1, \ldots$. We adopt the convention that $\mathsf{V}^{\otimes 0} = \mathsf{G}^{(0)}$, the trivial G-module. The representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ (also called the McKay graph or McKay quiver) is the graph with vertices labeled by elements of $\Lambda(\mathsf{G})$ with $a_{\lambda,\mu}$ edges directed from λ to μ if the decomposition of $\mathsf{G}^\lambda \otimes \mathsf{V}$ into irreducible G-modules is given by

$$\mathsf{G}^\lambda \otimes \mathsf{V} = \bigoplus_{\mu \in \Lambda(\mathsf{G})} a_{\lambda,\mu}\, \mathsf{G}^\mu. \tag{1.4}$$

The following properties of $\mathcal{R}_\mathsf{V}(\mathsf{G})$ hold for all finite subgroups $\mathsf{G} \subseteq \mathsf{SU}_2$ (see [St]), and we will assume that they hold for the groups considered here:

1. $a_{\lambda,\mu} = a_{\mu,\lambda}$ for all pairs $\lambda, \mu \in \Lambda(\mathsf{G})$.

2. $a_{\lambda,\lambda} = 0$ for all $\lambda \in \Lambda(\mathsf{G})$, $\lambda \neq 0$.

3. If $\mathsf{G} \neq \{\mathbf{1}\}, \{\mathbf{1}, -\mathbf{1}\}$, where $\mathbf{1}$ is the 2×2 identity matrix, then $a_{\lambda,\mu} \in \{0, 1\}$ for all $\lambda, \mu \in \Lambda(\mathsf{G})$.

Thus, $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is an undirected, simple graph. Since V is faithful (being the defining module for G), all irreducible G-modules occur in some $\mathsf{V}^{\otimes k}$ when G is finite, and thus $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is connected. Moreover, if $c_{\lambda,\mu} = 2\delta_{\lambda,\mu} - a_{\lambda,\mu}$ for $\lambda, \mu \in \Lambda(\mathsf{G})$, where $\delta_{\lambda,\mu}$ is the Kronecker delta, then McKay [Mc] observed that $\mathcal{C}(\mathsf{G}) = [c_{\lambda,\mu}]$ is the Cartan matrix corresponding to the simply-laced affine Dynkin diagram of type $\hat{\mathsf{A}}_{\mathsf{n}-1}$, $\hat{\mathsf{D}}_{\mathsf{n}+2}$, $\hat{\mathsf{E}}_6$ $\hat{\mathsf{E}}_7$, $\hat{\mathsf{E}}_8$ when G is one of the finite groups $\mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$, respectively. The trivial module $\mathsf{G}^{(0)}$ corresponds to the affine node in those cases.

1.3 Tensor powers and Bratteli diagrams

For $k \geq 1$, the k-fold tensor power $\mathsf{V}^{\otimes k}$ is 2^k-dimensional and has a basis of simple tensors

$$\mathsf{V}^{\otimes k} = \mathsf{span}_\mathbb{C} \left\{\, \mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k} \;\middle|\; r_j \in \{-1, 1\} \,\right\}.$$

If $\mathsf{r} = (r_1, \ldots, r_k) \in \{-1, 1\}^k$, we adopt the notation

$$\mathsf{v}_\mathsf{r} = \mathsf{v}_{(r_1,\ldots,r_k)} = \mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k} \tag{1.5}$$

as a shorthand. Group elements $g \in \mathsf{G}$ act on simple tensors by the diagonal action

$$g(\mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k}) = g\mathsf{v}_{r_1} \otimes g\mathsf{v}_{r_2} \otimes \cdots \otimes g\mathsf{v}_{r_k}. \tag{1.6}$$

Let

$$\Lambda_k(\mathsf{G}) = \{\ \lambda \in \Lambda(\mathsf{G}) \mid \mathsf{G}^\lambda \text{ appears as a summand in the decomposition of } \mathsf{V}^{\otimes k}\} \tag{1.7}$$

index the irreducible G-modules occurring in $\mathsf{V}^{\otimes k}$. Then, since $\mathcal{R}_\mathsf{V}(\mathsf{G})$ encodes the tensor product rule (1.4), $\Lambda_k(\mathsf{G})$ is the set of vertices in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ that can be reached by paths of length k starting from 0. Furthermore,

$$\Lambda_k(\mathsf{G}) \subseteq \Lambda_{k+2}(\mathsf{G}), \qquad \text{for all } k \geq 0, \tag{1.8}$$

since if a node can be reached in k steps, then it can also be reached in $k+2$ steps.

The Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{G})$ is the infinite graph with vertices labeled by $\Lambda_k(\mathsf{G})$ on level k and $b_{\lambda,\mu}$ edges from vertex $\lambda \in \Lambda_k(\mathsf{G})$ to vertex $\mu \in \Lambda_{k+1}(\mathsf{G})$. The Bratteli diagram for SU_2 is shown in Figure 1, and the Bratteli diagrams corresponding to $\mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$, as well as to the infinite subgroups $\mathbf{C}_\infty$, $\mathbf{D}_\infty$, are displayed in Section 4.2.

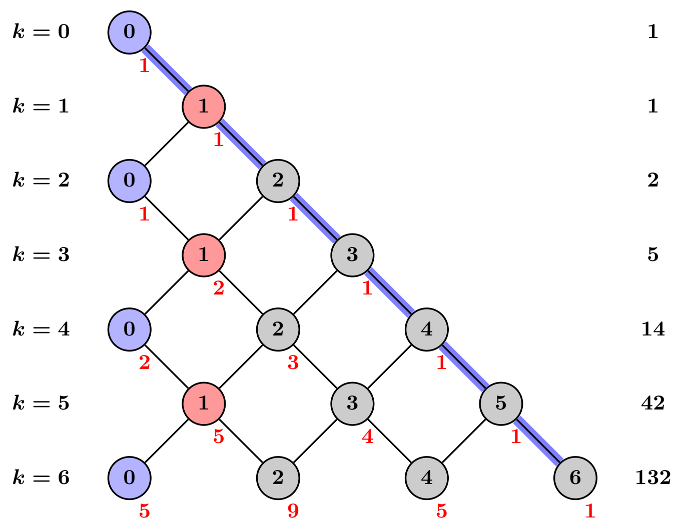


Figure 1: First 6 levels of the Bratteli Diagram for SU_2.

A walk of length k on the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ from 0 to $\lambda \in \Lambda(\mathsf{G})$, is a sequence $\left(0, \lambda^1, \lambda^2, \ldots, \lambda^k = \lambda\right)$ starting at $\lambda^0 = 0$, such that $\lambda^j \in \Lambda(\mathsf{G})$ for each $1 \leq j \leq k$, and λ^{j-1} is connected to λ^j by an edge in $\mathcal{R}_\mathsf{V}(\mathsf{G})$. Such a walk is equivalent to a unique path of length k on the Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{G})$ from $0 \in \Lambda_0(\mathsf{G})$ to $\lambda \in \Lambda_k(\mathsf{G})$. Let $\mathcal{W}_k^\lambda(\mathsf{G})$ denote the set of walks on $\mathcal{R}_\mathsf{V}(\mathsf{G})$ of length k from $0 \in \Lambda_k(\mathsf{G})$ to $\lambda \in \Lambda_k(\mathsf{G})$, and let $\mathcal{P}_k^\lambda(\mathsf{G})$ denote the set of paths on $\mathcal{B}_\mathsf{V}(\mathsf{G})$ of length k from $0 \in \Lambda_0(\mathsf{G})$ to $\lambda \in \Lambda_k(\mathsf{G})$. Thus, $|\mathcal{P}_k^\lambda(\mathsf{G})| = |\mathcal{W}_k^\lambda(\mathsf{G})|$.

A pair of walks of length k from 0 to λ corresponds uniquely (by reversing the second walk) to a walk of length $2k$ beginning and ending at 0. Hence,

$$|\mathcal{W}_{2k}^0(\mathsf{G})| = \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{W}_k^\lambda(\mathsf{G})|^2 = \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{P}_k^\lambda(\mathsf{G})|^2 = |\mathcal{P}_{2k}^0(\mathsf{G})|. \tag{1.9}$$

Let m_k^λ denote the multiplicity of G^λ in $\mathsf{V}^{\otimes k}$. Then, by induction on (1.4) and the observations in the previous paragraph, we see that this multiplicity is enumerated as

$$\begin{aligned} m_k^\lambda &= |\mathcal{W}_k^\lambda(\mathsf{G})| &&= \#(\text{walks on } \mathcal{R}_\mathsf{V}(\mathsf{G}) \text{ of length } k \text{ from at } 0 \text{ to } \lambda) \\ &= |\mathcal{P}_k^\lambda(\mathsf{G})| &&= \#(\text{paths in } \mathcal{B}_\mathsf{V}(\mathsf{G}) \text{ of length } k \text{ from } 0 \in \Lambda_0(\mathsf{G}) \text{ to } \lambda \in \Lambda_k(\mathsf{G})). \end{aligned} \tag{1.10}$$

The first six rows of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathsf{SU}_2)$ for SU_2 are displayed in Figure 1, and the labels below vertex r on level k give the number of paths from the top of the diagram to r, which is also multiplicity of $\mathsf{V}(r)$ in $\mathsf{V}^{\otimes k}$. These numbers also give the number of walks to r on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{SU}_2)$ of length k from 0 to r. The column to the right contains the sum of the squares of the multiplicities.

1.4 Schur-Weyl duality

The *centralizer of* G *on* $\mathsf{V}^{\otimes k}$ is the algebra

$$\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k}) = \left\{ a \in \mathsf{End}(\mathsf{V}^{\otimes k}) \;\middle|\; a(gw) = ga(w) \text{ for all } g \in \mathsf{G},\, w \in \mathsf{V}^{\otimes k} \right\}. \tag{1.11}$$

If the group G is apparent from the context, we will simply write Z_k for $\mathsf{Z}_k(\mathsf{G})$. Since $\mathsf{V}^{\otimes 0} = \mathsf{G}^{(0)}$, we have $\mathsf{Z}_0(\mathsf{G}) = \mathbb{C}1$. There is a natural embedding $\iota : \mathsf{Z}_k(\mathsf{G}) \hookrightarrow \mathsf{Z}_{k+1}(\mathsf{G})$ given by

$$\begin{array}{rccc} \iota: & \mathsf{Z}_k(\mathsf{G}) & \rightarrow & \mathsf{Z}_{k+1}(\mathsf{G}) \\ & a & \mapsto & a \otimes \mathbf{1} \end{array} \tag{1.12}$$

where $a \otimes \mathbf{1}$ acts as a on the first k tensor factors and $\mathbf{1}$ acts as the identity in the $(k+1)$st tensor position. Iterating this embedding gives an infinite tower of algebras

$$\mathsf{Z}_0(\mathsf{G}) \subseteq \mathsf{Z}_1(\mathsf{G}) \subseteq \mathsf{Z}_2(\mathsf{G}) \subseteq \cdots. \tag{1.13}$$

By classical double-centralizer theory (see for example [CR, Secs. 3B and 68]), we know the following:

- $\mathsf{Z}_k(\mathsf{G})$ is a semisimple associative $\mathbb{C}$-algebra whose irreducible modules $\left\{\mathsf{Z}_k^\lambda \,\middle|\, \lambda \in \Lambda_k(\mathsf{G})\right\}$ are labeled by $\Lambda_k(\mathsf{G})$.
- $\dim \mathsf{Z}_k^\lambda = m_k^\lambda = |\mathcal{W}_k^\lambda(\mathsf{G})| = |\mathcal{P}_k^\lambda(\mathsf{G})|$.
- The edges from level k to level $k-1$ in $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ represent the restriction and induction rules for $\mathsf{Z}_{k-1}(\mathsf{G}) \subseteq \mathsf{Z}_k(\mathsf{G})$.
- If $d^\lambda = \dim \mathsf{G}^\lambda$, then the tensor space $\mathsf{V}^{\otimes k}$ has the following decomposition

$$\begin{aligned} \mathsf{V}^{\otimes k} &\cong \bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} m_k^\lambda\, \mathsf{G}^\lambda && \text{as a } \mathsf{G}\text{-module}, \\ &\cong \bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} d^\lambda\, \mathsf{Z}_k^\lambda && \text{as a } \mathsf{Z}_k(\mathsf{G})\text{-module}, \\ &\cong \bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} \left(\mathsf{G}^\lambda \otimes \mathsf{Z}_k^\lambda\right) && \text{as a } (\mathsf{G}, \mathsf{Z}_k(\mathsf{G}))\text{-bimodule}. \end{aligned} \tag{1.14}$$

As an immediate consequence of these isomorphisms, we have from counting dimensions that

$$2^k = \sum_{\lambda \in \Lambda_k(\mathsf{G})} d^\lambda m_k^\lambda. \tag{1.15}$$

- By general Wedderburn theory, the dimension of $\mathsf{Z}_k(\mathsf{G})$ is the sum of the squares of the dimensions of its irreducible modules,

$$\dim \mathsf{Z}_k(\mathsf{G}) = \sum_{\lambda \in \Lambda_k(\mathsf{G})} (m_k^\lambda)^2 = \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{W}_k^\lambda(\mathsf{G})|^2 = \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{P}_k^\lambda(\mathsf{G})|^2. \tag{1.16}$$

Therefore, it follows from (1.9) that

$$\dim \mathsf{Z}_k(\mathsf{G}) = \sum_{\lambda\in\Lambda_k(\mathsf{G})} (m_k^\lambda)^2 = m_{2k}^0 = \left|\mathcal{W}_{2k}^0(\mathsf{G})\right| = \dim \mathsf{Z}_{2k}^{(0)}, \tag{1.17}$$

the number of walks on $\mathcal{R}_\mathsf{V}(\mathsf{G})$ (which is the associated Dynkin diagram when G is a finite subgroup) of length $2k$ that begin and end at 0.

1.5 The Temperley-Lieb algebras

Let S_k denote the symmetric group of permutations on $\{1, 2, \dots, k\}$, and let $\sigma \in \mathsf{S}_k$ act on a simple tensor by place permutation as follows:

$$\sigma \cdot (\mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k}) = \mathsf{v}_{\sigma(r_1)} \otimes \mathsf{v}_{\sigma(r_2)} \otimes \cdots \otimes \mathsf{v}_{\sigma(r_k)}.$$

It is well known, and easy to verify, that under this action S_k commutes with SU_2 on $\mathsf{V}^{\otimes k}$. Thus, there is a representation $\Phi_k : \mathbb{C}\mathsf{S}_k \to \mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k})$; however, this map is injective only for $k \leq 2$.

For $1 \leq i \leq k-1$, let $s_i = (i\ i+1) \in \mathsf{S}_k$ be the simple transposition that exchanges i and $i+1$, and set

$$e_i = 1 - s_i. \tag{1.18}$$

Then e_i acts on tensor space as

$$\mathsf{e}_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{i-1 \text{ factors}} \otimes \mathsf{e} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k-i-1 \text{ factors}}, \tag{1.19}$$

where $\mathbf{1}$ is the 2×2 identity matrix, which we identify with the identity map id_V of V, and $\mathsf{e} : \mathsf{V} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{V}$ acts in tensor positions i and $i+1$ by

$$\mathsf{e}(\mathsf{v}_i \otimes \mathsf{v}_j) = \mathsf{v}_i \otimes \mathsf{v}_j - \mathsf{v}_j \otimes \mathsf{v}_i, \qquad i, j \in \{-1, 1\}. \tag{1.20}$$

For any $\mathsf{G} \subseteq \mathsf{SU}_2$, the vector space $\mathsf{V}^{\otimes 2} = \mathsf{V} \otimes \mathsf{V}$ decomposes into G-modules as

$$\mathsf{V}^{\otimes 2} = \mathsf{A}(\mathsf{V}^{\otimes 2}) \oplus \mathsf{S}(\mathsf{V}^{\otimes 2}),$$

where $\mathsf{A}(\mathsf{V}^{\otimes 2}) = \mathsf{span}_\mathbb{C}\{\mathsf{v}_{-1}\otimes\mathsf{v}_1 - \mathsf{v}_1\otimes\mathsf{v}_{-1}\}$ are the antisymmetric tensors and $\mathsf{S}(\mathsf{V}^{\otimes 2}) = \mathsf{span}_\mathbb{C}\{\mathsf{v}_{-1}\otimes \mathsf{v}_{-1}, \mathsf{v}_{-1} \otimes \mathsf{v}_1 + \mathsf{v}_1 \otimes \mathsf{v}_{-1}, \mathsf{v}_1 \otimes \mathsf{v}_1\}$ are the symmetric tensors. The operator $\mathsf{e} : \mathsf{V}^{\otimes 2} \to \mathsf{V}^{\otimes 2}$ projects onto the G-submodule $\mathsf{A}(\mathsf{V}^{\otimes 2})$ and $\frac{1}{2}\mathsf{e}$ is the corresponding idempotent.

The image $\mathsf{im}(\Phi_k)$ of the representation $\Phi_k : \mathbb{C}\mathsf{S}_k \to \mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k})$ can be identified with the Temperley-Lieb algebra $\mathsf{TL}_k(2)$. Recall that the Temperley-Lieb algebra $\mathsf{TL}_k(2)$ is the unital associative algebra with generators $\mathsf{e}_1, \dots, \mathsf{e}_{k-1}$ and relations

$$\begin{array}{lll} \text{(TL1)} & \mathsf{e}_i^2 = 2\mathsf{e}_i, & 1 \leq i \leq k-1, \\ \text{(TL2)} & \mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i, & 1 \leq i \leq k-1, \\ \text{(TL3)} & \mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i, & |i-j| > 1, \end{array} \tag{1.21}$$

(see [TL] and [GHJ]). Since the generator e_i in $\mathsf{TL}_k(2)$ is identified with the map in (1.19), we are using the same notation for them. If $\Psi_k : \mathsf{SU}_2 \to \mathsf{End}(\mathsf{V}^{\otimes k})$ is the tensor-product representation, then SU_2 and $\mathsf{TL}_k(2)$ generate full centralizers of each other in $\mathsf{End}(\mathsf{V}^{\otimes k})$, so that

$$\mathsf{TL}_k(2) \cong \mathsf{im}(\Phi_k) = \mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k}) \qquad \text{and} \qquad \mathsf{im}(\Psi_k) = \mathsf{End}_{\mathsf{TL}_k(2)}(\mathsf{V}^{\otimes k}). \tag{1.22}$$

Since $\mathsf{TL}_k(2) \cong \mathsf{Z}_k(\mathsf{SU}_2) = \mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k})$, the set

$$\Lambda_k(\mathsf{SU}_2) = \begin{cases} \{0, 2, \dots, k\}, & \text{if } k \text{ is even,} \\ \{1, 3, \dots, k\}, & \text{if } k \text{ is odd} \end{cases}$$

also indexes the irreducible $\mathsf{TL}_k(2)$-modules. The number of walks of length k from 0 to $k - 2\ell \in \Lambda_k(\mathsf{SU}_2)$ on $\mathcal{R}_\mathsf{V}(\mathsf{SU}_2)$ is equal to the number of walks from 0 to $k - 2\ell$ on the natural numbers $\mathbb{N}$ and is known to be (see [Wb, p. 545], [Jo, Sec. 5])

$$\left\{ {k \atop \ell} \right\} := \binom{k}{\ell} - \binom{k}{\ell - 1}.$$

For each $k - 2\ell \in \Lambda_k(\mathsf{SU}_2)$, where $\ell = 0, 1, \dots, \lfloor k/2 \rfloor$, let $\mathsf{TL}_k^{(k-2\ell)} = \mathsf{Z}_k^{(k-2\ell)}$ be the irreducible $\mathsf{TL}_k(2)$ module labeled by $k - 2\ell$. Then $\mathsf{TL}_k^{(k-2\ell)}$ has dimension $\left\{ {k \atop \ell} \right\}$, and these modules are constructed explicitly in [Wb]. Moreover,

$$\begin{aligned} \mathsf{V}^{\otimes k} &\cong \bigoplus_{k-2\ell \in \Lambda_k(\mathsf{SU}_2)} \left\{ {k \atop \ell} \right\} \mathsf{V}(k - 2\ell), && \text{as an } \mathsf{SU}_2\text{-module,} \\ &\cong \bigoplus_{k-2\ell \in \Lambda_k(\mathsf{SU}_2)} (k - 2\ell + 1) \mathsf{TL}_k^{(k-2\ell)}, && \text{as a } \mathsf{TL}_k(2)\text{-module,} \\ &\cong \bigoplus_{k-2\ell \in \Lambda_k(\mathsf{SU}_2)} \left(\mathsf{V}(k - 2\ell) \otimes \mathsf{TL}_k^{(k-2\ell)} \right), && \text{as an } (\mathsf{SU}_2, \mathsf{TL}_k(2))\text{-bimodule.} \end{aligned} \tag{1.23}$$

The dimension of $\mathsf{TL}_k(2)$ is given by the Catalan number $\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}$, as can be seen in the right-hand column of the Bratteli diagram for SU_2 in Figure 1.

1.6 The Jones Basic Construction

Let G be a subgroup of SU_2 such that $\mathsf{G} \neq \{\mathbf{1}\}, \{-\mathbf{1}, \mathbf{1}\}$. Any transformation that commutes with SU_2 on $\mathsf{V}^{\otimes k}$ also commutes with G. Thus, we have the reverse inclusion of centralizers $\mathsf{TL}_k(2) = \mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k}) \subseteq \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k}) = \mathsf{Z}_k(\mathsf{G})$ and identify the subalgebra of $\mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$ generated by the e_i in (1.19) with $\mathsf{TL}_k(2)$. In this section, we use the Jones basic construction to find additional generators for the centralizer algebra $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$ for each k. The construction uses the natural embedding of Z_k into Z_{k+1} given by $a \mapsto a \otimes \mathbf{1}$, which holds for any $k \geq 1$.

In what follows, if $\mathsf{q} = (q_1, \dots, q_k) \in \{-1, 1\}^k$ and $\mathsf{r} = (r_1, \dots, r_\ell) \in \{-1, 1\}^\ell$ for some $k, \ell \geq 1$, then $[\mathsf{q}, \mathsf{r}] = (q_1, \dots, q_k, r_1, \dots, r_\ell) \in \{-1, 1\}^{k+\ell}$ is the concatenation of the two tuples. In particular, if $t \in \{-1, 1\}$, then $[\mathsf{q}, t] = (q_1, \dots, q_k, t)$.

Now if $a \in \mathsf{End}(\mathsf{V}^{\otimes k})$, say $a = \sum_{\mathsf{s} \in \{-1,1\}^k} a_\mathsf{s}^\mathsf{r} \mathsf{E}_{\mathsf{r},\mathsf{s}}$, where $\mathsf{E}_{\mathsf{r},\mathsf{s}}$ is the standard matrix unit, then under the embedding $a \mapsto a \otimes \mathbf{1}$,

$$a_{[\mathsf{s}, s_{k+1}]}^{[\mathsf{r}, r_{k+1}]} = (a \otimes \mathbf{1})_{[\mathsf{s}, s_{k+1}]}^{[\mathsf{r}, r_{k+1}]} = \delta_{r_{k+1}, s_{k+1}} a_\mathsf{s}^\mathsf{r}, \tag{1.24}$$

where $r_{k+1}, s_{k+1} \in \{-1, 1\}$.

Note in this section we are writing a_s^r rather than $a_{\mathsf{r},\mathsf{s}}$ to simplify the notation.

Define a map $\varepsilon_k : \mathsf{End}(\mathsf{V}^{\otimes k}) \to \mathsf{End}(\mathsf{V}^{\otimes (k-1)})$, $\varepsilon_k(a) = \sum_{\mathsf{p},\mathsf{q} \in \{-1,1\}^{k-1}} \varepsilon_k(a)_\mathsf{q}^\mathsf{p} \mathsf{E}_{\mathsf{p},\mathsf{q}}$, called the *conditional expectation*, such that

$$\varepsilon_k(a)_\mathsf{q}^\mathsf{p} = \frac{1}{2} \left(a_{[\mathsf{q},-1]}^{[\mathsf{p},-1]} + a_{[\mathsf{q},1]}^{[\mathsf{p},1]} \right) \tag{1.25}$$

for all $\mathsf{p}, \mathsf{q} \in \{-1, 1\}^{k-1}$ and all $a \in \mathsf{End}(\mathsf{V}^{\otimes k})$.

Proposition 1.26. *Assume $k \geq 1$.*

(a) *If $a \in \mathsf{End}(\mathsf{V}^{\otimes k}) \subseteq \mathsf{End}(\mathsf{V}^{\otimes (k+1)})$, then $\mathsf{e}_k a \mathsf{e}_k = 2\varepsilon_k(a) \otimes \mathsf{e} = 2\varepsilon_k(a)\mathsf{e}_k$.*

(b) *If $a \in \mathsf{Z}_k$, then $\varepsilon_k(a) \in \mathsf{Z}_{k-1}$, so that $\varepsilon_k : \mathsf{Z}_k \to \mathsf{Z}_{k-1}$.*

(c) *$\varepsilon_k : \mathsf{Z}_k \to \mathsf{Z}_{k-1}$ is a $(\mathsf{Z}_{k-1}, \mathsf{Z}_{k-1})$-bimodule map; that is, $\varepsilon_k(a_1 b a_2) = a_1 \varepsilon_k(b) a_2$ for all $a_1, a_2 \in \mathsf{Z}_{k-1} \subseteq \mathsf{Z}_k$, $b \in \mathsf{Z}_k$. In particular, $\varepsilon_k(a) = a$ for all $a \in \mathsf{Z}_{k-1}$.*

(d) *Let tr_k denote the usual (nondegenerate) trace on $\mathsf{End}(\mathsf{V}^{\otimes k})$. Then for all $a \in \mathsf{Z}_k$ and $b \in \mathsf{Z}_{k-1}$, we have $\mathsf{tr}_k(ab) = \mathsf{tr}_k(\varepsilon_k(a)b)$.*

Proof. (a) It suffices to show that these expressions have the same action on a simple tensor $\mathsf{v_r}$ where $\mathsf{r} = (r_1, \dots, r_k, r_{k+1}) \in \{-1, 1\}^{k+1}$. If $r_k = r_{k+1}$, then both $\mathsf{e}_k a \mathsf{e}_k$ and $2\varepsilon_k(a) \otimes \mathsf{e}$ act as 0 on $\mathsf{v_r}$. So we suppose that $(r_k, r_{k+1}) = (-1, 1)$, and let $\mathsf{p} = (r_1, \dots, r_{k-1})$. Now

$$
\begin{aligned}
(\mathsf{e}_k a \mathsf{e}_k)\mathsf{v_r} &= \mathsf{e}_k a \left(\mathsf{v_p} \otimes \mathsf{v}_{-1} \otimes \mathsf{v}_1 - \mathsf{v_p} \otimes \mathsf{v}_1 \otimes \mathsf{v}_{-1}\right) \\
&= \mathsf{e}_k \sum_{\mathsf{q} \in \{-1,1\}^k} a^{\mathsf{q}}_{[\mathsf{p},-1]} \mathsf{v_q} \otimes \mathsf{v}_1 - \mathsf{e}_k \sum_{\mathsf{q} \in \{-1,1\}^k} a^{\mathsf{q}}_{[\mathsf{p},1]} \mathsf{v_q} \otimes \mathsf{v}_{-1} \\
&= \sum_{\mathsf{n} \in \{-1,1\}^{k-1}} a^{[\mathsf{n},-1]}_{[\mathsf{p},-1]} \left(\mathsf{v_n} \otimes \mathsf{v}_{-1} \otimes \mathsf{v}_1 - \mathsf{v_n} \otimes \mathsf{v}_1 \otimes \mathsf{v}_{-1}\right) \\
&\qquad - \sum_{\mathsf{n} \in \{-1,1\}^{k-1}} a^{[\mathsf{n},1]}_{[\mathsf{p},1]} \left(\mathsf{v_n} \otimes \mathsf{v}_1 \otimes \mathsf{v}_{-1} - \mathsf{v_n} \otimes \mathsf{v}_{-1} \otimes \mathsf{v}_1\right).
\end{aligned}
$$

Thus, the coefficient of $\mathsf{v_n} \otimes \mathsf{v}_{-1} \otimes \mathsf{v}_1$ is $a^{[\mathsf{n},-1]}_{[\mathsf{p},-1]} + a^{[\mathsf{n},1]}_{[\mathsf{p},1]}$, and the coefficient of $\mathsf{v_n} \otimes \mathsf{v}_1 \otimes \mathsf{v}_{-1}$ is the negative of that expression. These are exactly the coefficients that we get when $2\varepsilon_k(a) \otimes \mathsf{e}$ acts on $\mathsf{v_r}$. The proof when $(r_k, r_{k+1}) = (1, -1)$ is completely analogous.

(b) When $a \in \mathsf{Z}_k \subseteq \mathsf{Z}_{k+1}$, then $\mathsf{e}_k a \mathsf{e}_k \in \mathsf{Z}_{k+1}$, so from part (a) it follows that $\varepsilon_k(a) \otimes \mathsf{e} \in \mathsf{Z}_{k+1}$. Therefore, if $g \in \mathsf{G}$, then $\varepsilon_k(a) \otimes \mathsf{e} \in \mathsf{Z}_{k+1}$ commutes with $g^{\otimes (k+1)} = g^{\otimes (k-1)} \otimes g^{\otimes 2}$ on $\mathsf{V}^{(k-1)} \otimes \mathsf{V}^{\otimes 2}$. Since these actions occur independently on the first $k-1$ and last 2 tensor slots, $\varepsilon_k(a)$ commutes with $g^{\otimes (k-1)}$ for all $g \in \mathsf{G}$. Hence $\varepsilon_k(a) \in \mathsf{Z}_{k-1}$.

(c) Let $a_1, a_2 \in \mathsf{Z}_{k-1}$ and $b \in \mathsf{Z}_k$. Then using (1.24) and (1.25) we have for $\mathsf{p}, \mathsf{q} \in \{-1, 1\}^{k-1}$,

$$
\begin{aligned}
\varepsilon_k(a_1 b a_2)^{\mathsf{p}}_{\mathsf{q}} &= \frac{1}{2} \sum_{t \in \{-1,1\}} (a_1 b a_2)^{[\mathsf{p},t]}_{[\mathsf{q},t]} = \frac{1}{2} \sum_{t \in \{-1,1\}} \sum_{\mathsf{m},\mathsf{n} \in \{-1,1\}^{k-1}} (a_1)^{[\mathsf{p},t]}_{[\mathsf{n},t]} b^{[\mathsf{n},t]}_{[\mathsf{m},t]} (a_2)^{[\mathsf{m},t]}_{[\mathsf{q},t]} \\
&= \frac{1}{2} \sum_{t \in \{-1,1\}} \sum_{\mathsf{m},\mathsf{n} \in \{-1,1\}^{k-1}} (a_1)^{\mathsf{p}}_{\mathsf{n}} b^{[\mathsf{n},t]}_{[\mathsf{m},t]} (a_2)^{\mathsf{m}}_{\mathsf{q}} \\
&= \sum_{\mathsf{m},\mathsf{n} \in \{-1,1\}^{k-1}} (a_1)^{\mathsf{p}}_{\mathsf{n}} \left(\frac{1}{2} \sum_{t \in \{-1,1\}} b^{[\mathsf{n},t]}_{[\mathsf{m},t]} \right) (a_2)^{\mathsf{m}}_{\mathsf{q}} \\
&= (a_1 \varepsilon_k(b) a_2)^{\mathsf{p}}_{\mathsf{q}}.
\end{aligned}
$$

(d) Let $a \in \mathsf{Z}_k$ and $b \in \mathsf{Z}_{k-1} \subseteq \mathsf{Z}_k$. Then applying (1.24) gives

$$\begin{aligned}
\mathsf{tr}_k(ab) &= \sum_{\mathsf{r}\in\{-1,1\}^k} \sum_{\mathsf{s}\in\{-1,1\}^k} a_{\mathsf{s}}^{\mathsf{r}} b_{\mathsf{r}}^{\mathsf{s}} = \sum_{\mathsf{r}=[\mathsf{r}',r_k]\in\{-1,1\}^k} \sum_{\mathsf{s}=[\mathsf{s}',s_k]\in\{-1,1\}^k} a_{\mathsf{s}}^{\mathsf{r}} \delta_{r_k,s_k} b_{\mathsf{r}'}^{\mathsf{s}'} \\
&= \sum_{\mathsf{r}'\in\{-1,1\}^{k-1}} \sum_{\mathsf{s}'\in\{-1,1\}^{k-1}} \left(a_{[\mathsf{s}',-1]}^{[\mathsf{r}',-1]} + a_{[\mathsf{s}',1]}^{[\mathsf{r}',1]} \right) b_{\mathsf{r}'}^{\mathsf{s}'} \\
&= \sum_{\mathsf{r}'\in\{-1,1\}^{k-1}} \sum_{\mathsf{s}'\in\{-1,1\}^{k-1}} 2\varepsilon_k(a)_{\mathsf{s}'}^{\mathsf{r}'} b_{\mathsf{r}'}^{\mathsf{s}'} \\
&= \sum_{\mathsf{r}=[\mathsf{r}',r_k]\in\{-1,1\}^k} \sum_{\mathsf{s}=[\mathsf{s}',s_k]\in\{-1,1\}^k} \delta_{r_k,s_k} \varepsilon_k(a)_{\mathsf{s}}^{\mathsf{r}} b_{\mathsf{r}}^{\mathsf{s}} \\
&= \mathsf{tr}_k(\varepsilon_k(a)b).
\end{aligned}$$

□

Relative to the inner product $\langle , \rangle : \mathsf{Z}_k \times \mathsf{Z}_k \to \mathbb{C}$ defined by $\langle a, b\rangle = \mathsf{tr}_k(ab)$, for all $a, b \in \mathsf{Z}_k$, the conditional expectation ε_k is the orthogonal projection $\varepsilon_k : \mathsf{Z}_k \to \mathsf{Z}_{k-1}$ with respect to $\langle , \rangle$ since $\langle a - \varepsilon_k(a), b\rangle = \mathsf{tr}_k(ab) - \mathsf{tr}_k(\varepsilon_k(a)b) = 0$ by Proposition 1.26 (d). Its values are uniquely determined by the nondegeneracy of the trace.

Proposition 1.26 (a) tells us that $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ is a subalgebra of Z_{k+1}. Indeed, for a_1, a_2, a_1', a_2' in Z_k, we have

$$(a_1\mathsf{e}_k a_2)(a_1'\mathsf{e}_k a_2') = a_1\mathsf{e}_k(a_2a_1')\mathsf{e}_k a_2' = 2a_1\varepsilon_k(a_2a_1')\mathsf{e}_k a_2' \in \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k.$$

Part (b) of the next result says that in fact $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ is an ideal of Z_{k+1}.

Proposition 1.27. *For all $k \geq 0$,*

(a) *For $a \in \mathsf{End}(\mathsf{V}^{\otimes(k+1)})$ there is a unique $b \in \mathsf{End}(\mathsf{V}^{\otimes k})$ so $a\mathsf{e}_k = (b \otimes \mathbf{1})\mathsf{e}_k$, and for all $\mathsf{r}, \mathsf{s} \in \{-1, 1\}^k$,*

$$b_{\mathsf{s}}^{\mathsf{r}} = \frac{1}{2}\left(a_{[\mathsf{s},-s_k]}^{[\mathsf{r},-s_k]} - a_{[\mathsf{s}',-s_k,s_k]}^{[\mathsf{r},-s_k]} \right), \tag{1.28}$$

where $\mathsf{r}' = (r_1, \ldots, r_{k-1})$, $\mathsf{r} = [\mathsf{r}', r_k]$, $\mathsf{s}' = (s_1, \ldots, s_{k-1})$, and $\mathsf{s} = [\mathsf{s}', s_k]$. If $a \in \mathsf{Z}_{k+1}$, then $b \in \mathsf{Z}_k$.

(b) *$\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k = \mathsf{Z}_{k+1}\mathsf{e}_k\mathsf{Z}_{k+1}$ is an ideal of Z_{k+1}.*

(c) *The map $\mathsf{Z}_k \to \mathsf{Z}_k\mathsf{e}_k \subseteq \mathsf{Z}_{k+1}$ given by $a \mapsto a\mathsf{e}_k$ is injective.*

Proof. (a) First note that for all $\mathsf{p}, \mathsf{q} \in \{-1, 1\}^{k+1}$,

$$(\mathsf{e}_k)_{\mathsf{q}}^{\mathsf{p}} = \frac{1}{4}\left(\prod_{j=1}^{k-1} \delta_{p_j,q_j}\right)(p_{k+1} - p_k)(q_{k+1} - q_k). \tag{1.29}$$

Now assume $a \in \mathsf{End}(\mathsf{V}^{\otimes(k+1)})$, $b \in \mathsf{End}(\mathsf{V}^{\otimes k})$ and $\mathsf{n}, \mathsf{q} \in \{-1, 1\}^{k+1}$, and let $\mathsf{q}'' = (q_1, \ldots, q_{k-1})$. Then

$$\begin{aligned}
(a\mathsf{e}_k)^{\mathsf{n}}_{\mathsf{q}} &= \sum_{\mathsf{p}\in\{-1,1\}^{k+1}} a^{\mathsf{n}}_{\mathsf{p}}\,(\mathsf{e}_k)^{\mathsf{p}}_{\mathsf{q}} \\
&= \frac{1}{4}\sum_{\mathsf{p}\in\{-1,1\}^{k+1}} a^{\mathsf{n}}_{\mathsf{p}}\left(\prod_{j=1}^{k-1}\delta_{p_j,q_j}\right)(p_{k+1}-p_k)(q_{k+1}-q_k), \\
&= \frac{1}{4}\sum_{p_k,p_{k+1}\in\{-1,1\}} a^{\mathsf{n}}_{[\mathsf{q}'',p_k,p_{k+1}]}(p_{k+1}-p_k)(q_{k+1}-q_k), \\
&= \frac{1}{2}(q_{k+1}-q_k)\left(a^{\mathsf{n}}_{[\mathsf{q}'',-1,1]}-a^{\mathsf{n}}_{[\mathsf{q}'',1,-1]}\right), \qquad \text{while}
\end{aligned}$$

$$\begin{aligned}
((b\otimes\mathbf{1})\mathsf{e}_k)^{\mathsf{n}}_{\mathsf{q}} &= \sum_{\mathsf{p}\in\{-1,1\}^{k+1}} (b\otimes\mathbf{1})^{\mathsf{n}}_{\mathsf{p}}\,(\mathsf{e}_k)^{\mathsf{p}}_{\mathsf{q}} \\
&= \frac{1}{4}\sum_{\mathsf{p}\in\{-1,1\}^{k+1}} (b\otimes\mathbf{1})^{\mathsf{n}}_{\mathsf{p}}\left(\prod_{j=1}^{k-1}\delta_{p_j,q_j}\right)(p_{k+1}-p_k)(q_{k+1}-q_k) \\
&= \frac{1}{4}(q_{k+1}-q_k)\sum_{p_k\in\{-1,1\}} (b\otimes\mathbf{1})^{\mathsf{n}'}_{[\mathsf{q}'',p_k]}(n_{k+1}-p_k) \text{ where } \mathsf{n}'=(n_1,\dots,n_k) \\
&= \frac{1}{2}(q_{k+1}-q_k)\left(b^{\mathsf{n}'}_{[\mathsf{q}'',-1]}(n_{k+1}+1)+b^{\mathsf{n}'}_{[\mathsf{q}'',1]}(n_{k+1}-1)\right).
\end{aligned}$$

Therefore $a\mathsf{e}_k=(b\otimes\mathbf{1})\mathsf{e}_k$ if and only if $b^{\mathsf{n}'}_{\mathsf{q}'}=\frac{1}{2}\left(a^{[\mathsf{n}',-q_k]}_{[\mathsf{q}'',q_k,-q_k]}-a^{[\mathsf{n}',q_k]}_{[\mathsf{q}'',-q_k,q_k]}\right)$ for all $\mathsf{n},\mathsf{q}\in\{-1,1\}^k$. Setting $\mathsf{r}=\mathsf{n}'$ and $\mathsf{s}=(s_1,\dots,s_k)=\mathsf{q}'$ gives the expression in (1.28). Now assume $a\in\mathsf{Z}_{k+1}$ and b is the unique element in $\mathsf{End}(\mathsf{V}^{\otimes k})$ so that $a\mathsf{e}_k=(b\otimes\mathbf{1})\mathsf{e}_k$. Then $a\mathsf{e}_k\in\mathsf{Z}_{k+1}$ so that $a\mathsf{e}_k=ga\mathsf{e}_kg^{-1}$ for all $g\in\mathsf{G}$. It follows that $(b\otimes\mathbf{1})\mathsf{e}_k=g(b\otimes\mathbf{1})\mathsf{e}_kg^{-1}=g(b\otimes\mathbf{1})g^{-1}\mathsf{e}_k=(gbg^{-1}\otimes\mathbf{1})\mathsf{e}_k$. The uniqueness of b forces $gbg^{-1}=b$ to hold for all $g\in\mathsf{G}$, so that $b\in\mathsf{Z}_k$ as claimed in (a).

(b) Part (a) implies that $\mathsf{Z}_{k+1}\mathsf{e}_k=\mathsf{Z}_k\mathsf{e}_k$ (where Z_k is identified with $\mathsf{Z}_k\otimes\mathbf{1}$). A symmetric argument gives $\mathsf{e}_k\mathsf{Z}_{k+1}=\mathsf{e}_k\mathsf{Z}_k$, and it follows that $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k=\mathsf{Z}_{k+1}\mathsf{e}_k\mathsf{Z}_{k+1}$ is an ideal of Z_{k+1}. Part (c) is a consequence of the uniqueness of b in the above proof. □

The Jones basic construction for $\mathsf{Z}_k\subseteq\mathsf{Z}_{k+1}$ is based on the ideal $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ of Z_{k+1} and the fact that $\Lambda_{k-1}(\mathsf{G})\subseteq\Lambda_{k+1}(\mathsf{G})$, and it involves the following two key ideas.

(1) When decomposing $\mathsf{V}^{\otimes(k+1)}$ let

$$\mathsf{V}^{\otimes(k+1)}_{\text{old}} = \bigoplus_{\lambda\in\Lambda_{k-1}(\mathsf{G})} m^{\lambda}_{k+1}\,\mathsf{G}^{\lambda} \tag{1.30}$$

$$\mathsf{V}^{\otimes(k+1)}_{\text{new}} = \bigoplus_{\lambda\in\Lambda_{k+1}(\mathsf{G})\setminus\Lambda_{k-1}(\mathsf{G})} m^{\lambda}_{k+1}\,\mathsf{G}^{\lambda}. \tag{1.31}$$

Thus, $\mathsf{V}^{\otimes(k+1)}=\mathsf{V}^{\otimes(k+1)}_{\text{old}}\oplus\mathsf{V}^{\otimes(k+1)}_{\text{new}}$. Using the fact that $\frac{1}{2}\mathsf{e}_k$ corresponds to the projection onto the trivial G-module in the last two tensor slots of $\mathsf{V}^{\otimes(k+1)}$, Wenzl ([W3, Prop. 4.10],[W4, Prop. 2.2]) proves that $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k\cong\mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes(k+1)}_{\text{old}})$. Applying the decomposition $\mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes(k+1)})\cong\mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes(k+1)}_{\text{old}})\oplus\mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes(k+1)}_{\text{new}})$ then gives

$$\mathsf{Z}_{k+1}\cong\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k\oplus\mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes(k+1)}_{\text{new}}). \tag{1.32}$$

(2) There is an algebra isomorphism $\mathsf{Z}_{k-1} \cong \mathsf{e}_k \mathsf{Z}_k \mathsf{e}_k$ via the map that sends $a \in \mathsf{Z}_{k-1}$ to $\mathsf{e}_k a \mathsf{e}_k = 2a\mathsf{e}_k = 2\mathsf{e}_k a$. Viewing $\mathsf{Z}_k\mathsf{e}_k$ as a module for $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ and for $\mathsf{Z}_{k-1} \cong \mathsf{e}_k\mathsf{Z}_k\mathsf{e}_k$ by multiplication on the left and right, respectively, we have that these actions commute and centralize one another:

$$\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k \cong \mathsf{End}_{\mathsf{Z}_{k-1}}(\mathsf{Z}_k\mathsf{e}_k) \qquad \text{and} \qquad \mathsf{Z}_{k-1} \cong \mathsf{End}_{\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k}(\mathsf{Z}_k\mathsf{e}_k).$$

Double-centralizer theory (e.g., [CR, Secs. 3B and 68]) then implies that the simple summands of the semisimple algebras $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ and Z_{k-1} (hence their irreducible modules) can be indexed by the same set $\Lambda_{k-1}(\mathsf{G})$.

As before, let $\mathsf{Z}_k^\lambda, \lambda \in \Lambda_k(\mathsf{G})$, denote the irreducible Z_k-modules. By restriction, Z_k^λ is a Z_{k-1}-module and

$$\mathsf{Res}^{\mathsf{Z}_k}_{\mathsf{Z}_{k-1}}(\mathsf{Z}_k^\lambda) = \bigoplus_{\mu \in \Lambda_{k-1}} \Theta_{\lambda,\mu}\, \mathsf{Z}_{k-1}^\mu,$$

where $\Theta_{\lambda,\mu}$ is the multiplicity of Z_{k-1}^μ in Z_k^λ. The $|\Lambda_k| \times |\Lambda_{k-1}|$ matrix Θ whose (λ,μ)-entry is $\Theta_{\lambda,\mu}$ is the *inclusion matrix* for $\mathsf{Z}_{k-1} \subseteq \mathsf{Z}_k$. For all of the groups G in this paper, the restriction is "multiplicity free" meaning that each $\Theta_{\lambda,\mu}$ is either 0 or 1.

General facts from double-centralizer theory imply that the inclusion matrix for $\mathsf{Z}_{k-1} \subseteq \mathsf{Z}_k$ is the transpose of the inclusion matrix for $\mathsf{End}_{\mathsf{Z}_k}(\mathsf{Z}_k\mathsf{e}_k) \subseteq \mathsf{End}_{\mathsf{Z}_{k-1}}(\mathsf{Z}_k\mathsf{e}_k)$, which implies the following:

> *In the Bratteli diagram for the tower of algebras Z_k, the edges between levels k and $k+1$ corresponding to $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k \subseteq \mathsf{Z}_{k+1}$ are the reflection over level k of the edges between $k-1$ and k corresponding to $\mathsf{Z}_{k-1} \subseteq \mathsf{Z}_k$.*

In Section 4.2, we have highlighted the edges of the Bratteli diagrams that are *not* reflections over level k and left unhighlighted the edges corresponding to the Jones basic construction.

> *The highlighted edges give a copy of the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (i.e. the Dynkin diagram) embedded in the Bratteli diagram.*

This will be discussed further in Examples 1.46.

1.7 Projection mappings

The Jones-Wenzl idempotents (see [W1], [Jo, Sec. 3], [FK]) in $\mathsf{TL}_k(2)$ are defined recursively by setting $\mathsf{f}_1 = \mathbf{1}$ and letting

$$\mathsf{f}_n = \mathsf{f}_{n-1} - \frac{n-1}{n}\mathsf{f}_{n-1}\mathsf{e}_{n-1}\mathsf{f}_{n-1}, \qquad 1 < n \le k. \tag{1.33}$$

These idempotents satisfy the following properties (see [W1], [FK], [CJ] for proofs),

$$\begin{array}{lll}
\text{(JW1)} & \mathsf{f}_n^2 = \mathsf{f}_n, & 1 \le n \le k-1, \\
\text{(JW2)} & \mathsf{e}_i\mathsf{f}_n = \mathsf{f}_n\mathsf{e}_i = 0, & 1 \le i < n \le k, \\
\text{(JW3)} & \mathsf{e}_i\mathsf{f}_n = \mathsf{f}_n\mathsf{e}_i, & 1 \le n < i \le k-1, \\
\text{(JW4)} & \mathsf{e}_n\mathsf{f}_n\mathsf{e}_n = \frac{n+1}{n}\mathsf{f}_{n-1}\mathsf{e}_n, & 1 \le n \le k-1, \\
\text{(JW5)} & 1 - \mathsf{f}_n \in \langle \mathsf{e}_1, \ldots, \mathsf{e}_{n-1} \rangle, & \\
\text{(JW6)} & \mathsf{f}_m\mathsf{f}_n = \mathsf{f}_n\mathsf{f}_m & 1 \le m, n \le k,
\end{array} \tag{1.34}$$

where $\langle \mathsf{e}_1, \ldots, \mathsf{e}_{n-1} \rangle$ stands for the subalgebra of $\mathsf{TL}_k(2)$ generated by $\mathsf{e}_1, \ldots, \mathsf{e}_{n-1}$. An expression for f_n in the $\mathsf{TL}_k(2)$ basis of words in the generators $\mathsf{e}_1, \ldots, \mathsf{e}_{k-1}$ can be found in [FK, Mo].

The simple SU_2-module $\mathsf{V}(k)$ appears in $\mathsf{V}^{\otimes k}$ with multiplicity 1, and it is does not appear as a simple summand of $\mathsf{V}^{\otimes \ell}$ for any $\ell < k$. To locate $\mathsf{V}(k)$ inside $\mathsf{V}^{\otimes k}$, let $\mathsf{r} = (r_1, \dots, r_k) \in \{-1, 1\}^k$ for some $k \geq 1$, and set

$$|\mathsf{r}| = |\{ r_i \mid r_i = -1 \}| . \tag{1.35}$$

Then the totally symmetric tensors $\mathsf{S}(\mathsf{V}^{\otimes k})$ form the $(k+1)$-dimensional subspace of $\mathsf{V}^{\otimes k}$ spanned by the vectors $\mathsf{w}_0, \mathsf{w}_1, \dots, \mathsf{w}_k$, where

$$\mathsf{w}_t = \sum_{|\mathsf{r}|=t} \mathsf{v}_\mathsf{r}, \qquad 0 \leq t \leq k. \tag{1.36}$$

It is well known (see for example [FH, Sec. 11.1]) that $\mathsf{S}(\mathsf{V}^{\otimes k}) \cong \mathsf{V}(k)$ as an SU_2-module, and that $\mathsf{f}_k(\mathsf{V}^{\otimes k}) = \mathsf{S}(\mathsf{V}^{\otimes k})$ ([FK, Prop. 1.3, Cor. 1.4]). In particular,

$$\mathsf{S}(\mathsf{V}^{\otimes 2}) = \mathsf{span}_\mathbb{C}\{\mathsf{w}_0 = \mathsf{v}_1 \otimes \mathsf{v}_1,\ \mathsf{w}_1 = \mathsf{v}_{-1} \otimes \mathsf{v}_1 + \mathsf{v}_1 \otimes \mathsf{v}_{-1},\ \mathsf{w}_2 = \mathsf{v}_{-1} \otimes \mathsf{v}_{-1}\} \cong \mathsf{V}(2).$$

Observe that $\mathsf{f}_2 = \mathbf{1} - \frac{1}{2}\mathsf{e}_1$ and $\mathsf{f}_2(\mathsf{w}_t) = \mathsf{w}_t$ for $t = 0, 1, 2$.

1.8 Projections related to branch nodes

A *branch node* in the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is any vertex of degree greater than 2. Let $\mathsf{br}(\mathsf{G})$ denote the branch node in $\mathcal{R}_\mathsf{V}(\mathsf{G})$, and in the case of $\mathbf{D}_n (n > 2)$, which has 2 branch nodes, set $\mathsf{br}(\mathbf{D}_n) = 1$. In the special case of $\mathcal{R}_\mathsf{V}(\mathbf{C}_n)$ for $n \leq \infty$, we consider the affine node itself to be the branch node, so that $\mathsf{br}(\mathbf{C}_n) = 0$. When $\mathsf{G} = \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}, \mathbf{C}_\infty$, or $\mathbf{D}_\infty$, we say that the *diameter* of $\mathcal{R}_\mathsf{V}(\mathsf{G})$, denoted by $\mathsf{di}(\mathsf{G})$, is the maximum distance between any vertex $\lambda \in \Lambda(\mathsf{G})$ and $0 \in \Lambda(\mathsf{G})$. In particular, $\mathsf{di}(\mathsf{G}) = \infty$ for $\mathsf{G} = \mathbf{C}_\infty$ or $\mathbf{D}_\infty$. For $\mathsf{G} = \mathbf{C}_n$, we let $\mathsf{di}(\mathsf{G}) = \tilde{n}$, where $\tilde{n}$ is as in (1.37).

G	SU_2	$\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{T}$	$\mathbf{O}$	$\mathbf{I}$	$\mathbf{C}_\infty$	$\mathbf{D}_\infty$
$\mathsf{di}(\mathsf{G})$	∞	$\tilde{n}$	n	4	6	7	∞	∞
$\mathsf{br}(\mathsf{G})$	$\emptyset$	0	1	2	3	5	0	1

$$\text{where} \quad \tilde{n} = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases} \tag{1.37}$$

In this section, we develop a recursive procedure for constructing the idempotents f_ν that project onto the irreducible G-summands G^ν of $\mathsf{V}_{\mathsf{new}}^{\otimes k}$. Assume $k \leq \ell$, where $\ell = \mathsf{br}(\mathsf{G})$. Then $\mathsf{V}_{\mathsf{new}}^{\otimes k} = \mathsf{G}^{(k)} = \mathsf{V}(k)$. In this case, the projection of $\mathsf{V}^{\otimes k}$ onto $\mathsf{G}^{(k)}$ is given by $\mathsf{f}_{(k)} := \mathsf{f}_k$, where f_k is the Jones-Wenzl idempotent. The irreducible SU_2-module $\mathsf{V}(\ell+1)$ is reducible as a G-module. Suppose $\mathsf{deg}(\ell)$ is the degree of the branch node indexed by ℓ in the representation graph of G, so that $\mathsf{deg}(\ell) = 3$, except when $\mathsf{G} = \mathbf{D}_2$ where $\mathsf{deg}(\ell) = 4$. We assume the decomposition into irreducible G-modules is given by $\mathsf{V}_{\mathsf{new}}^{\otimes(\ell+1)} = \mathsf{V}(\ell+1) = \bigoplus_{j=1}^{\mathsf{deg}(\ell)-1} \mathsf{G}^{\beta_j}$. For example, when $\mathsf{G} = \mathbf{O}$, then $\ell = 3$ and $\mathsf{V}_{\mathsf{new}}^{\otimes 4} = \mathsf{V}(4) = \mathsf{G}^{(4^+)} \oplus \mathsf{G}^{(4^-)}$; and when $\mathsf{G} = \mathbf{D}_2$, then $\ell = 1$ and $\mathsf{V}_{\mathsf{new}}^{\otimes 2} = \mathsf{V}(2) = \mathsf{G}^{(0')} \oplus \mathsf{G}^{(2)} \oplus \mathsf{G}^{(2')}$, where the labels of the irreducible G-modules are as in Section 4.1.

Let

$$\mathsf{f}_{\ell+1} = \sum_{j=1}^{\mathsf{deg}(\ell)-1} \mathsf{f}_{\beta_j} \tag{1.38}$$

be the decomposition of the Jones-Wenzl idempotent $\mathsf{f}_{\ell+1}$ into minimal orthogonal idempotents that commute with G and project $\mathsf{V}_{\mathsf{new}}^{\otimes(\ell+1)}$ onto the irreducible G-summands G^{β_j}. For finite subgroups G, these idempotents can be constructed using the corresponding irreducible characters χ_{β_j} as

$$\mathsf{f}_{\beta_j} = \frac{\dim \mathsf{G}^{\beta_j}}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \overline{\chi_{\beta_j}(g)}\, g^{\otimes(\ell+1)}, \tag{1.39}$$

where $g^{\otimes(\ell+1)}$ is the matrix of g on $\mathsf{V}^{\otimes(\ell+1)}$ and "$-$" denotes complex conjugate. (See for example, [FH, (2.32)].) We will not need these explicit expressions in this paper.

Lemma 1.40. *Let $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$ for all k. Let $\lambda \in \Lambda_k(\mathsf{G})$, and assume $\mathsf{f}_\lambda \in \mathsf{Z}_k$ projects $\mathsf{V}^{\otimes k}$ onto the irreducible G-module G^λ in $\mathsf{V}^{\otimes k}_{\mathsf{new}}$. Let $d^\lambda = \dim \mathsf{G}^\lambda$, and suppose $\mathsf{G}^\lambda \otimes \mathsf{V} = \bigoplus_i \mathsf{G}^{\mu_i}$. Let f_{μ_i} be the orthogonal idempotents in Z_{k+1} that project $\mathsf{V}^{\otimes(k+1)}$ onto the irreducible G-modules G^{μ_i}, and assume $d^{\mu_i} = \dim \mathsf{G}^{\mu_i}$. Then the following hold for all i such that G^{μ_i} is a summand of $\mathsf{V}^{\otimes(k+1)}_{\mathsf{new}}$; i.e., for all $\mu_i \in \Lambda_{k+1}(\mathsf{G}) \setminus \Lambda_{k-1}(\mathsf{G})$:*

(i) $\mathsf{f}_\lambda \mathsf{f}_{\mu_i} = \mathsf{f}_{\mu_i} = \mathsf{f}_{\mu_i}\mathsf{f}_\lambda$;

(ii) f_{μ_i} *commutes with* e_j *for* $j > k+1$;

(iii) $\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) = \frac{d^{\mu_i}}{2d^\lambda}\mathsf{f}_\lambda$ *; and*

(iv) $\mathsf{e}_{k+1}\mathsf{f}_{\mu_i}\mathsf{e}_{k+1} = 2\varepsilon_{k+1}(\mathsf{f}_{\mu_i})\mathsf{e}_{k+1} = \frac{d^{\mu_i}}{d^\lambda}\mathsf{f}_\lambda\mathsf{e}_{k+1}$.

Proof. (i) Now $\mathsf{f}_{\mu_i}\mathsf{f}_\lambda\left(\mathsf{V}^{\otimes(k+1)}\right) = \mathsf{f}_{\mu_i}(\mathsf{G}^\lambda \otimes \mathsf{V}) = \mathsf{G}^{\mu_i} = \mathsf{f}_{\mu_i}(\mathsf{V}^{\otimes(k+1)})$, so $\mathsf{f}_{\mu_i}\mathsf{f}_\lambda = \mathsf{f}_{\mu_i}$. For the product in the other order, we have $\mathsf{f}_\lambda\mathsf{f}_{\mu_i}\left(\mathsf{V}^{\otimes(k+1)}\right) = \mathsf{f}_\lambda(\mathsf{G}^{\mu_i})$. Since G^{μ_i} is contained in $\mathsf{G}^\lambda \otimes \mathsf{V}$ and f_λ acts as the identity on that space, $\mathsf{f}_\lambda(\mathsf{G}^{\mu_i}) = \mathsf{G}^{\mu_i} = \mathsf{f}_{\mu_i}(\mathsf{V}^{\otimes(k+1)})$, which implies the result.

(ii) This is clear, because $\mathsf{f}_{\mu_i} \in \mathsf{Z}_{k+1}$, and Z_{k+1} commutes with e_j for $j > k+1$.

(iii) From (i) and part (c) of Proposition 1.26, we have for each i,

$$\mathsf{f}_\lambda\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) = \varepsilon_{k+1}(\mathsf{f}_\lambda\mathsf{f}_{\mu_i}) = \varepsilon_{k+1}(\mathsf{f}_{\mu_i}) = \varepsilon_{k+1}(\mathsf{f}_{\mu_i}\mathsf{f}_\lambda) = \varepsilon_{k+1}(\mathsf{f}_{\mu_i})\mathsf{f}_\lambda. \tag{1.41}$$

Suppose $\mathsf{W} = \mathsf{V}^{\otimes k}$, and let $\mathsf{W} = \mathsf{W}_0 \oplus \mathsf{W}_1$ be the eigenspace decomposition of f_λ on W so that $\mathsf{f}_\lambda w = jw$ for $w \in \mathsf{W}_j$, $j = 0,1$. Since f_λ and $\varepsilon_{k+1}(\mathsf{f}_{\mu_i})$ commute by (1.41), $\varepsilon_{k+1}(\mathsf{f}_{\mu_i})$ maps W_j into itself for $j = 0,1$. Now if $w \in \mathsf{W}_0$, then $\varepsilon_{k+1}(\mathsf{f}_{\mu_i})w = \varepsilon_{k+1}(\mathsf{f}_{\mu_i})\mathsf{f}_\lambda w = 0$, so that $\varepsilon_{k+1}(\mathsf{f}_{\mu_i})$ is 0 on W_0. Since $\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) \in \mathsf{Z}_k$, we have $\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) \in \mathsf{End}_\mathsf{G}(\mathsf{W}_1) = \mathbb{C}\mathsf{id}_{\mathsf{W}_1} = \mathbb{C}\mathsf{f}_\lambda$ by Schur's lemma, as $\mathsf{W}_1 = \mathsf{G}^\lambda$, an irreducible G-module. Therefore, there exists $\xi_i \in \mathbb{C}$ such that $\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) = \xi_i\mathsf{f}_\lambda$ on W_1. But since these transformations agree on W_0 (as both equal 0 on W_0), we have $\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) = \xi_i\mathsf{f}_\lambda$ on $\mathsf{V}^{\otimes k}$. Taking traces and using (d) of Proposition 1.26 gives

$$d^{\mu_i} = \mathsf{tr}_{k+1}(\mathsf{f}_{\mu_i}) = \mathsf{tr}_{k+1}\left(\varepsilon_{k+1}(\mathsf{f}_{\mu_i})\right) = \xi_i\mathsf{tr}_{k+1}(\mathsf{f}_\lambda) = 2\xi_i\mathsf{tr}_k(\mathsf{f}_\lambda) = 2\xi_i d^\lambda.$$

Therefore $\xi_i = \frac{d^{\mu_i}}{2d^\lambda}$ so that $\varepsilon_{k+1}(\mathsf{f}_{\mu_i}) = \frac{d^{\mu_i}}{2d^\lambda}\mathsf{f}_\lambda$, as asserted in (iii). Part (iv) follows immediately from (iii) and Proposition 1.26 (a). □

Proposition 1.42. *Assume the notation of Lemma 1.40, and let $\mu = \mu_i \in \Lambda_{k+1}(\mathsf{G}) \setminus \Lambda_{k-1}(\mathsf{G})$ for some i. Suppose $\mathsf{G}^\mu \otimes \mathsf{V} = \mathsf{G}^\lambda \oplus \mathsf{G}^\nu$, where G^ν is an irreducible G-module of dimension d^ν and $\nu \in \Lambda_{k+2}(\mathsf{G}) \setminus \Lambda_k(\mathsf{G})$. Set*

$$\mathsf{f}_\nu = \mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu.$$

Then the following hold:

(i) f_ν *is an idempotent in* Z_{k+2};

(ii) $\mathsf{f}_\mu\mathsf{f}_\nu = \mathsf{f}_\nu = \mathsf{f}_\nu\mathsf{f}_\mu$;

(iii) f_ν *commutes with* e_j *for* $j > k+2$;

(iv) $\mathsf{e}_{k+2}\mathsf{f}_\nu\mathsf{e}_{k+2} = \frac{d^\nu}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+2}$, *so that* $\varepsilon_{k+2}(\mathsf{f}_\nu) = \frac{d^\nu}{2d^\mu}\mathsf{f}_\mu$.

(v) f_ν *projects* $\mathsf{V}^{\otimes(k+2)}$ *onto* G^ν.

Proof. (i) We have

$$\begin{aligned}
\mathsf{f}_\nu^2 &= \left(\mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu\right)^2 = \mathsf{f}_\mu - 2\frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu + \frac{(d^\lambda)^2}{(d^\mu)^2}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu \\
&= \mathsf{f}_\mu - 2\frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu + \frac{(d^\lambda)^2}{(d^\mu)^2}\frac{d^\mu}{d^\lambda}\mathsf{f}_\mu\mathsf{f}_\lambda\mathsf{e}_{k+1}\mathsf{f}_\mu \qquad \text{by (iv) of Lemma 1.40} \\
&= \mathsf{f}_\mu - 2\frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu + \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu \qquad \text{by (i) of Lemma 1.40} \\
&= \mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu = \mathsf{f}_\nu.
\end{aligned}$$

By construction, $\mathsf{f}_\nu \in \mathsf{Z}_{k+2}$.

Part (ii) follows easily from the definition of f_ν and the fact that f_μ is an idempotent.

(iii) Since $\mathsf{f}_\mu \in \mathsf{Z}_{k+1}$ commutes with e_j for $j > k+1$, and e_{k+1} commutes with e_j for $j > k+2$, f_ν commutes with e_j for $j > k+2$.

For (iv) we compute

$$\begin{aligned}
\mathsf{e}_{k+2}\mathsf{f}_\nu\mathsf{e}_{k+2} &= \mathsf{e}_{k+2}\left(\mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu\right)\mathsf{e}_{k+2} \\
&= \mathsf{e}_{k+2}^2\mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+2}\mathsf{e}_{k+1}\mathsf{e}_{k+2}\mathsf{f}_\mu \qquad \text{using (ii) of Lemma 1.40} \\
&= 2\mathsf{e}_{k+2}\mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+2}\mathsf{f}_\mu \\
&= \left(2 - \frac{d^\lambda}{d^\mu}\right)\mathsf{f}_\mu\mathsf{e}_{k+2} = \frac{2d^\mu - d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+2} = \frac{d^\nu}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+2}.
\end{aligned}$$

This equation along with Proposition 1.26 (a) implies that $\varepsilon_{k+2}(\mathsf{f}_\mu) = \frac{d^\nu}{2d^\mu}\mathsf{f}_\mu \in \mathsf{Z}_{k+1}$.

(v) From (1.32) with $k+2$ instead of $k+1$, we have $\mathsf{Z}_{k+2} = \mathsf{Z}_{k+1}\mathsf{e}_{k+1}\mathsf{Z}_{k+1} \oplus \mathsf{End}_\mathsf{G}(\mathsf{V}_{\mathsf{new}}^{\otimes(k+2)})$. Now

$$\mathsf{f}_\mu \otimes \mathbf{1} = \mathsf{f}_\mu = \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_u + \mathsf{f}_\nu \in \mathsf{Z}_{k+2}, \tag{1.43}$$

and the idempotent $\mathsf{f}_\mu \otimes \mathbf{1}$ projects $\mathsf{V}^{\otimes(k+2)}$ onto $\mathsf{G}^\mu \otimes \mathsf{V} = \mathsf{G}^\lambda \oplus \mathsf{G}^\nu$. Observe that $\mathsf{p} := \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu \in \mathsf{Z}_{k+1}\mathsf{e}_{k+1}\mathsf{Z}_{k+1}$, and

$$\mathsf{p}^2 = \left(\frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu\right)^2 = \frac{(d^\lambda)^2}{(d^\mu)^2}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu = \frac{(d^\lambda)^2}{(d^\mu)^2}\frac{d^\mu}{d^\lambda}\mathsf{f}_\mu\mathsf{f}_\lambda\mathsf{e}_{k+1}\mathsf{f}_\mu = \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu = \mathsf{p}$$

using Lemma 1.40 (iv), so we can conclude that p is an idempotent once we know it is nonzero. But if $\mathsf{p} = 0$, then

$$0 = 2\mathsf{e}_{k+1}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu\mathsf{e}_{k+1} = (\mathsf{e}_{k+1}\mathsf{f}_\mu\mathsf{e}_{k+1})^2 = \left(\frac{d^\mu}{d^\lambda}\mathsf{f}_\lambda e_{k+1}\right)^2 = 2\frac{(d^\mu)^2}{(d^\lambda)^2}\mathsf{f}_\lambda\mathsf{e}_{k+1}$$

by Lemma 1.40 (iv). Since f_λ and e_{k+1} act on different tensor slots and both are nonzero, we have reached a contradiction. Thus, p is an idempotent. Moreover, $\mathsf{f}_\nu\mathsf{p} = (\mathsf{f}_\mu - \mathsf{p})\mathsf{p} = 0$. Therefore, (1.43) gives the decomposition of $\mathsf{f}_\mu \otimes \mathbf{1}$ into orthogonal idempotents, with the first idempotent $\mathsf{p} = \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_{k+1}\mathsf{f}_\mu$ in $\mathsf{Z}_{k+1}\mathsf{e}_{k+1}\mathsf{Z}_{k+1} = \mathsf{End}_\mathsf{G}(\mathsf{V}_{\mathsf{old}}^{\otimes(k+2)})$. Then $\mathsf{G}^\mu \otimes \mathsf{V} = \mathsf{p}\left(\mathsf{V}^{\otimes(k+2)}\right) \oplus \mathsf{f}_\nu\left(\mathsf{V}^{\otimes(k+2)}\right)$ is a decomposition of $\mathsf{G}^\mu \otimes \mathsf{V} = \mathsf{G}^\lambda \oplus \mathsf{G}^\nu$ into G-submodules such that $\mathsf{p}\left(\mathsf{V}^{\otimes(k+2)}\right) \in \mathsf{V}_{\mathsf{old}}^{\otimes(k+2)}$. Hence, $\mathsf{p}\left(\mathsf{V}^{\otimes(k+2)}\right) = \mathsf{G}^\lambda$ and $\mathsf{f}_\nu\left(\mathsf{V}^{\otimes(k+2)}\right) = \mathsf{G}^\nu$. □

This procedure can be applied recursively to produce the projection idempotents for all $k \leq \mathsf{di}(\mathsf{G})$ that do not come from (1.39), as illustrated in the next series of examples. Our labeling of the irreducible G-modules is as in Section 4.1.

Examples 1.44. • *The* $\mathsf{G} = \mathbf{T}, \mathbf{O}, \mathbf{I}$ *cases:* Let $\ell = \mathsf{br}(\mathsf{G})$ (so $\ell = 2, 3, 5$, respectively) and set $\mathsf{f}_{(k)} = \mathsf{f}_k$ for $0 \leq k \leq \ell$, where f_k is given by (1.33). Let $\mathsf{f}_{((\ell+1)^+)}$ and $\mathsf{f}_{((\ell+1)^-)}$ be the projections onto $\mathsf{G}^{((\ell+1)^+)}$ and $\mathsf{G}^{((\ell+1)^-)}$, respectively, which can be constructed using (1.39). When $\mathsf{G} = \mathbf{T}$, applying Proposition 1.42 with $\lambda = (2)$, $\mu = (3^{\pm})$, and $\nu = (4^{\pm})$ produces the idempotents $\mathsf{f}_{(4^{\pm})} = \mathsf{f}_{(3^{\pm})} - \frac{3}{2}\mathsf{f}_{(3^{\pm})}\mathsf{e}_3\mathsf{f}_{(3^{\pm})}$ that project $\mathsf{V}^{\otimes 4}$ onto $\mathsf{G}^{(4^{\pm})}$. When $\mathsf{G} = \mathbf{O}$, first taking $\lambda = (3)$, $\mu = (4^+)$, $\nu = (5)$ and then taking $\lambda = (4^+)$, $\mu = (5)$, $\nu = (6)$ in Proposition 1.42 will construct the two remaining idempotents $\mathsf{f}_{(5)}$ and $\mathsf{f}_{(6)}$. Similarly, for $\mathsf{G} = \mathbf{I}$, applying the procedure to $\lambda = (5)$, $\mu = (6^+)$, and $\nu = (7)$, will produce the last idempotent $\mathsf{f}_{(7)}$.

- *The* $\mathsf{G} = \mathbf{C}_n$*,* $n \leq \infty$ *case:* Let $\mathsf{f}_{(1)}$ and $\mathsf{f}_{(-1)}$ project V onto the one-dimensional modules $\mathsf{G}^{(1)}$ and $\mathsf{G}^{(-1)}$, respectively. Applying Proposition 1.42 with $\lambda = (0)$ (i.e. with $\mathsf{f}_{(0)} = \mathbf{1}$), $\mu = (\pm 1)$, and $\nu = (\pm 2)$ begins the recursive process and constructs $\mathsf{f}_{(\pm 2)}$. (We are adopting the conventions that $(+j)$ stands for (j) and $\mathsf{G}^{(j)} \cong \mathsf{G}^{(i)}$ whenever $n < \infty$ and $j \equiv i \bmod n$.) Then assuming we have constructed $\mathsf{f}_{(\pm j)}$ for all $1 \leq j \leq k$, we obtain from the proposition that $\mathsf{f}_{(\pm(k+1))} = \mathsf{f}_{(\pm k)} - \mathsf{f}_{(\pm k)}\mathsf{e}_k\mathsf{f}_{(\pm k)}$ projects $\mathsf{V}_{\mathsf{new}}^{\otimes(k+1)}$ onto $\mathsf{G}^{(\pm(k+1))}$. In the case that $\mathsf{G} = \mathbf{C}_\infty$, iterations of this process produce the idempotent projections onto $\mathsf{V}_{\mathsf{new}}^{\otimes(\pm(k+1))}$ for all $k \geq 1$.

  When $n < \infty$, the diameter is $\tilde{n}$, and we proceed as above to construct the idempotents $\mathsf{f}_{(\pm k)}$ for $k \leq \tilde{n}$. Now $\mathsf{V}_{\mathsf{new}}^{\otimes \tilde{n}} = \mathsf{G}^{(\tilde{n})} \oplus \mathsf{G}^{(\tilde{n})}$ when n is even, as $-\tilde{n} \equiv \tilde{n} \bmod n$; and $\mathsf{V}_{\mathsf{new}}^{\otimes \tilde{n}} = \mathsf{G}^{(0)} \oplus \mathsf{G}^{(0)}$ when n is odd, as $\tilde{n} = n$. The idempotent $\mathsf{f}_{(\pm \tilde{n})}$ projects onto the space $\mathbb{C}\mathsf{v}_{\pm \underline{1}}$, where $\underline{1}$ is the $\tilde{n}$ tuple of all 1s, and $\mathsf{v}_{\pm \underline{1}}$ has $\tilde{n}$ tensor factors equal to $\mathsf{v}_{\pm 1}$. In the centralizer algebra $\mathsf{Z}_{\tilde{n}}$ there is a corresponding 2×2 matrix block. The idempotents $\mathsf{f}_{(\tilde{n})}$, $\mathsf{f}_{(-\tilde{n})}$ act as the diagonal matrix units $\mathsf{E}_{\underline{1},\underline{1}}$, $\mathsf{E}_{-\underline{1},-\underline{1}}$, respectively. The remaining basis elements of the matrix block are the matrix units $\mathsf{E}_{\underline{1},-\underline{1}}$, $\mathsf{E}_{-\underline{1},\underline{1}}$.

- *The* $\mathsf{G} = \mathbf{D}_n$*,* $2 \leq n \leq \infty$ *case:* Suppose first that $n \geq 3$. The symmetric tensors in $\mathsf{V}^{\otimes 2}$ are reducible in the $\mathbf{D}_n$-case and decompose into a direct sum of the one-dimensional G-module $\mathsf{G}^{(0')}$ and the two-dimensional irreducible G-module $\mathsf{G}^{(2)}$. Let $\mathsf{f}_{(0')}$ and $\mathsf{f}_{(2)}$ denote the corresponding projections. Note that $\mathsf{f}_{(0')} + \mathsf{f}_{(2)} = \mathsf{f}_2 = \mathbf{1} - \frac{1}{2}\mathsf{e}_1$. Starting with $\lambda = (1)$ (so $\mathsf{f}_{(1)} = \mathbf{1}$), $\mu = (2)$ and $\nu = (3)$, and applying the recursive procedure, we obtain the idempotents $\mathsf{f}_{(k)} = \mathsf{f}_{(k-1)} - \mathsf{f}_{(k-1)}\mathsf{e}_{k-1}\mathsf{f}_{(k-1)}$ for all $k = 3, 4, \ldots$ in the $\mathbf{D}_\infty$-case, and for $3 \leq k \leq n-1$ in the $\mathbf{D}_n$-case. Now $\mathsf{G}^{(n-1)} \otimes \mathsf{V} = \mathsf{G}^{(n-2)} \oplus \mathsf{G}^{(n)} \oplus \mathsf{G}^{(n')}$, where $\mathsf{G}^{(n)}$ and $\mathsf{G}^{(n')}$ are one-dimensional G-modules. Denoting the projections onto them by $\mathsf{f}_{(n)}$ and $\mathsf{f}_{(n')}$ (they can be constructed using (1.39)), we have $\mathsf{f}_{(n-2)} + \mathsf{f}_{(n)} + \mathsf{f}_{(n')} = \mathsf{f}_{(n-1)} \otimes \mathbf{1}$. Thus, $\mathsf{Z}_n = \langle \mathsf{Z}_{n-1}, \mathsf{e}_n, \mathsf{f}_{(n)} \rangle$.

  Now when $\mathsf{G} = \mathbf{D}_2$, then $\mathsf{br}(\mathsf{G}) = 1$, and the branch node $\mathsf{br}(\mathsf{G})$ has degree 4 in the corresponding Dynkin diagram. In this case

  $$1 - \frac{1}{2}\mathsf{e}_1 = \mathsf{f}_2 = \mathsf{f}_{(0')} + \mathsf{f}_{(2')} + \mathsf{f}_{(2)},$$

  where the 3 summands on the right are mutually orthogonal idempotents giving the projections onto the one-dimensional irreducible G-modules $\mathsf{G}^{(0')}$, $\mathsf{G}^{(2')}$, and $\mathsf{G}^{(2)}$, respectively. Thus $\mathsf{Z}_2 = \mathbb{C}\mathbf{1} \oplus \mathbb{C}\mathsf{e}_1 \oplus \mathbb{C}\mathsf{f}_{(0')} \oplus \mathbb{C}\mathsf{f}_{(2')}$.

Note that when $2 \le n < \infty$, and $\mu = (n)$ or (n') (or $(0')$ when $n = 2$), then by Lemma 1.40,

$$\varepsilon_n(\mathsf{f}_\mu) = \frac{1}{4}\mathsf{f}_{(n-1)}, \qquad \mathsf{e}_n\mathsf{f}_\mu\mathsf{e}_n = 2\varepsilon_n(\mathsf{f}_\mu)\mathsf{e}_n = \frac{1}{2}\mathsf{f}_{(n-1)}\mathsf{e}_n,$$

(where $\mathsf{f}_{(n-1)} = \mathsf{f}_{(1)} = \mathsf{f}_1 = \mathbf{1}$ when $n = 2$).

Theorem 1.45. *Let* G, $\mathsf{br}(\mathsf{G})$, *and* $\mathsf{di}(\mathsf{G})$ *be as in* (1.37), *and let* $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$. *Then* $\mathsf{Z}_1 = \mathbb{C}\mathbf{1} \cong \mathsf{Z}_0$. *Moreover,*

(a) *if* $1 \le k < \mathsf{di}(\mathsf{G})$, *and* $k \ne \tilde{n} - 1$ *in the case* $\mathsf{G} = \mathbf{C}_n$, *then* $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k \oplus \mathsf{End}_\mathsf{G}(\mathsf{V}_{\text{new}}^{\otimes k})$, *where* $\mathsf{End}_\mathsf{G}(\mathsf{V}_{\text{new}}^{\otimes k})$ *is a commutative subalgebra of dimension equal to the number of nodes in* $\mathcal{R}_\mathsf{V}(\mathsf{G})$ *a distance* k *from the trivial node;*

(b) *if* $k \ge \mathsf{di}(\mathsf{G})$, *then* $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$*;*

(c) *if* $1 \le k < \mathsf{di}(\mathsf{G})$, $k \ne \mathsf{br}(\mathsf{G})$, *and* $k \ne n-1$ *in the case* $\mathsf{G} = \mathbf{D}_n$, *then* $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k \rangle$*;*

(d) *if* $k = \mathsf{br}(\mathsf{G})$ *and* $\mathsf{G} \ne \mathbf{D}_2$, *then* $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k, \mathsf{f}_\mu \rangle$, *where* μ *is either of the two elements in* $\Lambda_{k+1}(\mathsf{G}) \setminus \Lambda_{k-1}(\mathsf{G})$, *and* f_μ *is the projection of* $\mathsf{V}_{\mathsf{new}}^{\otimes(k+1)}$ *onto* G^μ,

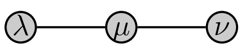
(e) *if* $\mathsf{G} = \mathbf{C}_n$ $(n < \infty)$, *then* $\mathsf{Z}_{\tilde{n}} = \langle \mathsf{Z}_{\tilde{n}-1}, \mathsf{e}_{\tilde{n}-1}, \mathsf{E}_{\mathsf{p},\mathsf{q}}, \text{for } \mathsf{p}, \mathsf{q} \in \{-\underline{1}, \underline{1}\} \rangle$, *where* $\mathsf{E}_{\mathsf{p},\mathsf{q}}$ *is the matrix unit in Examples 1.44.*

(f) *if* $\mathsf{G} = \mathbf{D}_n$ $(2 < n < \infty)$ *then* $\mathsf{Z}_n = \langle \mathsf{Z}_{n-1}, \mathsf{e}_{n-1}, \mathsf{f}_\mu \rangle$, *where* $\mu \in \{(n), (n')\}$

(g) *if* $\mathsf{G} = \mathbf{D}_2$, *then* $\mathsf{Z}_2 = \langle \mathsf{Z}_1, \mathsf{e}_1, \mathsf{f}_{\mu_1}, \mathsf{f}_{\mu_2} \rangle$ *where* $\mu_1, \mu_2 \in \{(0'), (2), (2')\}$, $\mu_1 \ne \mu_2$.

Proof. (a) Since $k < \mathsf{di}(\mathsf{G})$, $\mathsf{V}_{\text{new}}^{\otimes(k+1)}$ is a direct sum of irreducible G-modules each with multiplicity 1 (except for the case where $\mathsf{G} = \mathbf{C}_n$ and $k = \tilde{n} - 1$ which is handled in part (e)), and the centralizer $\mathsf{End}_\mathsf{G}(\mathsf{V}_{\text{new}}^{\otimes(k+1)})$ is commutative and spanned by central idempotents which project onto the irreducible summands. Thus, the dimension equals the number of new modules that appear at level $k + 1$.

(b) When $k \ge \mathsf{di}(\mathsf{G})$, $\mathsf{V}_{\text{new}}^{\otimes(k+1)} = 0$, and $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ follows from (1.32).

(c) If 

represents the neighborhood of a node μ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ with μ a distance k from the trivial node and $\mathsf{deg}(\mu) = 2$, then $\mathsf{G}^{(\nu)}$ has multiplicity 1 in $\mathsf{V}^{\otimes(k+1)}$. By Proposition 1.42, the central projection from $\mathsf{V}^{\otimes(k+1)}$ onto G^ν is given by $\mathsf{f}_\nu = \mathsf{f}_\mu - \frac{d^\lambda}{d^\mu}\mathsf{f}_\mu\mathsf{e}_k\mathsf{f}_\mu$, where $\mathsf{f}_\mu \in \mathsf{Z}_k$ projects $\mathsf{V}^{\otimes k}$ to G^μ. Thus $\mathsf{f}_\nu \in \langle \mathsf{Z}_k, \mathsf{e}_k \rangle$, and the set of f_ν, for ν a distance $k+1$ from the trivial node in $\mathcal{R}_\mathsf{V}(\mathsf{G})$, generate $\mathsf{End}_\mathsf{G}(\mathsf{V}_{\text{new}}^{\otimes(k+1)})$. The result then follows from part (a).

(d) if $k = \mathsf{br}(\mathsf{G})$ and $\mathsf{G} \ne \mathbf{D}_2$, then $\mathsf{V}_{\mathsf{new}}^{\otimes(k+1)} \cong \mathsf{V}(k+1) \cong \mathsf{G}^{\beta_1} \oplus \mathsf{G}^{\beta_2}$ where $\{\beta_1, \beta_2\} = \Lambda_{k+1}(\mathsf{G}) \setminus \Lambda_{k-1}(\mathsf{G})$. The Jones-Wenzl idempotent decomposes as $\mathsf{f}_{k+1} = \mathsf{f}_{\beta_1} + \mathsf{f}_{\beta_2}$ as in (1.38) (where the f_{β_j} can be constructed as in (1.39)). We know from (1.34) that $\mathbf{1} - \mathsf{f}_{k+1} \in \langle \mathsf{e}_1, \ldots, \mathsf{e}_k \rangle \subseteq \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$, and we have $\mathsf{f}_{\beta_1} + \mathsf{f}_{\beta_2} = \mathbf{1} - (\mathbf{1} - \mathsf{f}_{k+1})$. Thus, Z_{k+1} is generated by $\mathsf{Z}_k, \mathsf{e}_k, \mathsf{f}_{\beta_j}$ for $j = 1$ or $j = 2$.

(e) If $\mathsf{G} = \mathbf{C}_n$ and $k = \tilde{n} - 1$, then $\mathsf{G}^{(\tilde{n})}$ has multiplicity 2 in $\mathsf{V}^{\otimes\tilde{n}}$, where $\mathsf{G}^{(\tilde{n})} := \mathsf{G}^{(0)}$ if n is odd. In this case, $\mathsf{End}_\mathsf{G}(\mathsf{V}_{\text{new}}^{\otimes\tilde{n}})$ is 4-dimensional with a basis of matrix units as in Examples 1.44.

(f) If $\mathsf{G} = \mathbf{D}_n$ with $2 < n < \infty$, then $\mathsf{V}_{\text{new}}^{\otimes n} \cong \mathsf{G}^{(n)} \oplus \mathsf{G}^{(n')}$, and we let $\mathsf{f}_{(n)}$ and $\mathsf{f}_{(n')}$ project $\mathsf{V}_{\text{new}}^{\otimes n}$ onto $\mathsf{G}^{(n)}$ and $\mathsf{G}^{(n')}$, respectively. As in part (d), these minimal central idempotents can be constructed using (1.39); the only difference in this case is that $\mathsf{f}_{(n)} + \mathsf{f}_{(n')}$ does not equal the Jones-Wenzl idempotent f_n; however, $\mathsf{f}_{(n-1)} = \mathsf{f}_{(n-2)} + \mathsf{f}_{(n)} + \mathsf{f}_{(n')}$ holds.

(g) When $\mathsf{G} = \mathbf{D}_2$, $\mathsf{V}_{\text{new}}^{\otimes 2} \cong \mathsf{G}^{(0')} \oplus \mathsf{G}^{(2)} \oplus \mathsf{G}^{(2')}$ and the corresponding central idempotents $\mathsf{f}_{(0')}, \mathsf{f}_{(2)}, \mathsf{f}_{(2')}$ can be constructed as in (1.39). Furthermore $\mathsf{f}_{(0')} + \mathsf{f}_{(2)} + \mathsf{f}_{(2')} = \mathsf{f}_2 = 1 - \frac{1}{2}\mathsf{e}_1$, so Z_2 is generated by e_1, Z_1, and any two of $\mathsf{f}_{(0')}, \mathsf{f}_{(2)}, \mathsf{f}_{(2')}$. □

Examples 1.46. For all $\mathsf{G} \neq \mathbf{C}_n$ for $2 \leq n \leq \infty$, we have $\mathsf{Z}_1 = \mathbb{C}\mathbf{1} \cong \mathsf{Z}_0$.

- If $\mathsf{G} = \mathbf{O}$, then from Theorem 1.45 we deduce the following: $\mathsf{Z}_2 = \mathsf{Z}_1\mathsf{e}_1\mathsf{Z}_1 + \mathbb{C}\mathsf{f}_2 = \mathbb{C}\mathsf{e}_1 + \mathbb{C}\mathsf{f}_2 \cong \mathsf{TL}_2(2)$, where $\mathsf{f}_2 = 1 - \frac{1}{2}\mathsf{e}_1$; $\mathsf{Z}_3 = \mathsf{Z}_2\mathsf{e}_2\mathsf{Z}_2 + \mathbb{C}\mathsf{f}_3 \cong \mathsf{TL}_3(2)$ where $\mathsf{f}_3 = \mathsf{f}_2 - \frac{2}{3}\mathsf{f}_2\mathsf{e}_2\mathsf{f}_2$; $\mathsf{Z}_4 = \langle \mathsf{Z}_3, \mathsf{e}_3, \mathsf{f}_{(4+)}\rangle = \langle \mathsf{Z}_3, \mathsf{e}_3, \mathsf{f}_{(4-)}\rangle$ where $\mathsf{f}_{(4+)} + \mathsf{f}_{(4-)} = \mathsf{f}_4 = \mathsf{f}_3 - \frac{3}{4}\mathsf{f}_3\mathsf{e}_3\mathsf{f}_3$; $\mathsf{Z}_5 = \langle \mathsf{Z}_4, \mathsf{e}_4\rangle$; $\mathsf{Z}_6 = \langle \mathsf{Z}_5, \mathsf{e}_5\rangle$; and $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ for all $k \geq 7$. When $2 \leq k \leq \mathsf{di}(\mathbf{O}) = 6$, there is exactly one new idempotent added each time, except for $k = 4$, where the two idempotents $\frac{1}{2}\mathsf{e}_3$ and $\mathsf{f}_{(4+)}$ must be adjoined to Z_3 to get Z_4. Each added idempotent corresponds to a highlighted edge in the Bratteli diagram of $\mathbf{O}$ (see Section 4.2). The highlighted edges together with the nodes attached to them give the Dynkin diagram of $\hat{\mathsf{E}}_7$. (In fact, for all groups G, the added idempotents give the corresponding Dynkin diagram as a subgraph of the Bratteli diagram.)

- If $\mathsf{G} = \mathbf{D}_n$ for $n > 2$. Then $\mathsf{Z}_2 = \langle \mathsf{Z}_1, \mathsf{e}_1, \mathsf{f}_{(0')}\rangle = \langle \mathsf{Z}_1, \mathsf{e}_1, \mathsf{f}_{(2)}\rangle$ where $\mathsf{f}_{(0')} + \mathsf{f}_{(2)} = \mathsf{f}_2 = 1 - \frac{1}{2}\mathsf{e}_1$; $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k\rangle$ for $2 \leq k \leq n-1$; $\mathsf{Z}_n = \langle \mathsf{Z}_{n-1}, \mathsf{e}_{n-1}, \mathsf{f}_{(n)}, \mathsf{f}_{(n')}\rangle$; $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ for all $k \geq n$. (In particular when $n = \infty$, $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k\rangle$ for all $k \geq 2$.)

1.9 Relations

Recall that $\mathsf{Z}_k(\mathsf{G}) \supseteq \mathsf{TL}_k(2)$ for all $k \geq 0$ and that $\mathsf{TL}_k(2)$ has generators e_i $(1 \leq i < k)$, which satisfy the following relations from (1.21):

(a) $\mathsf{e}_i^2 = 2\mathsf{e}_i, \qquad \mathsf{e}_i\mathsf{e}_{i\pm 1}\mathsf{e}_i = \mathsf{e}_i, \qquad$ and $\qquad \mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i, \quad$ for $|i-j| > 1$.

In the next two results, we identify additional generators needed to generate $\mathsf{Z}_k(\mathsf{G})$ and the relations they satisfy. In most instances, these are not minimal sets of generators (as is evident from Theorem 1.45), but rather the generators are chosen because they satisfy some reasonably nice relations.

Proposition 1.47. *Let* $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$ *for all* $k \geq 0$, $\ell = \mathsf{br}(\mathsf{G})$, *and* $\mathsf{di}(\mathsf{G}) = \ell + m$. *Suppose* $\nu_0 = (\ell), \nu_1, \ldots, \nu_m$ *is a sequence of distinct nodes from the branch node* (ℓ) *to the node* ν_m *a distance* $\mathsf{di}(\mathsf{G})$ *from* 0. *Set* $\mathsf{b}_j := \mathsf{f}_{\nu_j}$ *(the projection of* $\mathsf{V}_{\text{new}}^{\otimes(\ell+j)}$ *onto* G^{ν_j}*), and let* $d_j = d^{\nu_j} = \dim \mathsf{G}^{\nu_j}$ *for* $j = 0, 1, \ldots, m$. *Then the following hold:*

(i) *If* $k \leq \ell$, *then* $\mathsf{Z}_k = \mathsf{TL}_k(2)$, *and* Z_k *has generators* e_i $(1 \leq i < k)$ *which satisfy* (a).

(ii) *If* $\ell < k \leq \ell + m = \mathsf{di}(\mathsf{G})$, *and* $k \neq \mathsf{di}(\mathsf{G})$ *for* $\mathsf{G} = \mathbf{C}_n, \mathbf{D}_n$, $n < \infty$, *then* Z_k *has generators* e_i $(1 \leq i < k)$ *and* b_j $(1 \leq j \leq k - \ell)$ *which satisfy* (a) *and*

  (b) $\mathsf{b}_i\mathsf{b}_j = \mathsf{b}_j = \mathsf{b}_j\mathsf{b}_i, \quad$ *for all* $0 \leq i \leq j \leq k - \ell$;

  (c) $\mathsf{b}_{j+1} = \mathsf{b}_j - \frac{d_{j-1}}{d_j}\mathsf{b}_j\mathsf{e}_{\ell+j}\mathsf{b}_j$, *for all* $j = 1, \ldots, k - \ell - 1$;

  (d) $\mathsf{e}_i\mathsf{b}_j = 0 = \mathsf{b}_j\mathsf{e}_i, \quad$ *for all* $1 \leq i < \ell + j$, *and* $\mathsf{e}_i\mathsf{b}_j = \mathsf{b}_j\mathsf{e}_i$, *for all* $\ell + j < i \leq k$;

  (e) $\mathsf{e}_{\ell+j}\mathsf{b}_j\mathsf{e}_{\ell+j} = \frac{d_j}{d_{j-1}}\mathsf{b}_{j-1}\mathsf{e}_{\ell+j}$, *for all* $j = 1, \ldots, k - \ell$.

(iii) *If* $k > \mathsf{di}(\mathsf{G})$, *and* $\mathsf{G} \neq \mathbf{C}_n, \mathbf{D}_n$ *for* $n < \infty$, *then* Z_k *has generators* e_i $(1 \leq i < k)$ *and* b_j $(1 \leq j \leq m)$ *which satisfy* (a)-(e) *and*

  (f) $\mathsf{b}_m = \frac{d_{m-1}}{d_m}\mathsf{b}_m\mathsf{e}_{\ell+m}\mathsf{b}_m$.

Proof. For (i) we have $\mathsf{TL}_k(2) \subseteq \mathsf{Z}_k$ for all $k \geq 0$, and thus the e_i satisfy the relations in (a) by (1.21). The equality $\mathsf{TL}_k(2) = \mathsf{Z}_k$ is proved by comparing dimensions for $k \leq \ell$. Part (ii) follows from Theorem 1.45, Lemma 1.40, and Proposition 1.42. (Note that the projecting idempotents for the other branch of the Dynkin diagram can be obtained from these generators using the recursive process and the fact that the two idempotents needed for $\mathsf{V}_{new}^{\otimes(\ell+1)}$ sum to the Jones-Wenzl idempotent f_ℓ where $1 - \mathsf{f}_\ell \in \langle \mathsf{e}_1, \ldots, \mathsf{e}_{\ell-1}\rangle$.) For (iii), we have from Theorem 1.45 that $\mathsf{Z}_k = \mathsf{Z}_{k-1}\mathsf{e}_{k-1}\mathsf{Z}_{k-1} = \mathsf{End}_\mathsf{G}(\mathsf{V}_\mathsf{old}^{\otimes k})$ for all $k > \mathsf{di}(\mathsf{G}) = \ell + m$. Thus, $\mathsf{e}_1, \ldots, \mathsf{e}_{k-1}, \ \mathsf{b}_1, \ldots, \mathsf{b}_m$ generate Z_k and satisfy (a)-(e). To show that (f) holds, consider $\mathsf{G}^{\nu_m} \otimes \mathsf{V} \cong \mathsf{G}^{\nu_{m-1}}$. We know $\frac{d_{m-1}}{d_m}\mathsf{b}_m\mathsf{e}_{\ell+m}\mathsf{b}_m$ is an idempotent in $\mathsf{End}_\mathsf{G}(\mathsf{G}^{\nu_m} \otimes \mathsf{V})$ (compare the argument for p in Proposition 1.42) and $\mathsf{G}^{\nu_m} \otimes \mathsf{V} \cong \mathsf{G}^{\nu_{m-1}}$ is an irreducible G-module. By Schur's lemma, $\frac{d_{m-1}}{d_m}\mathsf{b}_m\mathsf{e}_{\ell+m}\mathsf{b}_m$ has to be a multiple of the identity, which is $\mathsf{b}_m = \mathsf{b}_m \otimes \mathbf{1}$, but since both are idempotents, they must be equal. □

Remark 1.48. Relation (ii) (c) shows that only the e_i and b_1 are needed to generate Z_k for $k > \ell$, and the other generators b_j, $2 \leq j \leq m$, can be constructed recursively from them. However, then relation (f) in (iii) needs to be replaced with a complicated expression in the e_i and b_1.

Proposition 1.47 covers all cases except when $k \geq \mathsf{di}(\mathsf{G})$ for $\mathsf{G} = \mathbf{C}_n, \mathbf{D}_n$, $n < \infty$, which will be considered next.

Proposition 1.49. *Assume* $\mathsf{G} = \mathbf{C}_n, \mathbf{D}_n$ *for* $n < \infty$*, and let* $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$ *where* $k \geq \mathsf{di}(\mathsf{G})$*. Then we have the following:*

($\mathbf{C}_n$) Z_k *has generators* e_i $(1 \leq i < k)$ *and* $\mathsf{b}_j^\pm$ $(1 \leq j \leq \tilde{n} = \mathsf{di}(\mathsf{G}))$ *(where* $\mathsf{b}_j^\pm$ *is the projection of* $\mathsf{V}_\mathsf{new}^{\otimes j}$ *onto* $\mathsf{G}^{(\pm j)}$*), together with* $\mathsf{b}_+^- = \mathsf{E}_{-\underline{1},\underline{1}}$, $\mathsf{b}_-^+ = \mathsf{E}_{\underline{1},-\underline{1}}$*, such that the relations in* (a)-(e) *hold when* $\mathsf{b}_j = \mathsf{b}_j^+$ *or when* $\mathsf{b}_j = \mathsf{b}_j^-$*. In addition, the following relations hold:*

(f$_\mathbf{C}$) $(\mathsf{b}_{\tilde{n}}^\pm)^2 = \mathsf{b}_{\tilde{n}}^\pm$ *and* $\mathsf{b}_i^\pm\mathsf{b}_j^\mp = 0 = \mathsf{b}_j^\mp\mathsf{b}_i^\pm$*, for all* $1 \leq i, j \leq \tilde{n}$*; and*

(g$_\mathbf{C}$) *for* $\mathsf{b}_+^+ = \mathsf{b}_{\tilde{n}}^+$*, and* $\mathsf{b}_-^- = \mathsf{b}_{\tilde{n}}^-$*,*

$$\mathsf{b}_\gamma^\zeta\mathsf{b}_\eta^\vartheta = \delta_{\gamma,\vartheta}\mathsf{b}_\eta^\zeta \ \text{ for } \ \gamma,\zeta,\eta,\vartheta \in \{-,+\} \ \text{ and } \ \mathsf{b}_j^\pm\mathsf{b}_\gamma^\zeta = 0 = \mathsf{b}_\gamma^\zeta\mathsf{b}_j^\pm \quad \text{for} \quad \zeta \neq \gamma, \ 1 \leq j < \tilde{n}.$$

($\mathbf{D}_n$) Z_k *has generators* e_i $(1 \leq i < k)$*,* b_j $(1 \leq j < n = \mathsf{di}(\mathsf{G}))$*, and* b'*, where* b_j *is the projection of* $\mathsf{V}^{\otimes(j+1)}$ *onto* $\mathsf{G}^{(j+1)}$*, and* b' *is the projection of* $\mathsf{V}^{\otimes n}$ *onto* $\mathsf{G}^{(n')}$*. They satisfy the relations in* (a)-(f) *of Proposition 1.47, and additionally*

(g$_\mathbf{D}$) $\mathsf{b}_j\mathsf{b}' = \mathsf{b}' = \mathsf{b}'\mathsf{b}_j$*, for* $1 \leq j < n$*,* $(\mathsf{b}')^2 = \mathsf{b}'$*, and* $\mathsf{b}_{n-1}\mathsf{b}' = 0 = \mathsf{b}'\mathsf{b}_{n-1}$*;*

(h$_\mathbf{D}$) $\mathsf{e}_n\mathsf{b}'\mathsf{e}_n = \frac{1}{2}\mathsf{b}_{n-1}\mathsf{e}_n$*,* $\mathsf{e}_i\mathsf{b}' = 0 = \mathsf{b}'\mathsf{e}_i = 0$ *for* $1 < i < n$*, and* $\mathsf{e}_i\mathsf{b}' = \mathsf{b}'e_i$ *for* $i > n$*;*

(i$_\mathbf{D}$) $\mathsf{b}_{n-1} = 2\mathsf{b}_{n-1}\mathsf{e}_n\mathsf{b}_{n-1}$ *and* $\mathsf{b}' = 2\mathsf{b}'\mathsf{e}_n\mathsf{b}'$.

Proof. For $\mathsf{G} = \mathbf{C}_n$, the fact that $\mathsf{e}_1, \ldots, \mathsf{e}_{k-1}, \mathsf{b}_1^\pm, \ldots, \mathsf{b}_{\tilde{n}}^\pm, \mathsf{b}_+^-, \mathsf{b}_-^+$ generate Z_k follows from Theorem 1.45 parts (a), (d), and (e). Relations (a)-(e) hold as in Proposition 1.47. Using (c) of Proposition 1.47 and induction, it is straightforward to prove that $\mathsf{b}_k^\pm = \mathsf{E}_{\pm\underline{1},\pm\underline{1}} \in \mathsf{Z}_k$, where $\underline{1}$ is the k-tuple of all 1s. The relations in (f$_\mathbf{C}$) and (g$_\mathbf{C}$) then follow by multiplication of matrix units.

For $\mathsf{G} = \mathbf{D}_n$, the fact that $\mathsf{e}_1, \ldots, \mathsf{e}_{k-1}, \mathsf{b}_1, \ldots, \mathsf{b}_{n-1}, \mathsf{b}'$ generate Z_k follows from Theorem 1.45 parts (a), (d), and (f). In Z_2, the projection onto $\mathsf{G}^{(2)}$ is $\mathsf{b}_1 = \mathsf{E}_{(1,1),(1,1)} + \mathsf{E}_{(-1,-1),(-1,-1)}$. It follows easily by induction that for $2 \leq j < n-1$, $\mathsf{b}_j = \mathsf{b}_{j-1} - \mathsf{b}_{j-1}\mathsf{e}_j\mathsf{b}_{j-1} = \mathsf{E}_{\underline{1},\underline{1}} + \mathsf{E}_{-\underline{1},-\underline{1}}$ is projection of $\mathsf{V}^{\otimes j}$ onto $\mathsf{G}^{(j)}$, where $\underline{1}$ is the j-tuple of all 1s. When $j = n-1$ we have $\mathsf{b}_{n-1}\mathsf{e}_n\mathsf{b}_{n-1} = \mathsf{E}_{\underline{1},\underline{1}} + \mathsf{E}_{-\underline{1},-\underline{1}}$ is the projection onto $\mathsf{G}^{(n)} \oplus \mathsf{G}^{(n')}$, and this splits into projections $\mathsf{b}_{n-1} = \frac{1}{2}(\mathsf{E}_{\underline{1},\underline{1}} + \mathsf{E}_{-\underline{1},-\underline{1}} - \mathsf{E}_{\underline{1},-\underline{1}} - \mathsf{E}_{-\underline{1},\underline{1}})$ and $\mathsf{b}' = \frac{1}{2}(\mathsf{E}_{\underline{1},\underline{1}} + \mathsf{E}_{-\underline{1},-\underline{1}} + \mathsf{E}_{\underline{1},-\underline{1}} + \mathsf{E}_{-\underline{1},\underline{1}})$, which project onto $\mathsf{G}^{(n)}$ and $\mathsf{G}^{(n')}$, respectively. The other relations follow by multiplication of matrix units. □

Remark 1.50. The results of this section identify a set of generators for each centralizer algebra and relations they satisfy, but it is not shown here that they give a presentation.

2 The Cyclic Subgroups

Let $\mathbf{C}_n$ denote the cyclic subgroup of SU_2 generated by

$$g = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \in \mathsf{SU}_2,$$

where $\zeta = \zeta_n$, a primitive nth root of unity. The irreducible modules for $\mathbf{C}_n$ are all one-dimensional and are given by $\mathbf{C}_n^{(\ell)} = \mathbb{C}\mathsf{v}_\ell$ for $\ell = 0, 1, \ldots, n-1$, where $g\mathsf{v}_\ell = \zeta^\ell \mathsf{v}_\ell$, and $\mathbf{C}_n^{(\ell)} \otimes \mathbf{C}_n^{(m)} \cong \mathbf{C}_n^{(\ell+m)}$ (superscripts interpreted $\mod n$). Thus, we can assume that the labels for the irreducible $\mathbf{C}_n$-modules are (ℓ), where $\ell \in \Lambda(\mathbf{C}_n) = \{0, 1, \ldots, n-1\}$, with the understanding that $(j) = (\ell)$ whenever an integer j such that $j \equiv \ell \mod n$ occurs in some expression.

The natural $\mathbf{C}_n$-module V of 2×1 column vectors which $\mathbf{C}_n$ acts on by matrix multiplication can be identified with the module $\mathbf{C}_n^{(-1)} \oplus \mathbf{C}_n^{(1)}$. As before, we let $\mathsf{v}_{-1} = (1, 0)^{\mathsf{t}}$ and $\mathsf{v}_1 = (0, 1)^{\mathsf{t}}$.

2.1 The Centralizer algebra $\mathsf{Z}_k(\mathbf{C}_n)$

Our aim in this section is to understand the centralizer algebra $\mathsf{Z}_k(\mathbf{C}_n)$ of the $\mathbf{C}_n$-action on $\mathsf{V}^{\otimes k}$ and the representation theory of $\mathsf{Z}_k(\mathbf{C}_n)$. As in (1.37), let

$$\tilde{n} = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{2}n & \text{if } n \text{ is even.} \end{cases} \tag{2.1}$$

Assume $\mathsf{r} = (r_1, \ldots, r_k) \in \{-1, 1\}^k$, and set

$$|\mathsf{r}| = |\{\, r_i \mid r_i = -1 \,\}|\,. \tag{2.2}$$

Corresponding to $\mathsf{r} \in \{-1, 1\}^k$ is the vector $\mathsf{v}_\mathsf{r} = \mathsf{v}_{r_1} \otimes \cdots \otimes \mathsf{v}_{r_k} \in \mathsf{V}^{\otimes k}$, and

$$g\mathsf{v}_\mathsf{r} = \zeta^{k-2|\mathsf{r}|}\mathsf{v}_\mathsf{r}. \tag{2.3}$$

For two such k-tuples r and s,

$$k - 2|\mathsf{r}| \equiv k - 2|\mathsf{s}| \mod n \iff |\mathsf{r}| \equiv |\mathsf{s}| \mod \tilde{n}.$$

Recall that $\Lambda_k(\mathbf{C}_n)$ is the subset of $\Lambda(\mathbf{C}_n) = \{0, 1, \ldots, n-1\}$ of labels for the irreducible $\mathbf{C}_n$-modules occurring in $\mathsf{V}^{\otimes k}$. Now if $\ell \in \Lambda(\mathbf{C}_n)$, then

$$\ell \in \Lambda_k(\mathbf{C}_n) \iff k - 2|\mathsf{r}| \equiv \ell \mod n \text{ for some } \mathsf{r} \in \{-1, 1\}^k.$$

Thus,

$$\Lambda_k(\mathbf{C}_n) = \{\ell \in \Lambda(\mathbf{C}_n) \mid \ell \equiv k - 2a_\ell \mod n \text{ for some } a_\ell \in \{0, 1, \ldots, k\}\}\,. \tag{2.4}$$

We will always assume a_ℓ is the minimal value in $\{0, 1, \ldots, k\}$ with that property.

In particular, $k - \ell$ must be even when n is even. Hence there are at most $\tilde{n}$ distinct values in $\Lambda_k(\mathbf{C}_n)$. When $k \geq \tilde{n} - 1$, then for every $a \in \{0, 1, \ldots, \tilde{n} - 1\}$, there exists an $\ell \in \Lambda_k(\mathbf{C}_n)$ so that $k - 2a \equiv \ell \mod n$, and there are exactly $\tilde{n}$ distinct values in $\Lambda_k(\mathbf{C}_n)$.

Lemma 2.5. *Assume r and s are two k-tuples satisfying $|\mathsf{r}| \equiv |\mathsf{s}| \mod \tilde{n}$. Let $\mathsf{E}_{\mathsf{r},\mathsf{s}}$ be the transformation on $\mathsf{V}^{\otimes k}$ defined by*

$$\mathsf{E}_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{t} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{v}_\mathsf{r}. \tag{2.6}$$

Then $\mathsf{E}_{\mathsf{r},\mathsf{s}} \in \mathsf{Z}_k(\mathbf{C}_n)$.

Proof. Note that $g\mathsf{E}_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{t} = \delta_{\mathsf{s},\mathsf{t}}\zeta^{k-2|\mathsf{r}|}\mathsf{v}_\mathsf{r}$, while $\mathsf{E}_{\mathsf{r},\mathsf{s}}g\mathsf{v}_\mathsf{t} = \zeta^{k-2|\mathsf{t}|}\delta_{\mathsf{s},\mathsf{t}}\mathsf{v}_\mathsf{r} = \zeta^{k-2|\mathsf{s}|}\delta_{\mathsf{s},\mathsf{t}}\mathsf{v}_\mathsf{r}$. Consequently, $g\mathsf{E}_{\mathsf{r},\mathsf{s}} = \mathsf{E}_{\mathsf{r},\mathsf{s}}g$ for all such tuples r,s, and $\mathsf{E}_{\mathsf{r},\mathsf{s}} \in \mathsf{Z}_k(\mathbf{C}_n)$ by (2.4). □

Theorem 2.7. (a) *The set*

$$\mathcal{B}^k(\mathbf{C}_n) = \{\mathsf{E}_{\mathsf{r},\mathsf{s}} \mid \mathsf{r},\mathsf{s} \in \{-1,1\}^k,\ |\mathsf{r}| \equiv |\mathsf{s}| \bmod \tilde{n}\} \tag{2.8}$$

is a basis for the centralizer algebra $\mathsf{Z}_k(\mathbf{C}_n) = \mathsf{End}_{\mathbf{C}_n}(\mathsf{V}^{\otimes k})$.

(b) *The set* $\{z_\ell \mid \ell \in \Lambda_k(\mathbf{C}_n)\}$ *is a basis for the center of* $\mathsf{Z}_k(\mathbf{C}_n)$, *where*

$$z_\ell = \sum_{\substack{\mathsf{r}\in\{-1,1\}^k \\ k-2|\mathsf{r}|\equiv \ell \bmod n}} \mathsf{E}_{\mathsf{r},\mathsf{r}} \qquad \text{for} \quad \ell \in \Lambda_k(\mathbf{C}_n). \tag{2.9}$$

(c) *If* $n = 2\tilde{n}$ *and* $\tilde{n}$ *is odd, then* $\mathsf{Z}_k(\mathbf{C}_n) \cong \mathsf{Z}_k(\mathbf{C}_{\tilde{n}})$.

(d) *The dimension of the centralizer algebra* $\mathsf{Z}_k(\mathbf{C}_n)$ *is the coefficient of* z^k *in* $(1+z)^{2k}\big|_{z^{\tilde{n}}=1}$; *hence, it is given by*

$$\dim \mathsf{Z}_k(\mathbf{C}_n) = \sum_{\substack{0\le a,b\le k \\ a\equiv b \bmod \tilde{n}}} \binom{k}{a}\binom{k}{b}. \tag{2.10}$$

Remark 2.11. The notation $(1+z)^{2k}\big|_{z^{\tilde{n}}=1}$ used in the statement of this result can be regarded as saying consider $(1+z)^{2k}$ in the polynomial algebra $\mathbb{C}[z]$ modulo the ideal generated by $z^{\tilde{n}} - 1$ where $\tilde{n}$ is as in (2.1).

Proof. (a) For $X \in \mathsf{End}(\mathsf{V}^{\otimes k})$, suppose that $X\mathsf{v}_\mathsf{s} = \sum_\mathsf{r} X_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{r}$ for scalars $X_{\mathsf{r},\mathsf{s}} \in \mathbb{C}$, where r ranges over all the k-tuples in $\{-1,1\}^k$. Then $X \in \mathsf{Z}_k(\mathbf{C}_n)$ if and only if $g^{-1}Xg = X$ if and only if

$$g^{-1}Xg\mathsf{v}_\mathsf{s} = \sum_\mathsf{r} \zeta^{(k-2|\mathsf{s}|)-(k-2|\mathsf{r}|)}X_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{r} = \sum_\mathsf{r} X_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{r}.$$

Hence, for all r,s, with $X_{\mathsf{r},\mathsf{s}} \neq 0$, it must be that $\zeta^{2(|\mathsf{r}|-|\mathsf{s}|)} = 1$; that is, $|\mathsf{r}| \equiv |\mathsf{s}| \bmod \tilde{n}$ by (2.4). Thus, $X = \sum_{|\mathsf{r}|\equiv|\mathsf{s}| \bmod \tilde{n}} X_{\mathsf{r},\mathsf{s}}\mathsf{E}_{\mathsf{r},\mathsf{s}}$, and the transformations $\mathsf{E}_{\mathsf{r},\mathsf{s}}$ with $|\mathsf{r}| \equiv |\mathsf{s}| \bmod \tilde{n}$ span $\mathsf{Z}_k(\mathbf{C}_n)$. It is easy to see that the $\mathsf{E}_{\mathsf{r},\mathsf{s}}$ multiply like matrix units and are linearly independent, so they form a basis of $\mathsf{Z}_k(\mathbf{C}_n)$.

(b) For each $\ell \in \Lambda_k(\mathbf{C}_n)$, the basis elements $\mathsf{E}_{\mathsf{r},\mathsf{s}}$ with $|\mathsf{r}| = |\mathsf{s}| \equiv \frac{1}{2}(k-\ell) \bmod \tilde{n}$ form a matrix algebra, whose center is $\mathbb{C}z_\ell$ where z_ℓ is as in (2.9). Since $\mathsf{Z}_k(\mathbf{C}_n)$ is the direct sum of these matrix algebra ideals as ℓ ranges over $\Lambda_k(\mathbf{C}_n)$, the result follows.

When $n = 2\tilde{n}$ and $\tilde{n}$ is odd, the elements $\{\mathsf{E}_{\mathsf{r},\mathsf{s}} \mid |\mathsf{r}| \equiv |\mathsf{s}| \bmod \tilde{n}\}$ comprise a basis of both $\mathsf{Z}_k(\mathbf{C}_n)$ and $\mathsf{Z}_k(\mathbf{C}_{\tilde{n}})$ to give the assertion in (c).

(d) It follows from Lemma 2.5 that

$$\dim \mathsf{Z}_k(\mathbf{C}_n) = \sum_{\substack{0\le a,b\le k \\ a\equiv b \bmod \tilde{n}}} \binom{k}{a}\binom{k}{b} = \sum_{\substack{0\le a,b\le k \\ a\equiv b \bmod \tilde{n}}} \binom{k}{a}\binom{k}{k-b}.$$

Since $a + k - b \equiv k \bmod \tilde{n}$, this expression is the coefficient of z^k in $(1+z)^k(1+z)^k\big|_{z^{\tilde{n}}=1} = (1+z)^{2k}\big|_{z^{\tilde{n}}=1}$, as claimed. □

Example 2.12. Suppose $n = 8$ (so $\tilde{n} = 4$) and $k = 6$. Then

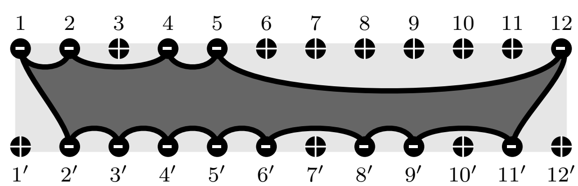
$$\begin{aligned}
\left|\{(\mathsf{r},\mathsf{s}) \mid |\mathsf{r}| \equiv |\mathsf{s}| \equiv 0 \bmod 4\}\right| &= \binom{6}{0}^2 + \binom{6}{4}^2 + 2\binom{6}{0}\binom{6}{4} = 256 \\
\left|\{(\mathsf{r},\mathsf{s}) \mid |\mathsf{r}| \equiv |\mathsf{s}| \equiv 1 \bmod 4\}\right| &= \binom{6}{1}^2 + \binom{6}{5}^2 + 2\binom{6}{1}\binom{6}{5} = 144 \\
\left|\{(\mathsf{r},\mathsf{s}) \mid |\mathsf{r}| \equiv |\mathsf{s}| \equiv 2 \bmod 4\}\right| &= \binom{6}{2}^2 + \binom{6}{6}^2 + 2\binom{6}{2}\binom{6}{6} = 256 \\
\left|\{(\mathsf{r},\mathsf{s}) \mid |\mathsf{r}| \equiv |\mathsf{s}| \equiv 3 \bmod 4\}\right| &= \binom{6}{3}^2 = 400.
\end{aligned}$$

Therefore $\dim \mathsf{Z}_6(\mathbf{C}_8) = 1056$. Now observe that when $k = 6$ and $n = 8$ that

$$\begin{aligned}
(1+z)^{2k}\big|_{z^{\tilde{n}}=1} &= (1+z)^{12}\big|_{z^4=1} \\
&= 1 + 12z + 66z^2 + 220z^3 + 495 + 792z + 924z^2 + 792z^3 \\
&\quad +495 + 220z + 66z^2 + 12z^3 + 1.
\end{aligned}$$

Since $k = 6 \equiv 2 \bmod 4$, by (c) of Theorem 2.7 we have that $\dim \mathsf{Z}_6(\mathbf{C}_8)$ is the coefficient of z^2 in this expression, so $\dim \mathsf{Z}_6(\mathbf{C}_8) = 66 + 924 + 66 = 1056$, in agreement with the above calculation.

Remark 2.13. The matrix units can be viewed diagrammatically. For example, if $k = 12$, $n = 6$, $\tilde{n} = 3$, and $\mathsf{r} = (-1,-1,1,-1,-1,1,1,1,1,1,1,-1) \in \{-1,1\}^{12}$, then $|\mathsf{r}| = 5 \equiv 2 \bmod 3$, and if $\mathsf{s} = (1,-1,-1,-1,-1,-1,1,-1,-1,1,-1,1)$, then $|\mathsf{s}| = 8 \equiv 2 \equiv |\mathsf{r}| \bmod 3$. In this case, we identify the matrix unit $\mathsf{E}_{\mathsf{r},\mathsf{s}}$ with the diagram below

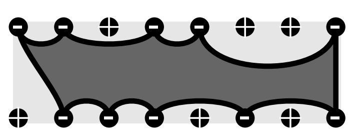
$\mathsf{E}_{\mathsf{r},\mathsf{s}} =$ 

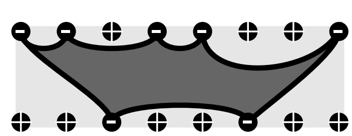
.

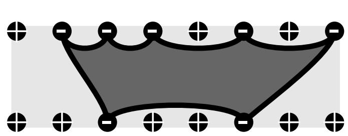
Each such two-rowed k-diagram d corresponds to two subsets, $t(d) \subseteq \{1, 2, \ldots, k\}$ and $b(d) \subseteq \{1', 2', \ldots, k'\}$, recording the positions of the -1s in the top and bottom rows of d, hence in r and s respectively, and $|t(d)| \equiv |b(d)| \bmod \tilde{n}$. Under this correspondence, diagrams multiply as matrix units. Thus, if d_1 and d_2 are diagrams, then

$$d_1 d_2 = \delta_{b(d_1),t(d_2)} d_3$$

where d_3 is the unique diagram given by $t(d_3) = t(d_1)$ and $b(d_3) = b(d_2)$. For example, if $\tilde{n} = 3$,

$d_1 =$ 

$=$  $= d_3.$

$d_2 =$ 

2.2 Irreducible Modules for $\mathsf{Z}_k(\mathbf{C}_n)$

For $\ell \in \Lambda_k(\mathbf{C}_n)$, set

$$\mathsf{Z}_k^{(\ell)} := \mathsf{span}_{\mathbb{C}}\{\mathsf{v_r} \in \mathsf{V}^{\otimes k} \mid k - 2|\mathsf{r}| \equiv \ell \bmod n\} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v_r} \in \mathsf{V}^{\otimes k} \mid |\mathsf{r}| \equiv a_\ell \bmod \tilde{n}\}, \tag{2.14}$$

where a_ℓ is as in (2.4). *When we need to emphasize that we are working with the group $\mathbf{C}_n$, we will write this as $\mathsf{Z}_k(\mathbf{C}_n)^{(\ell)}$.* Now g acts as the scalar ζ^ℓ on $\mathsf{Z}_k^{(\ell)}$, and these scalars are distinct for different values of $\ell \in \{0, 1, \ldots, n-1\}$. Therefore,

$$\mathsf{V}^{\otimes k} = \bigoplus_{\ell \in \Lambda_k(\mathbf{C}_n)} \mathsf{Z}_k^{(\ell)}$$

is a decomposition of $\mathsf{V}^{\otimes k}$ into $\mathbf{C}_n$-modules.

The mappings $\mathsf{E_{r,s}}$ with $|\mathsf{r}| \equiv |\mathsf{s}| \equiv a_\ell \bmod \tilde{n}$ act as matrix units on $\mathsf{Z}_k^{(\ell)}$ and act trivially on $\mathsf{Z}_k^{(m)}$ for $m \in \Lambda_k(\mathbf{C}_n),\ m \neq \ell$. In addition,

$$\mathsf{span}\{\mathsf{E_{r,s}} \mid |\mathsf{r}| \equiv |\mathsf{s}| \equiv a_\ell \bmod \tilde{n}\} = \mathsf{End}(\mathsf{Z}_k^{(\ell)}) = \mathsf{End}_{\mathbf{C}_n}(\mathsf{Z}_k^{(\ell)}).$$

As a consequence, we have that the spaces $\mathsf{Z}_k^{(\ell)}$ are modules for $\mathsf{Z}_k(\mathbf{C}_n)$. Since they are also invariant under the action of $\mathbf{C}_n$, they are modules both for $\mathbf{C}_n$ and for $\mathsf{Z}_k(\mathbf{C}_n)$. It is apparent that $\mathsf{Z}_k^{(\ell)}$ is irreducible as a $\mathsf{Z}_k(\mathbf{C}_n)$-module from the fact that the natural module for a matrix algebra is its unique irreducible module.

Examples 2.15. For any $m \geq 1$, let ζ_m be a primitive mth root of unity. Assume $k = 5$ and $n = 12$, so $\tilde{n} = 6$. Then $\mathsf{Z}_5(\mathbf{C}_{12})$ has 6 irreducible modules $\mathsf{Z}_5(\mathbf{C}_{12})^{(\ell)}$ for $\ell = 1, 3, 5, 7, 9, 11$. On them, the generator g of $\mathbf{C}_{12}$ acts by the scalars ζ_{12}, ζ_{12}^3, $\zeta_{12}^5, \zeta_{12}^7$, ζ_{12}^9, ζ_{12}^{11}, respectively.

The algebra $\mathsf{Z}_5(\mathbf{C}_6)$ has 3 irreducible modules $\mathsf{Z}_5(\mathbf{C}_6)^{(\ell)}$ for $\ell = 1, 3, 5$, on which the generator g' of $\mathbf{C}_6$ acts by the scalars ζ_6^1, ζ_6^3, ζ_6^5, respectively.

The algebra $\mathsf{Z}_5(\mathbf{C}_3)$ also has 3 irreducible modules $\mathsf{Z}_5(\mathbf{C}_3)^{(\ell)}$ for $\ell = 0, 1, 2$, on which the generator g'' of $\mathbf{C}_3$ acts by the scalars 1, ζ_3, ζ_3^2, respectively .

The vectors $\{\mathsf{v_r} \mid \mathsf{r} \in \{-1, 1\}^5,\ k - 2|\mathsf{r}| \equiv \ell \bmod 3\}$ form a basis for $\mathsf{Z}_5(\mathbf{C}_6)^{(1)}$ and $\mathsf{Z}_5(\mathbf{C}_3)^{(1)}$ when $\ell \equiv 1 \bmod 3$; for $\mathsf{Z}_5(\mathbf{C}_6)^{(3)}$ and $\mathsf{Z}_5(\mathbf{C}_3)^{(0)}$ when $\ell \equiv 0 \bmod 3$; and for $\mathsf{Z}_5(\mathbf{C}_6)^{(5)}$ and $\mathsf{Z}_5(\mathbf{C}_3)^{(2)}$ when $\ell \equiv 2 \bmod 3$.

The next result gives an expression for the dimension of the module $\mathsf{Z}_k^{(\ell)} = \mathsf{Z}_k(\mathbf{C}_n)^{(\ell)}$.

Theorem 2.16. *With $\tilde{n}$ as in (2.1), suppose $\ell \in \Lambda_k(\mathbf{C}_n)$ and $k - 2a_\ell \equiv \ell \bmod n$ as in (2.4). Then $\mathsf{Z}_k^{(\ell)} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v_r} \in \mathsf{V}^{\otimes k} \mid |\mathsf{r}| \equiv a_\ell \bmod \tilde{n}\}$ is an irreducible $\mathsf{Z}_k(\mathbf{C}_n)$-module and the following hold:*

(i) $\mathsf{dim}\,\mathsf{Z}_k^{(\ell)} = \sum_{\substack{0 \leq b \leq k \\ b \equiv a_\ell \bmod \tilde{n}}} \binom{k}{b}$, *which is the coefficient of z^{a_ℓ} in $(1+z)^k\big|_{z^{\tilde{n}}=1}$*

(ii) *As a bimodule for $\mathbf{C}_n \times \mathsf{Z}_k(\mathbf{C}_n)$*

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\ell \in \Lambda_k(\mathbf{C}_n)} \left(\mathbf{C}_n^{(\ell)} \otimes \mathsf{Z}_k^{(\ell)}\right). \tag{2.17}$$

(Here $\mathbf{C}_n$ acts only on the factors $\mathbf{C}_n^{(\ell)}$ and $\mathsf{Z}_k(\mathbf{C}_n)$ on the factors $\mathsf{Z}_k^{(\ell)}$. Since $\mathsf{Z}_k^{(\ell)}$ is also both a $\mathbf{C}_n$ and a $\mathsf{Z}_k(\mathbf{C}_n)$-module and the actions commute, the decomposition $\mathsf{V}^{\otimes k} = \bigoplus_{\ell \in \Lambda_k(\mathbf{C}_n)} \mathsf{Z}_k^{(\ell)}$ is also a $\mathbf{C}_n \times \mathsf{Z}_k(\mathbf{C}_n)$-bimodule decomposition.)

(iii) $\mathbf{C}_n^{(\ell)}$ *occurs as a summand in the* $\mathbf{C}_n$*-module* $\mathsf{V}^{\otimes k}$ *with multiplicity*

$$m_k^{(\ell)} = \sum_{\substack{0\le b\le k \\ b\equiv a_\ell \mod \tilde{n}}} \binom{k}{b} = \dim \mathsf{Z}_k^{(\ell)}.$$

(iv) *The number of walks on the affine Dynkin diagram of type* $\hat{\mathsf{A}}_{\mathsf{n-1}}$ *starting at node 0 and ending at node* ℓ *and taking* k *steps is the coefficient of* z^{a_ℓ} *in* $(1+z)^k\big|_{z^{\tilde{n}}=1}$*, which is* $m_k^{(\ell)} = \dim \mathsf{Z}_k^{(\ell)}$.

Proof. Part (i) follows readily from the fact that a basis for $\mathsf{Z}_k^{(\ell)}$ consists of the vectors v_r labeled by the tuples $\mathsf{r} \in \{-1,1\}^k$ with $|\mathsf{r}| = b \equiv a_\ell \mod \tilde{n}$ (see (2.14)), and the number of k-tuples r with b components equal to -1 is $\binom{k}{b}$. Each such vector v_r satisfies $g\mathsf{v}_\mathsf{r} = \zeta^{k-2|\mathsf{r}|}\mathsf{v}_\mathsf{r} = \zeta^\ell \mathsf{v}_\mathsf{r}$; hence $\mathbb{C}\mathsf{v}_\mathsf{r} \cong \mathbf{C}_n^{(\ell)}$ as a $\mathbf{C}_n$-module. The other statements are apparent from these. □

Examples 2.18. Consider the following special cases for $\mathbf{C}_n$, where $\tilde{n}$ is as in (2.1).

(i) If $k < \tilde{n}$, then $\dim \mathsf{Z}_k(\mathbf{C}_n) = \sum_{a=0}^{k} \binom{k}{a}^2 = \binom{2k}{k}$.

(ii) If $k = \tilde{n}$, then for $\ell \in \{0,1,\dots,n-1\}$ such that $k - 2\cdot 0 \equiv \ell \mod n$, we have $\dim \mathsf{Z}_{\tilde{n}}^{(\ell)} = \binom{\tilde{n}}{0} + \binom{\tilde{n}}{\tilde{n}} = 2$, and $\dim \mathsf{Z}_{\tilde{n}}(\mathbf{C}_n) = \sum_{a=1}^{\tilde{n}-1} \binom{\tilde{n}}{a}^2 + 2^2 = \binom{2\tilde{n}}{\tilde{n}} + 2$.

Example 2.19. Suppose $k = 6$ and $n = 8$, so $\tilde{n} = 4$. The irreducible $\mathbf{C}_8$-modules $\mathbf{C}_8^{(\ell)}$ occurring in $\mathsf{V}^{\otimes 6}$ have $\ell = 0,2,4,6$, and $a_\ell = 3,2,1,0$, respectively, where $k - \ell \equiv 2a_\ell \mod 8$. We have the following expressions for the number $m_8^{(\ell)}$ of times $\mathbf{C}_8^{(\ell)}$ occurs as a summand.

ℓ	a_ℓ	$m_6^{(\ell)}$
0	3	$m_6^{(0)} = \binom{6}{3} = 20 = \dim \mathsf{Z}_6^{(0)}$
2	2	$m_6^{(2)} = \binom{6}{2} + \binom{6}{6} = 16 = \dim \mathsf{Z}_6^{(2)}$
4	1	$m_6^{(4)} = \binom{6}{1} + \binom{6}{5} = 12 = \dim \mathsf{Z}_6^{(4)}$
6	0	$m_6^{(6)} = \binom{6}{0} + \binom{6}{4} = 16 = \dim \mathsf{Z}_6^{(6)}$

In particular

$$\dim \mathsf{Z}_6(\mathbf{C}_8) = \sum_{\ell\in\Lambda_k(n)} \left(\dim \mathsf{Z}_6^{(\ell)}\right)^2 = 20^2 + 16^2 + 12^2 + 16^2 = 1056,$$

exactly as in Example 2.12.

The number of walks on the Dynkin diagram of type $\hat{\mathsf{A}}_7$ with 6 steps starting and ending at 0 is the coefficient of z^3 in

$$(1+z)^6\big|_{z^4=1} = \quad 1 + 6z \, + \, 15z^2 + 20z^3 + 15 + 6z \, + \, z^2,$$

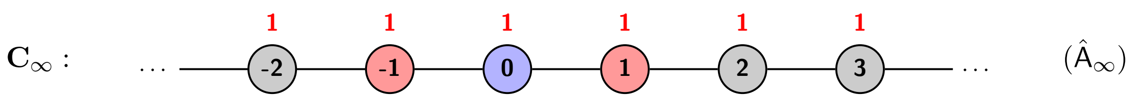
which is 20. The number of walks starting at 0 and ending at 4 is the coefficient of z in this expression, which is 6+6 = 12.

2.3 The cyclic subgroup $\mathbf{C}_\infty$

Let $\mathbf{C}_\infty$ denote the cyclic subgroup of SU_2 generated by

$$g = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \in \mathsf{SU}_2,$$

where $\zeta = e^{i\theta}$ for any $\theta \in \mathbb{R}$ such that ζ is not a root of unity. Then $\mathbf{C}_\infty$ has a natural action on $\mathsf{V}^{\otimes k}$, and the irreducible $\mathbf{C}_\infty$-modules occurring in the modules $\mathsf{V}^{\otimes k}$ are all one dimensional and are given by $\mathbf{C}_\infty^{(\ell)} = \mathbb{C}\mathsf{v}_\ell$ for some $\ell \in \mathbb{Z}$, where $g\mathsf{v}_\ell = \zeta^\ell \mathsf{v}_\ell$ and $\mathbf{C}_\infty^{(\ell)} \otimes \mathbf{C}_\infty^{(m)} \cong \mathbf{C}_\infty^{(\ell+m)}$. In particular, $\mathsf{V} = \mathbf{C}_\infty^{(-1)} \oplus \mathbf{C}_\infty^{(1)}$, and $\mathbf{C}_\infty^{(\ell)} \otimes \mathsf{V} = \mathbf{C}_\infty^{(\ell-1)} \oplus \mathbf{C}_\infty^{(\ell+1)}$ for all ℓ. Thus, the representation graph $\mathcal{R}_\mathsf{V}(\mathbf{C}_\infty)$ is the Dynkin diagram A_∞.



(2.20)

Now $g\mathsf{v}_\mathsf{r} = \zeta^{k-2|\mathsf{r}|}\mathsf{v}_\mathsf{r}$, for all $\mathsf{r} \in \{-1,1\}^k$, where $|\mathsf{r}|$ is as in (2.2). The arguments in the previous section can be easily adapted to show the following.

Theorem 2.21. *Let* $\mathsf{Z}_k = \mathsf{Z}_k(\mathbf{C}_\infty) = \mathsf{End}_{\mathbf{C}_\infty}(\mathsf{V}^{\otimes k})$.

(a) $\mathcal{B}^k(\mathbf{C}_\infty) = \left\{ \mathsf{E}_{\mathsf{r},\mathsf{s}} \mid \mathsf{r},\mathsf{s} \in \{-1,1\}^k, |\mathsf{r}| = |\mathsf{s}| \right\}$ *is a basis for* Z_k, *where* $\mathsf{E}_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{t} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{v}_\mathsf{r}$ *and* $\mathsf{E}_{\mathsf{r},\mathsf{s}}\mathsf{E}_{\mathsf{t},\mathsf{u}} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{E}_{\mathsf{r},\mathsf{u}}$ *for all* $\mathsf{r},\mathsf{s},\mathsf{t},\mathsf{u} \in \{-1,1\}^k$.

(b) *The irreducible modules for* Z_k *are labeled by* $\Lambda_k(\mathbf{C}_\infty) = \{k-2a \mid a = 0,1,\ldots,k\}$. *A basis for the irreducible* Z_k*-module* $\mathsf{Z}_k^{(k-2a)}$ *is* $\{\mathsf{v}_\mathsf{r} \mid \mathsf{r} \in \{-1,1\}^k,\ |\mathsf{r}| = a\}$, *and* $\dim \mathsf{Z}_k^{(k-2a)} = \binom{k}{a}$. *The module* $\mathsf{Z}_k^{(k-2a)}$ *is also a* $\mathbf{C}_\infty$*-module, hence a* $(\mathbf{C}_\infty \times \mathsf{Z}_k)$*-bimodule.*

(c) $\dim \mathsf{Z}_k = \sum_{a=0}^{k} \binom{k}{a}^2 = \binom{2k}{k} =$ coefficient of z^k in $(1+z)^{2k}$.

(d) Z_k *is isomorphic to the planar rook algebra* P_k.

Remark 2.22. A word of explanation about part (d) is in order. The planar rook algebra P_k was studied in [FHH], where it was shown (see [FHH, Prop. 3.3]) to have a basis of matrix units $\{\mathsf{X}_{\mathsf{R},\mathsf{S}} \mid \mathsf{R},\mathsf{S} \subseteq \{1,\ldots,k\},\ |\mathsf{R}| = |\mathsf{S}|\}$ such that $\mathsf{X}_{\mathsf{R},\mathsf{S}}\mathsf{X}_{\mathsf{T},\mathsf{U}} = \delta_{\mathsf{S},\mathsf{T}}\mathsf{X}_{\mathsf{R},\mathsf{U}}$. Identifying the subset R of $\{1,\ldots,k\}$ with the k-tuple $\mathsf{r} = (r_1,\ldots,r_k) \in \{-1,1\}^k$ such $r_j = -1$ if $j \in \mathsf{R}$ and $r_j = 1$ if $j \notin \mathsf{R}$, it is easy to see that $\mathsf{Z}_k(\mathbf{C}_\infty) \cong \mathsf{P}_k$ via the correspondence $\mathsf{E}_{\mathsf{r},\mathsf{s}} \mapsto \mathsf{X}_{\mathsf{R},\mathsf{S}}$.

Remark 2.23. As a module for the circle subgroup (maximal torus) $S^1 = \left\{ \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$ of SU_2, V has the same decomposition $\mathsf{V} = \mathbb{C}\mathsf{v}_{-1} \oplus \mathbb{C}\mathsf{v}_1$ into submodules (common eigenspaces) as it has for $\mathbf{C}_\infty$. Thus, the centralizer algebra $\mathsf{Z}_k(S^1) \cong \mathsf{Z}_k(\mathbf{C}_\infty)$, and its structure and representations are also given by this theorem.

3 The Binary Dihedral Subgroups

Let $\mathbf{D}_n$ denote the binary dihedral subgroup of SU_2 of order $4n$ generated by the elements $g, h \in \mathsf{SU}_2$, where

$$g = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3.1}$$

$\zeta = \zeta_{2n}$, a *primitive* $(2n)$*th root of unity* in $\mathbb{C}$, and $i = \sqrt{-1}$. The defining relations for $\mathbf{D}_n$ are

$$g^{2n} = 1, \quad g^n = h^2, \quad h^{-1}gh = g^{-1}. \tag{3.2}$$

Each of the nodes $\ell = 0, 0', 1, 2, \ldots, n-1, n, n'$ of the affine Dynkin diagram of type $\hat{\mathsf{D}}_{\mathsf{n+2}}$ (see Section 4.1) corresponds to an irreducible $\mathbf{D}_n$-module. For $\ell = 1, \ldots, n-1$, let $\mathbf{D}_n^{(\ell)}$ denote the two-dimensional $\mathbf{D}_n$-module on which the generators g, h have the following matrix representations,

$$g = \begin{pmatrix} \zeta^{-\ell} & 0 \\ 0 & \zeta^{\ell} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & i^{\ell} \\ i^{\ell} & 0 \end{pmatrix}.$$

relative to the basis $\{\mathsf{v}_{-\ell}, \mathsf{v}_{\ell}\}$. For $\ell = 0, 0', n, n'$, let the one-dimensional $\mathbf{D}_n$-module $\mathbf{D}_n^{(\ell)}$ be as follows:

$$\begin{array}{llll} \mathbf{D}_n^{(0)} &= \mathbb{C}\mathsf{v}_0, & g\mathsf{v}_0 = \mathsf{v}_0, & h\mathsf{v}_0 = \mathsf{v}_0 \\ \mathbf{D}_n^{(0')} &= \mathbb{C}\mathsf{v}_{0'}, & g\mathsf{v}_{0'} = \mathsf{v}_{0'}, & h\mathsf{v}_{0'} = -\mathsf{v}_{0'} \\ \mathbf{D}_n^{(n)} &= \mathbb{C}\mathsf{v}_n, & g\mathsf{v}_n = -\mathsf{v}_n, & h\mathsf{v}_n = i^n\mathsf{v}_n \\ \mathbf{D}_n^{(n')} &= \mathbb{C}\mathsf{v}_{n'}, & g\mathsf{v}_{n'} = -\mathsf{v}_{n'}, & h\mathsf{v}_{n'} = -i^n\mathsf{v}_{n'}. \end{array} \tag{3.3}$$

In each case, we refer to the given basis as the "standard basis" for $\mathbf{D}_n^{(\ell)}$. The modules $\mathbf{D}_n^{(\ell)}$ for $\ell = 0, 0', 1, 2, \ldots, n-1, n, n'$ give the complete list of irreducible $\mathbf{D}_n$-modules up to isomorphism.

Relative to the standard basis $\{\mathsf{v}_{-1}, \mathsf{v}_1\}$ for the module $\mathsf{V} := \mathbf{D}_n^{(1)}$, g and h have the matrix realizations displayed in (3.1), and V is the natural $\mathbf{D}_n$-module of 2×1 column vectors.

Proposition 3.4. *Tensor products of* V *with the irreducible modules* $\mathbf{D}_n^{(\ell)}$ *are given as follows:*

(a) $\mathbf{D}_n^{(\ell)} \otimes \mathsf{V} \cong \mathbf{D}_n^{(\ell-1)} \oplus \mathbf{D}_n^{(\ell+1)}$ *for* $1 < \ell < n-1$;

(b) $\mathbf{D}_n^{(1)} \otimes \mathsf{V} \cong \mathbf{D}_n^{(0')} \oplus \mathbf{D}_n^{(0)} \oplus \mathbf{D}_n^{(2)}$;

(c) $\mathbf{D}_n^{(n-1)} \otimes \mathsf{V} \cong \mathbf{D}_n^{(n-2)} \oplus \mathbf{D}_n^{(n)} \oplus \mathbf{D}_n^{(n')}$;

(d) $\mathbf{D}_n^{(0)} \otimes \mathsf{V} \cong \mathbf{D}_n^{(1)} = \mathsf{V}, \quad \mathbf{D}_n^{(0')} \otimes \mathsf{V} \cong \mathbf{D}_n^{(1)} = \mathsf{V}$;

(e) $\mathbf{D}_n^{(n)} \otimes \mathsf{V} \cong \mathbf{D}_n^{(n-1)}, \quad \mathbf{D}_n^{(n')} \otimes \mathsf{V} = \mathbf{D}_n^{(n-1)}$.

This can be readily checked using the standard bases above. They are exactly the tensor product rules given by the McKay correspondence.

Assume $\mathsf{s} = (s_1, \ldots, s_k) \in \{-1, 1\}^k$, and let $|\mathsf{s}| = |\{s_j \mid s_j = -1\}|$ as in (2.2). On the vector $\mathsf{v_s} = \mathsf{v}_{s_1} \otimes \cdots \otimes \mathsf{v}_{s_k} \in \mathsf{V}^{\otimes k}$, the generators g, h have the following action:

$$g\mathsf{v_s} = \zeta^{k-2|\mathsf{s}|}\mathsf{v_s}, \qquad h\mathsf{v_s} = i^k \mathsf{v}_{-\mathsf{s}}. \tag{3.5}$$

For two such k-tuples r and s,

$$k - 2|\mathsf{r}| \equiv k - 2|\mathsf{s}| \bmod 2n \iff |\mathsf{r}| \equiv |\mathsf{s}| \bmod n.$$

Let

$$\begin{aligned}\Lambda_k^\bullet(\mathbf{D}_n) &= \left\{\ell \in \{0,1,\ldots,n\} \;\middle|\; \ell \equiv k - 2a_\ell \ \mathsf{mod}\, 2n \ \text{ for some } \ a_\ell \in \{0,1,\ldots,k\}\right\} &(3.6)\\ \Lambda_k(\mathbf{D}_n) &= \Lambda_k^\bullet(\mathbf{D}_n) \cup \left\{\ell' \;\middle|\; \ell \in \Lambda_k^\bullet(\mathbf{D}_n) \cap \{0,n\}\right\}. &(3.7)\end{aligned}$$

We will always assume a_ℓ is minimal with that property. The set $\Lambda_k(\mathbf{D}_n)$ indexes the irreducible $\mathbf{D}_n$-modules in $\mathsf{V}^{\otimes k}$. In particular,

$$\text{for } \ \mathbf{D}_n^{(\ell)} \text{ to occur in } \ \mathsf{V}^{\otimes k} \ \text{ it is necessary that } \ k - \ell \equiv 0 \ \mathsf{mod}\, 2, \tag{3.8}$$

and when $k - \ell \equiv 0 \ \mathsf{mod}\, 2$ holds, then $i^{k-\ell} = i^{\ell - k} = 1$ or -1 depending on whether $k - \ell \equiv 0$ or $k - \ell \equiv 2 \ \mathsf{mod}\, 4$. Note also for $\ell = 0, n$ that $\mathbf{D}_n^{(\ell')}$ occurs in $\mathsf{V}^{\otimes k}$ with the same multiplicity as $\mathbf{D}_n^{(\ell)}$.

3.1 The centralizer algebra $\mathsf{Z}_k(\mathbf{D}_n)$

In this section, we investigate the centralizer algebra $\mathsf{Z}_k(\mathbf{D}_n) = \mathsf{End}_{\mathbf{D}_n}(\mathsf{V}^{\otimes k})$ for $\mathsf{V} = \mathbf{D}_n^{(1)} = \mathbb{C}\mathsf{v}_{-1} \oplus \mathbb{C}\mathsf{v}_1$. The element g in (3.1) generates a cyclic subgroup $\mathbf{C}_{2n}$ of order $2n$, which implies that that $\mathsf{End}_{\mathbf{D_n}}(\mathsf{V}^{\otimes k}) = \mathsf{Z}_k(\mathbf{D}_n) \subseteq \mathsf{Z}_k(\mathbf{C}_{2n}) = \mathsf{End}_{\mathbf{C}_{2n}}(\mathsf{V}^{\otimes k})$. We will exploit that fact in our considerations.

We impose the following order on k-tuples in $\{-1,1\}^k$.

Definition 3.9. Say $\mathsf{r} \succeq \mathsf{s}$ if $|\mathsf{r}| \le |\mathsf{s}|$, and if $|\mathsf{r}| = |\mathsf{s}|$, then r is greater than or equal to s in the lexicographic order coming from the relation $1 > -1$.

Example 3.10. $(1,1,-1,-1,-1,1) \succ (1,-1,1,-1,1,-1)$.

3.2 A basis for $\mathsf{Z}_k(\mathbf{D}_n)$

We determine the dimension of $\mathsf{Z}_k(\mathbf{D}_n)$ and a basis for it. It follows from Theorem 2.7 (a) that a basis for $\mathsf{Z}_k(\mathbf{C}_{2n})$ is given by

$$\mathcal{B}^k(\mathbf{C}_{2n}) = \left\{\mathsf{E}_{\mathsf{r},\mathsf{s}} \mid \mathsf{r},\mathsf{s} \in \{-1,1\}^{\mathsf{k}}, \ |\mathsf{r}| \equiv |\mathsf{s}| \ \mathsf{mod}\, \mathsf{n}\right\}, \tag{3.11}$$

where $|\mathsf{r}|, |\mathsf{s}| \in \{0,1,\ldots,k\}$. Now

$$|\mathsf{r}| = |\mathsf{s}| \ \mathsf{mod}\, n \iff |-\mathsf{r}| = k - |\mathsf{r}| \equiv k - |\mathsf{s}| = |-\mathsf{s}| \ \mathsf{mod}\, n, \tag{3.12}$$

so $\mathsf{E}_{\mathsf{r},\mathsf{s}} \in \mathcal{B}^k(\mathbf{C}_{2n})$ if and only if $\mathsf{E}_{-\mathsf{r},-\mathsf{s}} \in \mathcal{B}^k(\mathbf{C}_{2n})$.

Theorem 3.13. (a) $\mathsf{Z}_k(\mathbf{D}_n) = \{X \in \mathsf{Z}_k(\mathbf{C}_{2n}) \mid hX = Xh\}$.

(b) *A basis for $\mathsf{Z}_k(\mathbf{D}_n) = \mathsf{End}_{\mathbf{D_n}}(\mathsf{V}^{\otimes k})$ is the set*

$$\mathcal{B}^k(\mathbf{D}_n) = \left\{\mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}} \,\middle|\, \mathsf{r},\mathsf{s} \in \{-1,1\}^{\mathsf{k}}, \ \mathsf{r} \succ -\mathsf{r}, \ |\mathsf{r}| \equiv |\mathsf{s}| \ \mathsf{mod}\, \mathsf{n}\right\}. \tag{3.14}$$

(c) *The dimension of $\mathsf{Z}_k(\mathbf{D}_n)$ is given by*

$$\begin{aligned}\dim \mathsf{Z}_k(\mathbf{D}_n) &= \frac{1}{2}\dim \mathsf{Z}_k(\mathbf{C}_{2n}) = \frac{1}{2} \sum_{\substack{0 \le a,b \le k \\ a \equiv b \ \mathsf{mod}\, n}} \binom{k}{a}\binom{k}{b} &(3.15)\\ &= \frac{1}{2}\left(\text{coefficient of } \ z^k \text{ in } (1+z)^{2k} \,|_{z^n=1}\right).\end{aligned}$$

Proof. Since $\mathsf{Z}_k(\mathbf{D}_n) \subseteq \mathsf{Z}_k(\mathbf{C}_{2n})$, we may assume that $X \in \mathsf{Z}_k(\mathbf{D}_n)$ can be written as

$$X = \sum_{|\mathsf{r}|=|\mathsf{s}| \mod n} X_{\mathsf{r},\mathsf{s}} \mathsf{E}_{\mathsf{r},\mathsf{s}},$$

and that X commutes with the generator g of $\mathbf{D}_n$ in (3.1). In order for X to belong to $\mathsf{Z}_k(\mathbf{D}_n)$, $hX = Xh$ must hold, as asserted in (a). Applying both sides of $hX = Xh$ to v_s, we obtain

$$\sum_{|\mathsf{r}|=|\mathsf{s}| \mod n} i^k X_{\mathsf{r},\mathsf{s}} \mathsf{v}_{-\mathsf{r}} = \sum_{|\mathsf{t}|=|-\mathsf{s}| \mod n} i^k X_{\mathsf{t},-\mathsf{s}} \mathsf{v}_\mathsf{t}.$$

The coefficient $i^k X_{\mathsf{r},\mathsf{s}}$ of $\mathsf{v}_{-\mathsf{r}}$ on the left is nonzero if and only if the coefficient $i^k X_{-\mathsf{r},-\mathsf{s}}$ of $\mathsf{v}_{-\mathsf{r}}$ on the right is nonzero, and they are equal. Hence, $Xh = hX$ if and only if $X_{-\mathsf{r},-\mathsf{s}} = X_{\mathsf{r},\mathsf{s}}$ for all $|\mathsf{r}| = |\mathsf{s}| \mod n$. Therefore, we have

$$X = \sum_{\substack{|\mathsf{r}|=|\mathsf{s}| \mod n \\ \mathsf{r} \succ -\mathsf{r}}} X_{\mathsf{r},\mathsf{s}} (\mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}}).$$

Thus, the set $\mathcal{B}^k(\mathbf{D}_n)$ in (3.43) spans $\mathsf{Z}_k(\mathbf{D}_n)$, and since it is clearly linearly independent, it is a basis for $\mathsf{Z}_k(\mathbf{D}_n)$.

Part (c) is apparent from (3.11), part (b), and Theorem 2.7 (c), which says that $\dim \mathsf{Z}_k(\mathbf{C}_{2n})$ is the coefficient of z^k in $(1+z)^{2k} |_{z^n=1}$. □

Example 3.16. Assume $k = 4$ and $n = 5$. Then $\dim \mathsf{Z}_k(\mathbf{D}_n)$ is $\frac{1}{2}$ the coefficient of z^4 in

$$(1+z)^8 |_{z^5=1} = 1 + 8z + 28z^2 + 56z^3 + 70z^4 + 56 + 28z + 8z^2 + z^3,$$

so that $\dim \mathsf{Z}_4(\mathbf{D}_5) = \frac{1}{2} \cdot 70 = 35$. Since z^4 appears only once in $(1+z)^8 |_{z^n=1}$ for $n \geq 5$, in fact $\dim \mathsf{Z}_4(\mathbf{D}_n) = 35$ for all $n \geq 5$.

Now when $n = 4 = k$, $\dim \mathsf{Z}_k(\mathbf{D}_n)$ is $\frac{1}{2}$ the coefficient of $z^4 = z^0 = 1$ in

$$(1+z)^8 |_{z^4=1} = 1 + 8z + 28z^2 + 56z^3 + 70 + 56z + 28z^2 + 8z^3 + 1.$$

Thus, $\dim \mathsf{Z}_4(\mathbf{D}_4) = \frac{1}{2}(1 + 70 + 1) = 36$.

3.3 Copies of $\mathbf{D}_n^{(\ell)}$ in $\mathsf{V}^{\otimes k}$ for $\ell \in \{1, \ldots, n-1\}$

The next result locates copies of $\mathbf{D}_n^{(\ell)}$ inside $\mathsf{V}^{\otimes k}$ when $1 \leq \ell \leq n-1$.

Theorem 3.17. *Assume $\ell \in \Lambda_k(\mathbf{D}_n)$ and $1 \leq \ell \leq n-1$. Set*

$$\mathsf{K}_\ell = \left\{ \mathsf{r} \in \{-1, 1\}^k \;\middle|\; k - 2|\mathsf{r}| \equiv \ell \mod 2n,\ \mathsf{r} \succ -\mathsf{r} \right\}. \tag{3.18}$$

(i) *For each $\mathsf{r} \in \mathsf{K}_\ell$, the vectors $i^{\ell-k}\mathsf{v}_{-\mathsf{r}}$, v_r determine a standard basis for a copy of $\mathbf{D}_n^{(\ell)}$.*

(ii) *For $\mathsf{r}, \mathsf{s} \in \mathsf{K}_\ell$, the transformation $\mathsf{e}_{\mathsf{r},\mathsf{s}} := \mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}} \in \mathsf{Z}_k(\mathbf{D}_n) = \mathsf{End}_{\mathbf{D}_n}(\mathsf{V}^{\otimes k})$ satisfies*

$$\mathsf{e}_{\mathsf{r},\mathsf{s}}(\mathsf{v}_\mathsf{t}) = \delta_{\mathsf{s},\mathsf{t}} \mathsf{v}_\mathsf{r}, \qquad \mathsf{e}_{\mathsf{r},\mathsf{s}}(i^{\ell-k}\mathsf{v}_{-\mathsf{t}}) = \delta_{\mathsf{s},\mathsf{t}} i^{\ell-k}\mathsf{v}_{-\mathsf{r}}.$$

(iii) *For $\mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{u} \in \mathsf{K}_\ell$, $\mathsf{e}_{\mathsf{r},\mathsf{s}}\mathsf{e}_{\mathsf{t},\mathsf{u}} = \delta_{\mathsf{s},\mathsf{t}} \mathsf{e}_{\mathsf{r},\mathsf{u}}$ holds, so $\mathsf{span}_\mathbb{C}\{\mathsf{e}_{\mathsf{r},\mathsf{s}} \mid \mathsf{r}, \mathsf{s} \in \mathsf{K}_\ell\}$ can be identified with the matrix algebra $\mathcal{M}_{d_\ell}(\mathbb{C})$, where $d_\ell = |\mathsf{K}_\ell|$.*

(iv) $\mathsf{B}_k^{(\ell)} := \mathsf{span}_{\mathbb{C}}\{i^{\ell-k}\mathsf{v}_{-\mathsf{r}}, \mathsf{v}_\mathsf{r} \mid \mathsf{r} \in \mathsf{K}_\ell\}$ *is an irreducible bimodule for* $\mathbf{D}_n \times \mathcal{M}_{d_\ell}(\mathbb{C})$.

Proof. (i) Since $\mathsf{r} \in \mathsf{K}_\ell$ satisfies $k - 2|\mathsf{r}| \equiv \ell \bmod 2n$, we have

$$g(i^{\ell-k}\mathsf{v}_{-\mathsf{r}}) = \zeta^{-\ell}(i^{\ell-k}\mathsf{v}_{-\mathsf{r}}) \quad \text{and} \quad g\mathsf{v}_\mathsf{r} = \zeta^\ell \mathsf{v}_\mathsf{r},$$
$$h(i^{\ell-k}\mathsf{v}_{-\mathsf{r}}) = i^\ell \mathsf{v}_\mathsf{r} \quad \text{and} \quad h\mathsf{v}_\mathsf{r} = i^k\mathsf{v}_{-\mathsf{r}} = i^\ell(i^{k-\ell}\mathsf{v}_{-\mathsf{r}}) = i^\ell(i^{\ell-k}\mathsf{v}_{-\mathsf{r}}),$$

so $\{i^{\ell-k}\mathsf{v}_{-\mathsf{r}}, \mathsf{v}_\mathsf{r}\}$ forms a standard basis for a copy of $\mathbf{D}_n^{(\ell)}$.

Part (ii) can be verified by direct computation.

For (iii) note that if $\mathsf{s} \in \mathsf{K}_\ell$, then $-\mathsf{s} \notin \mathsf{K}_\ell$, as otherwise $-\ell = k - 2|-\mathsf{s}| \equiv \ell \bmod 2n$, which is impossible for $\ell \in \{1, \dots, n-1\}$. Thus, for $\mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{u} \in \mathsf{K}_\ell$,

$$\mathsf{e}_{\mathsf{r},\mathsf{s}}\mathsf{e}_{\mathsf{t},\mathsf{u}} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{e}_{\mathsf{r},\mathsf{u}} + \delta_{-\mathsf{s},\mathsf{t}}\mathsf{e}_{-\mathsf{r},\mathsf{u}} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{e}_{\mathsf{r},\mathsf{u}}.$$

For (iv), assume $\mathsf{S} \neq 0$ is a sub-bimodule of $\mathsf{B}_k^{(\ell)}$, and $0 \neq u = \sum_{\mathsf{r} \in \mathsf{K}_\ell} \left(\gamma_{-\mathsf{r}} i^{\ell-k}\mathsf{v}_{-\mathsf{r}} + \gamma_\mathsf{r}\mathsf{v}_\mathsf{r}\right) \in \mathsf{S}$. Then since $k - 2|\mathsf{r}| \equiv \ell \bmod 2n$, $gu - \zeta^{-\ell}u = (\zeta^\ell - \zeta^{-\ell})\sum_{\mathsf{r}\in\mathsf{K}_\ell}\gamma_\mathsf{r}\mathsf{v}_\mathsf{r} \in \mathsf{S}$. As $\zeta^\ell - \zeta^{-\ell} \neq 0$ for $\ell = 1, \dots, n-1$, we have that $w := \sum_{\mathsf{r}\in\mathsf{K}_\ell}\gamma_\mathsf{r}\mathsf{v}_\mathsf{r} \in \mathsf{S}$. If $\gamma_\mathsf{u} \neq 0$ for some $\mathsf{u} \in \mathsf{K}_\ell$, then $\mathsf{e}_{\mathsf{t},\mathsf{u}}w = \gamma_\mathsf{u}\mathsf{v}_\mathsf{t} \in \mathsf{S}$ for all $\mathsf{t} \in \mathsf{K}_\ell$. But then $h\mathsf{v}_\mathsf{t} = i^k\mathsf{v}_{-\mathsf{t}} \in \mathsf{S}$ for all such t as well. This implies $\mathsf{S} = \mathsf{B}_k^{(\ell)}$. If instead $\gamma_\mathsf{u} = 0$ for all $\mathsf{u} \in \mathsf{K}_\ell$, then $u = \sum_{\mathsf{r}\in\mathsf{K}_\ell}\gamma_{-\mathsf{r}}i^{\ell-k}\mathsf{v}_{-\mathsf{r}}$. There is some $\gamma_{-\mathsf{s}} \neq 0$, and applying $\mathsf{e}_{\mathsf{t},\mathsf{s}}$ to u shows that $\mathsf{v}_{-\mathsf{t}} \in \mathsf{S}$ for all $\mathsf{t} \in \mathsf{K}_\ell$. Applying h to those elements shows that $\mathsf{v}_\mathsf{t} \in \mathsf{S}$ for all $\mathsf{t} \in \mathsf{K}_\ell$. Hence $\mathsf{S} = \mathsf{B}_k^{(\ell)}$ in this case also, and $\mathsf{B}_k^{(\ell)}$ must be irreducible. □

3.4 Copies of $\mathbf{D}_n^{(\ell)}$ in $\mathsf{V}^{\otimes k}$ for $\ell \in \{0, 0', n, n'\}$

Throughout this section, let $\ell = 0, n$ and $\ell' = 0', n'$. Observe that if $k - 2|\mathsf{r}| \equiv \ell \bmod 2n$, for $\ell = 0, n$, then $k - 2|-\mathsf{r}| \equiv -\ell \equiv \ell \bmod 2n$.

Theorem 3.19. *Assume* $\ell = 0$ *or* n *and* $\ell \in \Lambda_k(\mathbf{D}_n)$. *Let*

$$\mathsf{K}_\ell = \left\{\mathsf{r} \in \{-1, 1\}^k \;\middle|\; k - 2|\mathsf{r}| \equiv \ell \mod 2n,\ \mathsf{r} \succ -\mathsf{r}\right\} \tag{3.20}$$

(i) $\mathbb{C}(\mathsf{v}_\mathsf{r} + i^{\ell-k}\mathsf{v}_{-\mathsf{r}}) \cong \mathbf{D}_n^{(\ell)}$ *and* $\mathbb{C}(\mathsf{v}_\mathsf{r} - i^{\ell-k}\mathsf{v}_{-\mathsf{r}}) \cong \mathbf{D}_n^{(\ell')}$ *for each* $\mathsf{r} \in \mathsf{K}_\ell$.

(ii) *For* $\mathsf{r}, \mathsf{s} \in \mathsf{K}_\ell$, *the transformations*

$$\mathsf{e}_{\mathsf{r},\mathsf{s}}^{\pm} := \frac{1}{2}\Big((\mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}}) \pm i^{\ell-k}(\mathsf{E}_{\mathsf{r},-\mathsf{s}} + \mathsf{E}_{-\mathsf{r},\mathsf{s}})\Big) \in \mathsf{Z}_k(\mathbf{D}_n)$$

satisfy

$$\begin{aligned} \mathsf{e}_{\mathsf{r},\mathsf{s}}^{\pm}(\mathsf{v}_\mathsf{t} \pm i^{\ell-k}\mathsf{v}_{-\mathsf{t}}) &= \delta_{\mathsf{s},\mathsf{t}}(\mathsf{v}_\mathsf{r} \pm i^{\ell-k}\mathsf{v}_{-\mathsf{r}}) \\ \mathsf{e}_{\mathsf{r},\mathsf{s}}^{\pm}(\mathsf{v}_\mathsf{t} \mp i^{\ell-k}\mathsf{v}_{-\mathsf{t}}) &= 0. \end{aligned}$$

(iii) *For* $\mathsf{r}, \mathsf{s}, \mathsf{t}, \mathsf{u} \in \mathsf{K}_\ell$, $\mathsf{e}_{\mathsf{r},\mathsf{s}}^{\pm}\mathsf{e}_{\mathsf{t},\mathsf{u}}^{\pm} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{e}_{\mathsf{r},\mathsf{u}}^{\pm}$ *and* $\mathsf{e}_{\mathsf{r},\mathsf{s}}^{\pm}\mathsf{e}_{\mathsf{t},\mathsf{u}}^{\mp} = 0$ *hold. Therefore* $\mathsf{span}_{\mathbb{C}}\{\mathsf{e}_{\mathsf{r},\mathsf{s}}^{\pm}\}$ *can be identified with the matrix algebra* $\mathcal{M}_{d_\ell}(\mathbb{C})^{\pm}$, *where* $d_\ell = |\mathsf{K}_\ell|$.

(iv) $\mathsf{B}_k^{(\ell)} := \mathsf{span}_{\mathbb{C}}\{\mathsf{v}_\mathsf{r} + i^{\ell-k}\mathsf{v}_{-\mathsf{r}} \mid \mathsf{r} \in \mathsf{K}_\ell\}$ *and* $\mathsf{B}_k^{(\ell')} := \mathsf{span}_{\mathbb{C}}\{\mathsf{v}_\mathsf{r} - i^{\ell-k}\mathsf{v}_{-\mathsf{r}} \mid \mathsf{r} \in \mathsf{K}_\ell\}$ *are irreducible bimodules for* $\mathbf{D}_n \times \mathcal{M}_{d_\ell}(\mathbb{C})^{\pm}$.

Proof. Part (i) follows directly from (3.5), and parts (ii) and (iii) can be verified by direct computation using the fact that if $\mathsf{r},\mathsf{s} \in \mathsf{K}_\ell$, then it cannot be the case that $\mathsf{r} = -\mathsf{s}$. Indeed, if $\mathsf{r} = -\mathsf{s}$, then $-\mathsf{s} = \mathsf{r} \succ -\mathsf{r} = \mathsf{s}$, which contradicts that s belongs to K_ℓ. Part (iv) can be deduced from the earlier parts and the fact that the maps $\mathsf{e}^{\pm}_{\mathsf{r},\mathsf{s}}$ belong to $\mathsf{Z}_k(\mathbf{D}_n)$, hence commute with the $\mathbf{D}_n$-action. The argument follows the proof of (iv) of Theorem 3.17 and is left to the reader. □

Corollary 3.21. (a) *The following set is a basis for* $\mathsf{Z}_k(\mathbf{D}_n)$ *which gives its decomposition into matrix summands:*

$$\mathcal{B}^k_{\mathsf{mat}}(\mathbf{D}_n) := \bigcup_{\ell=1,\dots,n-1} \{\mathsf{e}_{\mathsf{r},\mathsf{s}} \mid \mathsf{r},\mathsf{s} \in \mathsf{K}_\ell\} \ \cup \ \bigcup_{\ell=0,n} \left\{\mathsf{e}^{\pm}_{\mathsf{t},\mathsf{u}} \mid \mathsf{t},\mathsf{u} \in \mathsf{K}_\ell\right\}, \tag{3.22}$$

where $\ell \in \Lambda^{\bullet}_k(\mathbf{D}_n)$ *(see* (3.6)*);* $\mathsf{e}_{\mathsf{r},\mathsf{s}} = \mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}}$ *for* $\mathsf{r},\mathsf{s} \in \mathsf{K}_\ell$, $\ell = 1,\dots,n-1$, *and* K_ℓ *is as in* (3.18)*; and* $\mathsf{e}^{\pm}_{\mathsf{t},\mathsf{u}} := \frac{1}{2}\Big((\mathsf{E}_{\mathsf{t},\mathsf{u}} + \mathsf{E}_{-\mathsf{t},-\mathsf{u}}) \pm i^{\ell-k}(\mathsf{E}_{\mathsf{t},-\mathsf{u}} + \mathsf{E}_{-\mathsf{t},\mathsf{u}})\Big)$ *for* $\mathsf{t},\mathsf{u} \in \mathsf{K}_\ell$, $\ell = 0, n$, *and* K_ℓ *is as in* (3.20)*.*

(b) $\{z_\ell \mid \ell \in \{1,\dots,n-1\}\} \cup \left\{z^{\pm}_\ell \mid \ell = 0, n\right\}$, $\ell \in \Lambda^{\bullet}_k(\mathbf{D}_n)$, *is a basis for the center of* $\mathsf{Z}_k(\mathbf{D}_n)$, *where*

$$z_\ell \quad = \sum_{\mathsf{r}\in\mathsf{K}_\ell,\ \mathsf{r}\succ\mathsf{r}'} \mathsf{e}_{\mathsf{r},\mathsf{r}} \qquad \text{for} \quad \ell \in \{1,\dots,n-1\}, \tag{3.23}$$

$$z^{\pm}_\ell \quad = \sum_{\mathsf{r}\in\mathsf{K}_\ell} \mathsf{e}^{\pm}_{\mathsf{r},\mathsf{r}} \qquad \text{for} \quad \ell \in \{0,n\}. \tag{3.24}$$

3.5 Example $\mathsf{V}^{\otimes 4}$

Example 3.25. In this example, we decompose $\mathsf{V}^{\otimes 4}$ for $n \geq 4$.

First, suppose $n \geq 5$. Then using the results in Proposition 3.4, we have

$$\mathsf{V}^{\otimes 4} = 3 \cdot \mathbf{D}_n^{(0')} \oplus 3 \cdot \mathbf{D}_n^{(0)} \oplus 4 \cdot \mathbf{D}_n^{(2)} \oplus \mathbf{D}_n^{(4)}.$$

Correspondingly, $\mathsf{Z}_k(\mathbf{D}_n)$ decomposes into matrix blocks according to

$$\mathsf{Z}_k(\mathbf{D}_n) \cong \mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_4(\mathbb{C}) \oplus \mathcal{M}_1(\mathbb{C}),$$

and $\dim \mathsf{Z}_k(\mathbf{D}_n) = 3^2 + 3^2 + 4^2 + 1^2 = 35$ as in Example 3.16. More explicitly we have for $n \geq 5$ the following:

(i) Let $\ell = k = 4$, and set $\mathsf{q} = (1,1,1,1)$. Then $i^{\ell-k} = 1$, so $\{\mathsf{v}_{-\mathsf{q}}, \mathsf{v}_{\mathsf{q}}\}$ gives a standard basis for a copy of $\mathbf{D}_n^{(4)}$, and $\{\mathsf{e}_{\mathsf{q},\mathsf{q}} := \mathsf{E}_{\mathsf{q},\mathsf{q}} + \mathsf{E}_{-\mathsf{q},-\mathsf{q}}\}$ is a basis for $\mathcal{M}_1(\mathbb{C})$.

(ii) Let $\ell = 2$, and set $\mathsf{r} = (1,1,1,-1)$, $\mathsf{s} = (1,1,-1,1)$, $\mathsf{t} = (1,-1,1,1)$, and $\mathsf{u} = (-1,1,1,1)$. Then the pair $\{-\mathsf{v}_{-\lambda}, \mathsf{v}_\lambda\}$ for $\lambda = \mathsf{r},\mathsf{s},\mathsf{t},\mathsf{u}$ determines a standard basis for a copy of $\mathbf{D}_n^{(2)}$. The maps $\mathsf{e}_{\lambda,\mu} := \mathsf{E}_{\lambda,\mu} + \mathsf{E}_{-\lambda,-\mu}$ for $\lambda,\mu \in \{\mathsf{r},\mathsf{s},\mathsf{t},\mathsf{u}\}$ give a matrix unit basis for $\mathcal{M}_4(\mathbb{C})$.

(iii) Let $\ell = 0$, and set $\mathsf{v} = (1,1,-1,-1)$, $\mathsf{w} = (1,-1,1,-1)$, and $\mathsf{x} = (1,-1,-1,1)$. Then since $i^{\ell-k} = 1$, the vector $\mathsf{v}_\lambda - \mathsf{v}_{-\lambda}$ gives a basis for a copy of $\mathbf{D}_n^{(0')}$ for $\lambda \in \{\mathsf{v},\mathsf{w},\mathsf{x}\}$. The transformations $\mathsf{e}^{-}_{\lambda,\mu} = \frac{1}{2}\Big((\mathsf{E}_{\lambda,\mu} + \mathsf{E}_{-\lambda,-\mu}) - (\mathsf{E}_{\lambda,-\mu}+\mathsf{E}_{-\lambda,\mu})\Big)$ for $\lambda,\mu \in \{\mathsf{v},\mathsf{w},\mathsf{x}\}$ form a matrix unit basis for $\mathcal{M}_3(\mathbb{C})^- \cong \mathcal{M}_3(\mathbb{C})$ (in the notation of Theorem 3.19). Similarly, the vector $\mathsf{v}_\lambda + \mathsf{v}_{-\lambda}$ gives a basis for a copy of $\mathbf{D}_n^{(0)}$, and the maps $\mathsf{e}^{+}_{\lambda,\mu} = \frac{1}{2}\Big((\mathsf{E}_{\lambda,\mu} + \mathsf{E}_{-\lambda,-\mu}) + (\mathsf{E}_{\lambda,-\mu}+\mathsf{E}_{-\lambda,\mu})\Big)$ for $\lambda,\mu \in \{\mathsf{v},\mathsf{w},\mathsf{x}\}$ form a matrix unit basis for $\mathcal{M}_3(\mathbb{C})^+ \cong \mathcal{M}_3(\mathbb{C})$.

In (i) and (ii), the space $\mathsf{B}_4^{(\ell)} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v}_{\pm\lambda}\}$, as λ ranges over the appropriate indices, is an irreducible bimodule for $\mathbf{D}_n \times \mathsf{Z}_k(\mathbf{D}_n)$. In (iii), $\mathsf{B}_4^{(0)} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v}_\lambda + \mathsf{v}_{-\lambda}\}$ and $\mathsf{B}_4^{(0')} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v}_\lambda - \mathsf{v}_{-\lambda}\}$ are irreducible $(\mathbf{D}_n \times \mathsf{Z}_k(\mathbf{D}_n))$-bimodules in $\mathsf{V}^{\otimes 4}$. Therefore, as a $\mathbf{D}_n \times \mathsf{Z}_k(\mathbf{D}_n)$-bimodule $\mathsf{V}^{\otimes 4}$ decomposes into irreducible bimodules of dimensions $2, 8, 3, 3$.

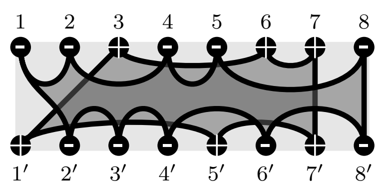
Suppose now that $n = 4$. Then $\mathsf{V}^{\otimes 4} = 3 \cdot \mathbf{D}_4^{(0')} \oplus 3 \cdot \mathbf{D}_4^{(0)} \oplus 4 \cdot \mathbf{D}_4^{(2)} \oplus \mathbf{D}_4^{(4)} \oplus \mathbf{D}_4^{(4')}$, and $\mathsf{Z}_k(\mathbf{D}_4) \cong \mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C}) \oplus \mathcal{M}_4(\mathbb{C}) \oplus \mathcal{M}_1(\mathbb{C}) \oplus \mathcal{M}_1(\mathbb{C})$, which has dimension 36 (compare Example 3.16). The only change is in (i), where $\{\mathsf{v}_\mathsf{q} - \mathsf{v}_{-\mathsf{q}}\}$ is a basis for a copy of $\mathbf{D}_4^{(4')}$ and $\{\mathsf{v}_\mathsf{q} + \mathsf{v}_{-\mathsf{q}}\}$ is a basis for a copy of $\mathbf{D}_4^{(4)}$. In the first case, $\mathsf{e}^-_{\mathsf{q},\mathsf{q}} = \frac{1}{2}\Big((\mathsf{E}_{\mathsf{q},\mathsf{q}} + \mathsf{E}_{-\mathsf{q},-\mathsf{q}}) - (\mathsf{E}_{\mathsf{q},-\mathsf{q}} + \mathsf{E}_{-\mathsf{q},\mathsf{q}})\Big)$ gives a basis for $\mathcal{M}_1(\mathbb{C})^-$, while in the second, $\mathsf{e}^+_{\mathsf{q},\mathsf{q}} = \frac{1}{2}\Big((\mathsf{E}_{\mathsf{q},\mathsf{q}} + \mathsf{E}_{-\mathsf{q},-\mathsf{q}}) + (\mathsf{E}_{\mathsf{q},-\mathsf{q}} + \mathsf{E}_{-\mathsf{q},\mathsf{q}})\Big)$ for $\mathcal{M}_1(\mathbb{C})^+$.

3.6 A diagram basis for $\mathbf{D}_n$

The basis $\{\mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}} \mid \mathsf{r},\mathsf{s} \in \{-1,1\}^\mathsf{k}, \mathsf{r} \succ -\mathsf{r}, |\mathsf{r}| \equiv |\mathsf{s}| \bmod n\}$ of matrix units can be viewed diagrammatically. If $\mathsf{r} = (r_1, \ldots, r_k) \in \{-1,1\}^k$ and $\mathsf{s} = (s_{1'}, \ldots, s_{k'}) \in \{-1,1\}^k$, let $d_{\mathsf{r},\mathsf{s}}$ be the diagram on two rows of k vertices labeled by $1, 2, \ldots, k$ on top and $1', 2', \ldots, k'$ on bottom. Mark the ith vertex on top with $+$ if $r_i = 1$ and with $-$ if $r_i = -1$. Similarly, mark the j'th vertex on the bottom with $+$ if $s_j = 1$ and with $-$ if $s_j = -1$. The positive and negative vertices correspond to a set partition of $\{1, \ldots, k, 1', \ldots, k'\}$ into two parts. Draw edges in the diagram so that the $+$ vertices form a connected component and the $-$ vertices form a connected component. For example, if

$$\mathsf{r} = (-1,-1,1,-1,-1,1,1,-1) \quad \text{and} \quad \mathsf{s} = (1,-1,-1,-1,1,-1,1,-1),$$

then the corresponding diagram and set partition are

$\mathsf{d}_{\mathsf{r},\mathsf{s}} =$ 

$= \Big\{\{1,2,4,5,8,2',3',4',6',8'\}, \{3,6,7,1',5',7'\}\Big\}.$

Two diagrams are equivalent if they correspond to the same underlying set partition. Furthermore, switching the $+$ signs and $-$ signs give the same diagram, since $\mathsf{d}_{\mathsf{r},\mathsf{s}} = \mathsf{d}_{-\mathsf{r},-\mathsf{s}}$.

If d is any diagram corresponding to a set partition of $\{1, \ldots, k, 1', \ldots, k'\}$ into two parts, then d acts on $\mathsf{V}^{\otimes k}$ by $d.\mathsf{v}_\mathsf{u} = \sum_{\mathsf{t} \in \{-1,1\}^k} d^\mathsf{t}_\mathsf{u} \mathsf{v}_\mathsf{t}$ for $\mathsf{t} = (u_1, \ldots, u_k)$ and $\mathsf{u} = (u_{1'}, \ldots, u_{k'}) \in \{-1,1\}^k$ with

$$d^\mathsf{t}_\mathsf{u} = \begin{cases} 1 & \text{if } u_a = u_b \textit{ if and only if } a \text{ and } b \text{ are in the same block of } d, \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

for $a, b \in \{1, \ldots, k, 1', \ldots, k'\}$. In this notation d^t_u denotes the (t,u)-entry of the matrix of d on $\mathsf{V}^{\otimes k}$ with respect to the basis of simple tensors. It follows from this construction that $\mathsf{e}_{\mathsf{r},\mathsf{s}} = \mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}}$ and $\mathsf{d}_{\mathsf{r},\mathsf{s}}$ have the same action on $\mathsf{V}^{\otimes k}$ and so are equal. Note that in the example above, we have

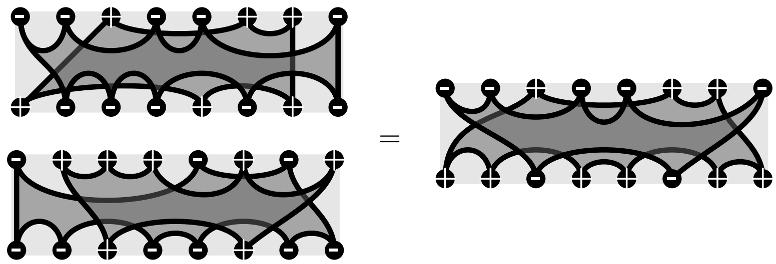
$$\mathsf{d}_{\mathsf{r},\mathsf{s}}.(\mathsf{v}_a \otimes \mathsf{v}_b \otimes \mathsf{v}_b \otimes \mathsf{v}_b \otimes \mathsf{v}_a \otimes \mathsf{v}_b \otimes \mathsf{v}_a \otimes \mathsf{v}_b) = \mathsf{v}_b \otimes \mathsf{v}_b \otimes \mathsf{v}_a \otimes \mathsf{v}_b \otimes \mathsf{v}_b \otimes \mathsf{v}_a \otimes \mathsf{v}_a \otimes \mathsf{v}_b,$$

for $\{a, b\} = \{-1, 1\}$ and $\mathsf{d}_{\mathsf{r},\mathsf{s}}$ acts as 0 on all other simple tensors.

The diagrams multiply as matrix units. Let $b(d)$ and $t(d)$ denote the set partitions imposed by d on the bottom and top rows of d, respectively. If d_1 and d_2 are diagrams, then

$$d_1 d_2 = \delta_{b(d_1), t(d_2)} d_3,$$

where d_3 is the diagram given by placing d_1 on top of d_2, identifying $b(d_1)$ with $t(d_2)$, and taking d_3 to be the connected components of the resulting diagram. For example,

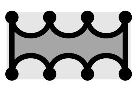
$d_1 =$ / $d_2 =$  $= d_3$.

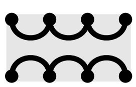
If the set partitions in the bottom of d_1 do not match *exactly* with the set partitions of the top of d_2 then this product is 0. Note that the set partitions must match, but the + and − labels might be reversed.

For $\mathsf{d}_{\mathsf{r},\mathsf{s}}$ to belong to the centralizer algebra $\mathsf{Z}_k(\mathbf{D}_n)$, we must have $\mathsf{r}, \mathsf{s} \in \mathsf{K}_\ell$. For a block B in a set partition diagram d we let $t(B) = |B \cap \{1, \ldots, k\}|$ and $b(B) = |B \cap \{1', \ldots, k'\}|$. Then $\mathsf{r}, \mathsf{s} \in \mathsf{K}_\ell$ if and only if $|\mathsf{r}| \equiv |\mathsf{s}| \equiv \frac{1}{2}(k - \ell) \bmod n$ which is equivalent to

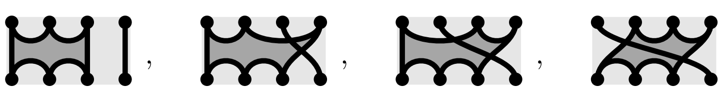
$$t(B) \equiv b(B) \bmod n \quad \text{for each block } B \text{ in } d. \tag{3.27}$$

Example 3.28. Recall from Example 3.16 that $\dim \mathsf{Z}_4(\mathbf{D}_4) = 36$, and the diagrammatic basis is given by diagrams on two rows of 4 vertices each that partition the vertices into ≤ 2 blocks B that satisfy $t(B) \equiv b(B) \bmod 4$. They are

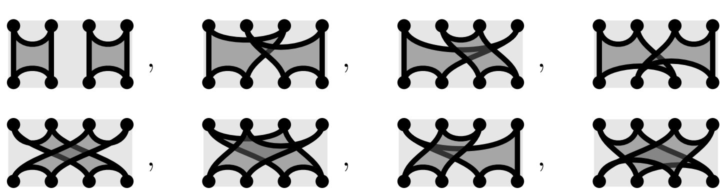
(a)  has one block B with $t(B) = b(B) = 4$.

(b)  has $t(B_1) = 4 \equiv 0 = b(B_1) \bmod 4$ and $t(B_2) = 0 \equiv 4 = b(B_2) \bmod 4$.

(c) There are 16 diagrams $d = B_1 \sqcup B_2$ in which B_1 has 3 vertices in each row and B_2 has 1 vertex in each row:

 , . . . (12 more) . . .

(d) There are 18 diagrams $d = B_1 \sqcup B_2$ in which both blocks have 2 vertices in each row:

 , . . . (5 more) . . . , . . . (5 more) . . .

3.7 Irreducible modules for $\mathsf{Z}_k(\mathbf{D}_n)$

Theorem 3.29. *Assume $\ell \in \Lambda_k^\bullet(\mathbf{D}_n)$, and let $a_\ell \in \{0, 1 \ldots, k\}$ be minimal such that $\ell \equiv k - 2a_\ell \bmod 2n$.*

(i) *For* $\ell \in \{1, \ldots, n-1\}$,

$$\mathsf{Z}_k^{(\ell)} = \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)} := \mathsf{span}_{\mathbb{C}}\{\mathsf{v_t} \mid \mathsf{t} \in \mathsf{K}_\ell\}, \tag{3.30}$$

where K_ℓ *is as in* (3.18)*, is an irreducible* $\mathsf{Z}_k(\mathbf{D}_n)$*-module, and*

$$\dim \mathsf{Z}_k^{(\ell)} = \sum_{\substack{0 \le b \le k \\ b \equiv a_\ell \mod n}} \binom{k}{b} = \text{coefficient of } z^{a_\ell} \text{ in } (1+z)^k\big|_{z^n=1}. \tag{3.31}$$

(ii) *For* $\ell \in \{0, n\}$,

$$\begin{aligned} \mathsf{Z}_k^{(\ell)} &= \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)} := \mathsf{span}_{\mathbb{C}}\{\mathsf{v_t} + i^{\ell-k}\mathsf{v_{-t}} \mid \mathsf{t} \in \mathsf{K}_\ell\}, \\ \mathsf{Z}_k^{(\ell')} &= \mathsf{Z}_k(\mathbf{D}_n)^{(\ell')} := \mathsf{span}_{\mathbb{C}}\{\mathsf{v_t} - i^{\ell-k}\mathsf{v_{-t}} \mid \mathsf{t} \in \mathsf{K}_\ell\}, \end{aligned} \tag{3.32}$$

where K_ℓ *is as in* (3.20)*, are irreducible* $\mathsf{Z}_k(\mathbf{D}_n)$*-modules, and*

$$\dim \mathsf{Z}_k^{(\ell')} = \dim \mathsf{Z}_k^{(\ell)} = \frac{1}{2} \sum_{\substack{0 \le b \le k \\ b \equiv a_\ell \mod n}} \binom{k}{b} = \frac{1}{2}\left(\text{coefficient of } z^{a_\ell} \text{ in } (1+z)^k\big|_{z^n=1}\right). \tag{3.33}$$

Up to isomorphism, the modules in (i) and (ii) are the only irreducible $\mathsf{Z}_k(\mathbf{D}_n)$*-modules.*

Proof. Assume initially that $\ell \in \{1, \ldots, n-1\}$ and $\ell \in \Lambda_k^{\bullet}(\mathbf{D}_n)$. Let $\mathsf{Z}_k^{(\ell)}$ be as in (3.30). For all $\mathsf{r,s,t} \in \mathsf{K}_\ell$, $\mathsf{e_{r,s}v_t} = \delta_{\mathsf{s,t}}\mathsf{v_r}$, where $\mathsf{e_{r,s}} = \mathsf{E_{r,s}} + \mathsf{E_{-r,-s}} \in \mathsf{Z}_k(\mathbf{D}_n)$ is as in Theorem 3.17 (ii). Thus, $\mathsf{Z}_k^{(\ell)}$ is the unique irreducible module (up to isomorphism) for the matrix subalgebra of $\mathsf{Z}_k(\mathbf{D}_n)$ having basis the matrix units $\mathsf{e_{r,s}}$, $\mathsf{r,s} \in \mathsf{K}_\ell$. Since the other basis elements in the basis $\mathcal{B}^k_{\mathsf{mat}}(\mathbf{D}_n)$ (in Corollary 3.21) act trivially on the vectors in $\mathsf{Z}_k^{(\ell)}$, we have that $\mathsf{Z}_k^{(\ell)}$ is an irreducible $\mathsf{Z}_k(\mathbf{D}_n)$-module. Since $k - 2|\mathsf{t}| \equiv \ell \ \mathsf{mod}\, 2n$ for all $\mathsf{t} \in \mathsf{K}_\ell$, we have $|\mathsf{t}| \equiv a_\ell \ \mathsf{mod}\, n$. Therefore $\dim \mathsf{Z}_k^{(\ell)} = \sum_{\substack{0 \le b \le k \\ b \equiv a_\ell \mod n}} \binom{k}{b}$ = coefficient of z^{a_ℓ} in $(1+z)^k\big|_{z^n=1}$, as claimed in (i). Observe $\mathsf{W}_k^{(\ell)} := \mathsf{span}_{\mathbb{C}}\{i^{\ell-k}\mathsf{v_{-t}} \mid \mathsf{t} \in \mathsf{K}_\ell\}$ is also a $\mathsf{Z}_k(\mathbf{D}_n)$-module isomorphic to $\mathsf{Z}_k^{(\ell)}$ by Theorem 3.17 (ii), and the $(\mathbf{D}_n \times \mathsf{Z}_k(\mathbf{D}_n))$-bimodule $\mathsf{B}_k^{(\ell)}$ in that theorem is the sum $\mathsf{B}_k^{(\ell)} = \mathsf{W}_k^{(\ell)} \oplus \mathsf{Z}_k^{(\ell)}$.

Now assume $\ell = 0, n$ and $\ell \in \Lambda_k^{\bullet}(\mathbf{D}_n)$. Let $\mathsf{Z}_k^{(\ell)}$ and $\mathsf{Z}_k^{(\ell')}$ be as in (3.32). Since according to Theorem 3.19,

$$\mathsf{e}^{\pm}_{\mathsf{r,s}}(\mathsf{v_t} \pm i^{\ell-k}\mathsf{v_{-t}}) = \delta_{\mathsf{s,t}}(\mathsf{v_r} \pm i^{\ell-k}\mathsf{v_{-r}}), \qquad \mathsf{e}^{\pm}_{\mathsf{r,s}}(\mathsf{v_t} \mp i^{\ell-k}\mathsf{v_{-t}}) = 0$$

for $\mathsf{e}^{\pm}_{\mathsf{r,s}} := \frac{1}{2}\Big((\mathsf{E_{r,s}} + \mathsf{E_{-r,-s}}) \pm i^{\ell-k}(\mathsf{E_{r,-s}} + \mathsf{E_{-r,s}})\Big)$ with $\mathsf{r,s} \in \mathsf{K}_\ell$, and since all other elements of the basis $\mathcal{B}^k_{\mathsf{mat}}(\mathbf{D}_n)$ act trivially on $\mathsf{Z}_k^{(\ell)}$ and $\mathsf{Z}_k^{(\ell')}$, we see that they are irreducible $\mathsf{Z}_k(\mathbf{D}_n)$-modules. Moreover, since $|\mathsf{t}| \equiv a_\ell$ for all $\mathsf{t} \in \mathsf{K}_\ell$, they have dimension

$$\frac{1}{2} \sum_{\substack{0 \le b \le k \\ b \equiv a_\ell \mod n}} \binom{k}{b} = \frac{1}{2}\left(\text{coefficient of } z^{a_\ell} \text{ in } (1+z)^k\big|_{z^n=1}\right).$$

As the sum of the squares of the dimensions of the modules in (i) and (ii) adds up to $\dim \mathsf{Z}_k(\mathbf{D}_n)$, these are all the irreducible $\mathsf{Z}_k(\mathbf{D}_n)$-modules up to isomorphism. □

Example 3.34. Assume $k = 4$ and $n = 5$. Then $\Lambda_k(\mathbf{D}_n) = \{0, 0', 2, 4\}$ and $a_\ell = \frac{1}{2}(k - \ell) = 2, 1, 0$, respectively. Thus,

$$\dim \mathsf{Z}_4^{(0')} = \dim \mathsf{Z}_4^{(0)} = \frac{1}{2} \sum_{\substack{0 \le b \le 4 \\ b \equiv 2 \mod 5}} \binom{4}{b} = \frac{1}{2}\binom{4}{2} = 3.$$

$$\dim \mathsf{Z}_4^{(\ell)} = \begin{cases} \displaystyle\sum_{\substack{0 \le b \le 4 \\ b \equiv 1 \mod 5}} \binom{4}{b} = \binom{4}{1} = 4 & \text{if } \ell = 2, \\ \displaystyle\sum_{\substack{0 \le b \le 4 \\ b \equiv 0 \mod 5}} \binom{4}{b} = \binom{4}{0} = 1 & \text{if } \ell = 4. \end{cases}$$

Then $\dim \mathsf{Z}_k(\mathbf{D}_n)$ is the sum of the squares of these numbers which is 35, as in Example 3.16.

Remark 3.35. For $\ell \in \{1, \dots, n-1\}$ we have

$$\dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(2n-\ell)} = \dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(\ell)} = \dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)}. \tag{3.36}$$

This can be seen from the observation that if $2n - \ell \equiv k - 2a \mod 2n$, for some $a \in \{0, 1, \dots, k\}$, then $\ell \equiv k - 2(k - a) \mod n$, so that

$$\begin{aligned} \dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(2n-\ell)} &= \sum_{\substack{0 \le b \le k \\ b \equiv a \mod n}} \binom{k}{b} = \sum_{\substack{0 \le k-b \le k \\ k-b \equiv k-a \mod n}} \binom{k}{k-b} = \sum_{\substack{0 \le c \le k \\ c \equiv k-a \mod n}} \binom{k}{c} \\ &= \dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(\ell)} = \dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)}. \end{aligned}$$

For $\ell \in \{0, n\}$, we have seen that

$$\dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell')} = \dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)} = \frac{1}{2}\dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(\ell)}.$$

Therefore,

$$\begin{aligned} \dim \mathsf{Z}_k(\mathbf{D}_n) &= \sum_{\ell=1}^{n-1} \left(\dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)}\right)^2 + \sum_{\ell \in \{0,n\}} \left(\left(\dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell)}\right)^2 + \left(\dim \mathsf{Z}_k(\mathbf{D}_n)^{(\ell')}\right)^2\right) \\ &= \frac{1}{2} \sum_{\ell=1, \ell \ne n}^{2n-1} \left(\dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(\ell)}\right)^2 + \frac{1}{2}\left(\dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(0)}\right)^2 + \frac{1}{2}\left(\dim \mathsf{Z}_k(\mathbf{C}_{2n})^{(n)}\right)^2 \\ &= \frac{1}{2}\dim \mathsf{Z}_k(\mathbf{C}_{2n}), \end{aligned}$$

as in Theorem 3.13(b).

3.8 The infinite binary dihedral subgroup $\mathbf{D}_\infty$ of SU_2

Let $\mathbf{D}_\infty$ denote the subgroup of SU_2 generated by

$$g = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3.37}$$

where $\zeta = e^{i\theta}$, $\theta \in \mathbb{R}$, $i = \sqrt{-1}$, and ζ is not a root of unity. Then the following relations hold in $\mathbf{D}_\infty$:

$$h^4 = 1, \quad h^{-1}gh = g^{-1}, \quad g^n \neq 1 \text{ for } n \neq 0. \tag{3.38}$$

For $\ell = 1, 2, \dots,$ let $\mathbf{D}_\infty^{(\ell)}$ denote the two-dimensional $\mathbf{D}_\infty$-module on which the generators g, h have the following matrix representations:

$$g = \begin{pmatrix} \zeta^{-\ell} & 0 \\ 0 & \zeta^{\ell} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & i^{\ell} \\ i^{\ell} & 0 \end{pmatrix}.$$

relative to the basis $\{\mathsf{v}_{-\ell}, \mathsf{v}_\ell\}$. In particular, $\mathsf{V} = \mathbf{D}_\infty^{(1)}$. For $\ell = 0, 0'$, let the one-dimensional $\mathbf{D}_\infty$-module $\mathbf{D}_n^{(\ell)}$ be given by

$$\mathbf{D}_\infty^{(0)} = \mathbb{C}\mathsf{v}_0, \qquad g\mathsf{v}_0 = \mathsf{v}_0, \quad h\mathsf{v}_0 = \mathsf{v}_0 \tag{3.39}$$

$$\mathbf{D}_\infty^{(0')} = \mathbb{C}\mathsf{v}_{0'}, \qquad g\mathsf{v}_{0'} = \mathsf{v}_{0'}, \quad h\mathsf{v}_{0'} = -\mathsf{v}_{0'}. \tag{3.40}$$

Proposition 3.41. *Tensor products of* V *with the irreducible modules* $\mathbf{D}_\infty^{(\ell)}$ *decompose as follows:*

(a) $\mathbf{D}_\infty^{(\ell)} \otimes \mathsf{V} \cong \mathbf{D}_\infty^{(\ell-1)} \oplus \mathbf{D}_\infty^{(\ell+1)}$ *for* $1 < \ell < \infty$;

(b) $\mathbf{D}_\infty^{(1)} \otimes \mathsf{V} \cong \mathbf{D}_\infty^{(0')} \oplus \mathbf{D}_\infty^{(0)} \oplus \mathbf{D}_\infty^{(2)}$;

(c) $\mathbf{D}_\infty^{(0)} \otimes \mathsf{V} \cong \mathbf{D}_\infty^{(1)} = \mathsf{V}, \quad \mathbf{D}_\infty^{(0')} \otimes \mathsf{V} \cong \mathbf{D}_\infty^{(1)} = \mathsf{V}.$

The representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{D}_\infty)$ of $\mathbf{D}_\infty$ is the Dynkin diagram D_∞ (see Section 4.1), where each of the nodes $\ell = 0, 0', 1, 2, \dots$ corresponds to one of these irreducible $\mathbf{D}_\infty$-modules. The arguments in previous sections can be adapted to show the next result.

Theorem 3.42. *Let* $\mathsf{Z}_k(\mathbf{D}_\infty) = \mathsf{End}_{\mathbf{D}_\infty}(\mathsf{V}^{\otimes k})$.

(a) $\mathsf{Z}_k(\mathbf{D}_\infty) = \{X \in \mathsf{Z}_k(\mathbf{C}_\infty) \mid hX = Xh\}$.

(b) *A basis for* $\mathsf{Z}_k(\mathbf{D}_\infty) = \mathsf{End}_{\mathbf{D}_\infty}(\mathsf{V}^{\otimes k})$ *is the set*

$$\mathcal{B}^k(\mathbf{D}_\infty) = \left\{ \mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}} \,\middle|\, \mathsf{r}, \mathsf{s} \in \{-1, 1\}^{\mathsf{k}},\ \mathsf{r} \succ -\mathsf{r},\ |\mathsf{r}| = |\mathsf{s}| \right\}, \tag{3.43}$$

where $\mathsf{r} \succ -\mathsf{r}$ *is the order in Definition 3.9.*

(c) *The dimension of* $\mathsf{Z}_k(\mathbf{D}_\infty)$ *is given by*

$$\begin{aligned} \dim \mathsf{Z}_k(\mathbf{D}_\infty) &= \frac{1}{2}\dim \mathsf{Z}_k(\mathbf{C}_\infty) = \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k} \\ &= \frac{1}{2}\left(\text{coefficient of } z^k \text{ in } (1+z)^{2k}\right). \end{aligned} \tag{3.44}$$

(d) *For* $\ell = 0, 1, 2, \dots, k$ *such that* $k - \ell$ *is even, let*

$$\tilde{\mathsf{K}}_\ell = \left\{ \mathsf{r} \in \{-1,1\}^k \,\middle|\, k - 2|\mathsf{r}| = \ell \right\} = \left\{ \mathsf{r} \in \{-1,1\}^k \,\middle|\, |\mathsf{r}| = \tfrac{1}{2}(k-\ell) \right\}. \tag{3.45}$$

(i) *When* $1 \le \ell \le k$, *then* $\mathsf{Z}_k^{(\ell)} = \mathsf{Z}_k(\mathbf{D}_\infty)^{(\ell)} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v}_\mathsf{t} \mid \mathsf{t} \in \tilde{\mathsf{K}}_\ell\}$ *is an irreducible* $\mathsf{Z}_k(\mathbf{D}_\infty)$-*module with*

$$\dim \mathsf{Z}_k^{(\ell)} = \binom{k}{b} \quad \text{where} \quad b = \frac{1}{2}(k-\ell). \tag{3.46}$$

For all $\mathsf{r}, \mathsf{s}, \mathsf{t} \in \mathsf{K}_\ell$, $\mathsf{e}_{\mathsf{r},\mathsf{s}}\mathsf{v}_\mathsf{t} = \delta_{\mathsf{s},\mathsf{t}}\mathsf{v}_\mathsf{r}$, *where* $\mathsf{e}_{\mathsf{r},\mathsf{s}} = \mathsf{E}_{\mathsf{r},\mathsf{s}} + \mathsf{E}_{-\mathsf{r},-\mathsf{s}} \in \mathsf{Z}_k(\mathbf{D}_\infty)$.

(ii) *When $\ell = 0$ (necessarily k is even), then*

$$\mathsf{Z}_k^{(0)} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v_t} + i^{-k}\mathsf{v_{-t}} \mid \mathsf{t} \in \tilde{\mathsf{K}}_0\} \text{ and } \mathsf{Z}_k^{(0')} = \mathsf{span}_{\mathbb{C}}\{\mathsf{v_t} - i^{-k}\mathsf{v_{-t}} \mid \mathsf{t} \in \tilde{\mathsf{K}}_0\},$$

are irreducible $\mathsf{Z}_k(\mathbf{D}_\infty)$-modules with dimensions

$$\dim \mathsf{Z}_k^{(0)} = \dim \mathsf{Z}_k^{(0')} = \frac{1}{2}\binom{k}{b} \quad \text{where} \quad b = \frac{1}{2}k, \tag{3.47}$$

and

$$\mathsf{e}^{\pm}_{\mathsf{r,s}}(\mathsf{v_t} \pm i^{-k}\mathsf{v_{-t}}) \;=\; \delta_{\mathsf{s,t}}(\mathsf{v_r} \pm i^{-k}\mathsf{v_{-r}}), \qquad \mathsf{e}^{\pm}_{\mathsf{r,s}}(\mathsf{v_t} \mp i^{-k}\mathsf{v_{-t}}) \;=\; 0$$

for $\mathsf{e}^{\pm}_{\mathsf{r,s}} := \frac{1}{2}\Big((\mathsf{E_{r,s}} + \mathsf{E_{-r,-s}}) \pm i^{-k}(\mathsf{E_{r,-s}} + \mathsf{E_{-r,s}})\Big)$ *with* $\mathsf{r,s} \in \tilde{\mathsf{K}}_0$.

(e) $\mathsf{Z}_k(\mathbf{D}_\infty) \cong \mathsf{Q}_k$*, where* Q_k *is the subalgebra of the planar rook algebra* P_k *having basis*

(i) $\{\mathsf{X_{R,S}} \mid 0 \leq |\mathsf{R}| = |\mathsf{S}| \leq \frac{1}{2}(k-1)\}$ *when k is odd;*

(ii) $\{\mathsf{X_{R,S}} \mid 0 \leq |\mathsf{R}| = |\mathsf{S}| \leq \frac{1}{2}(k-2)\} \cup \{\mathsf{X_{\pm R,\pm S}} \mid |\mathsf{R}| = |\mathsf{S}| = \frac{1}{2}k,\ \mathsf{R} \succ -\mathsf{R},\ \mathsf{S} \succ -\mathsf{S}\}$ *when k is even;*

where the $\mathsf{X_{R,S}}$ *for* $\mathsf{R,S} \subseteq \{1,\dots,k\}$ *are the matrix units in Remark 2.22, and $\succ$ is the order coming from the order in Definition 3.9 and the identification of a subset* R *with the tuple* $\mathsf{r} \in \{-1,1\}^k$.

Remark 3.48. In Remark 2.22, we identified a subset $\mathsf{R} \subseteq \{1,\dots,k\}$ with the k-tuple $\mathsf{r} = (r_1,\dots,r_k)$ such that $r_j = -1$ if $j \in \mathsf{R}$ and $r_j = 1$ if $j \notin \mathsf{R}$. The element $\mathsf{e_{r,s}} = \mathsf{E_{r,s}} + \mathsf{E_{-r,-s}}$ in part (b) of this theorem has the property that $|\mathsf{r}| = |\mathsf{s}|$ and $\mathsf{r} \succ -\mathsf{r}$. Therefore, $|\mathsf{R}| = |\mathsf{r}|$ (the number of -1s in r) is less than or equal to $|-\mathsf{r}|$ (the number of of -1s in $-\mathsf{r}$) by the definition of $\succ$, and $|\mathsf{R}| \leq \lfloor\frac{1}{2}k\rfloor$. Thus, when k is odd, sending each such $\mathsf{e_{r,s}}$ to the corresponding matrix unit $\mathsf{X_{R,S}}$ will give the desired isomorphism in part (e). When k is even,

$$\begin{aligned}\Big\{\mathsf{e_{r,s}} = \mathsf{E_{r,s}} + \mathsf{E_{-r,-s}} \;\Big|\; \mathsf{r,s} \in \{-1,1\}^k,\ \mathsf{r} \succ -\mathsf{r},\ |\mathsf{r}| = |\mathsf{s}| < \tfrac{1}{2}k\Big\} \\ \bigcup \;\; \Big\{\mathsf{e}^{\pm}_{\mathsf{r,s}} \;\Big|\; \mathsf{r,s} \in \{-1,1\}^k,\ \mathsf{r} \succ -\mathsf{r},\ |\mathsf{r}| = |\mathsf{s}| = \tfrac{1}{2}k\Big\}\end{aligned} \tag{3.49}$$

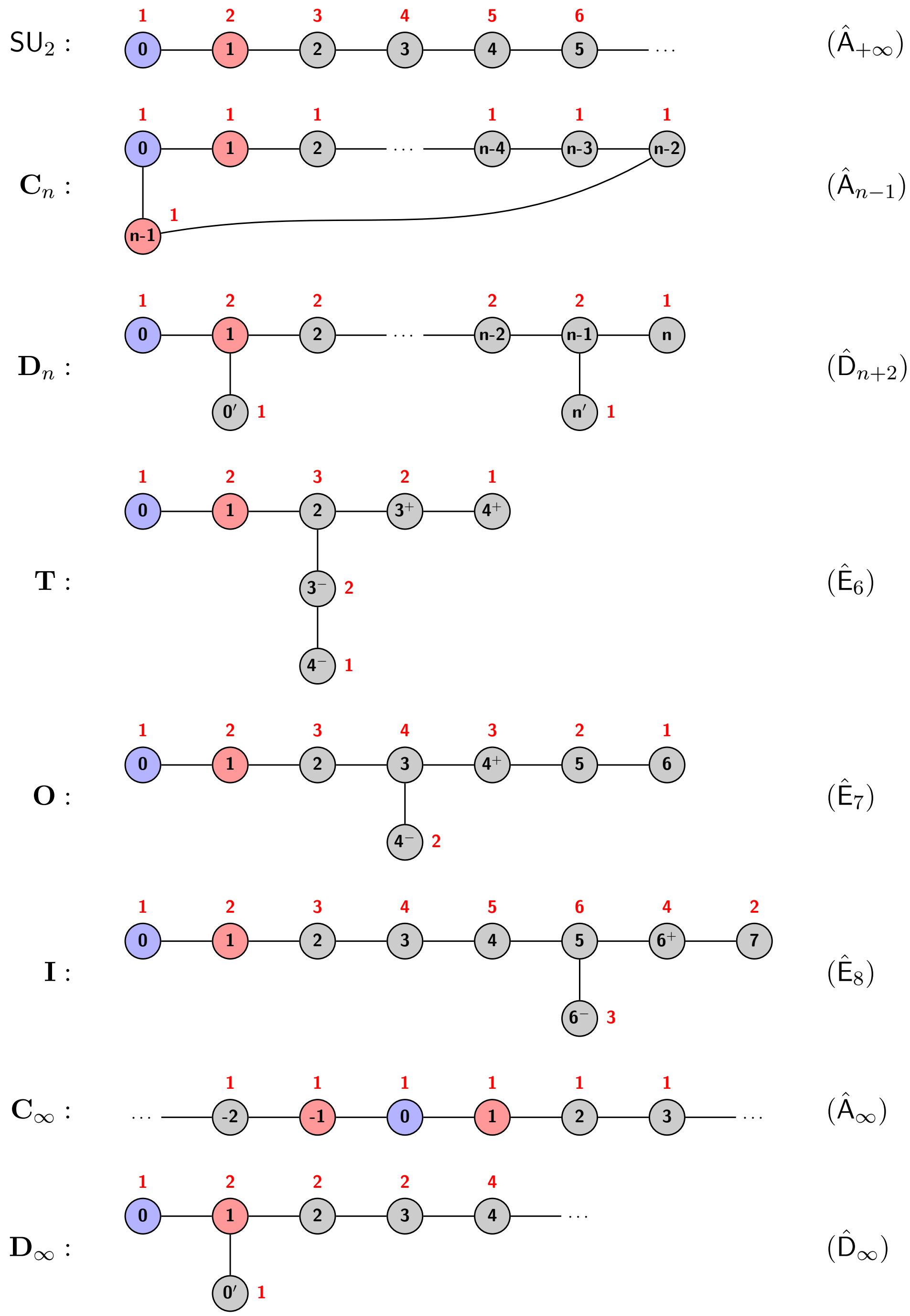
is a basis for $\mathsf{Z}_k(\mathbf{D}_\infty)$. Note that $\mathbb{C}\mathsf{e}^{+}_{\mathsf{r,s}} \oplus \mathbb{C}\mathsf{e}^{-}_{\mathsf{r,s}} = \mathbb{C}\mathsf{e}^{+}_{\mathsf{r,-s}} \oplus \mathbb{C}\mathsf{e}^{-}_{\mathsf{r,-s}}$, so we can assume $\mathsf{s} \succ -\mathsf{s}$. Then sending $\mathsf{e_{r,s}}$ to $\mathsf{X_{R,S}}$ when $|\mathsf{r}| = |\mathsf{s}| < \frac{1}{2}k$, and $\mathsf{e}^{\pm}_{\mathsf{r,s}}$ to $\mathsf{X_{\pm R,\pm S}}$ when $|\mathsf{r}| = |\mathsf{s}| = \frac{1}{2}k$ gives the isomorphism in (e).

In the planar rook algebra P_k, the elements $\mathsf{X_{R,S}}$ with $|\mathsf{R}| = |\mathsf{S}| = \frac{1}{2}k$ comprise a matrix unit basis of a matrix algebra of dimension $\binom{k}{b}^2$ where $b = \frac{1}{2}k$. The subsets $\{\mathsf{X_{R,S}} \mid \mathsf{R} \succ -\mathsf{R}, \mathsf{S} \succ -\mathsf{S}\}$ and $\{\mathsf{X_{-R,-S}} \mid \mathsf{R} \succ -\mathsf{R}, \mathsf{S} \succ -\mathsf{S}\}$ each give a basis of a matrix algebra of dimension $\big(\frac{1}{2}\binom{k}{b}\big)^2$. The sum of those two matrix subalgebras is included in Q_k.

4 Dynkin Diagrams, Bratteli Diagrams, and Dimensions

4.1 Representation Graphs and Dynkin Diagrams

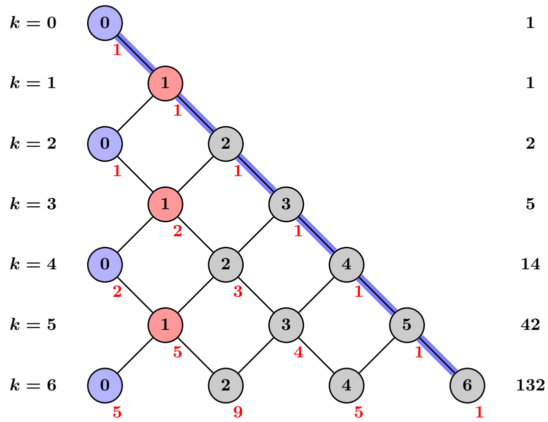
The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ for a finite subgroup G of SU_2 is the corresponding extended affine Dynkin diagram of type $\hat{\mathsf{A}}_{n-1}, \hat{\mathsf{D}}_{n+1}, \hat{\mathsf{E}}_6, \hat{\mathsf{E}}_7, \hat{\mathsf{E}}_8$. In the figures below, the label on the node is the index of the representation, and the label above the node is its dimension. The trivial module is shown in blue and the defining module V is shown in red.



4.2 Bratteli Diagrams

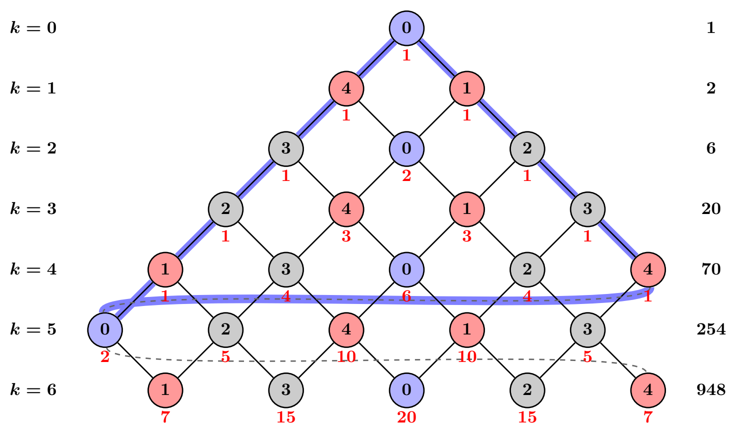
The first few rows of the Bratteli diagrams $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ for finite subgroups G of SU_2 are displayed here. The nodes label the irreducible G-modules that appear in $\mathsf{V}^{\otimes k}$. The label below each node at level k is the multiplicity of the corresponding G-module in $\mathsf{V}^{\otimes k}$, which is also the dimension of the corresponding module for $\mathsf{Z}_k(\mathsf{G})$ with that same label. The right-hand column contains the sum of the squares of these dimensions and equals $\dim \mathsf{Z}_k(\mathsf{G})$. An edge between level k and level $k+1$ is highlighted if it *cannot* be obtained as the reflection, over level k, of an edge between level $k-1$ and k. The non-highlighted edges correspond to the Jones basic construction discussed in Section 1.6. Note that the highlighted edges produce an embedding of the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ into the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$.

SU_2 :

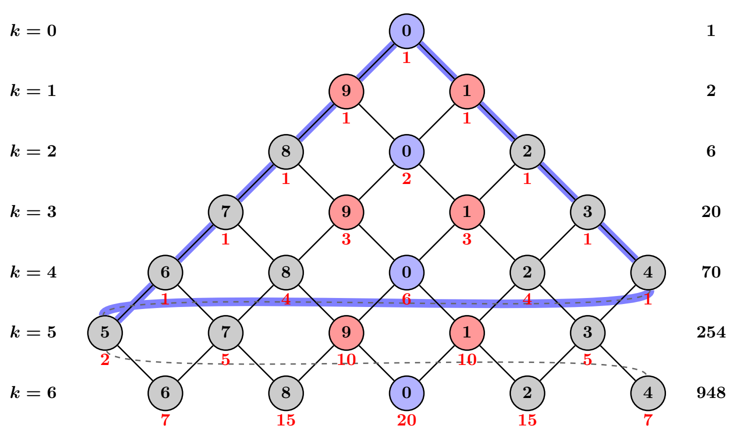


Observe that $\mathbf{C}_5$ and $\mathbf{C}_{10}$ have isomorphic Bratteli diagrams; they each correspond to Pascal's triangle on a cylinder of "diameter" 5:

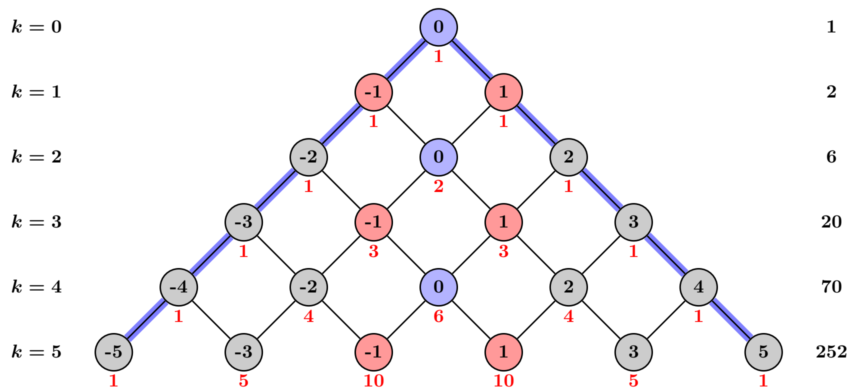
$\mathbf{C}_5$:



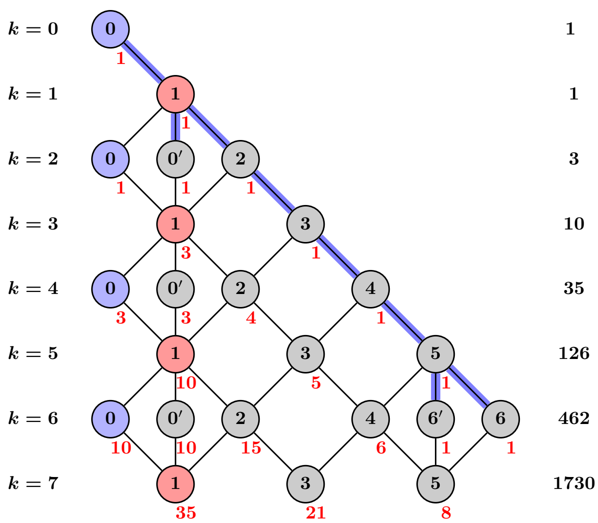
$\mathbf{C}_{10}$:



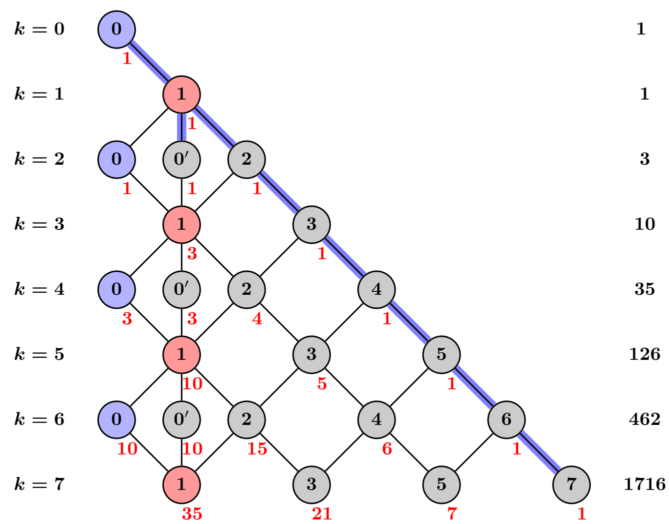
$\mathbf{C}_\infty$:



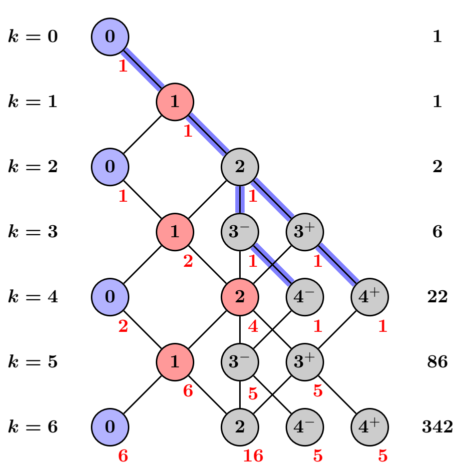
$\mathbf{D}_7$:



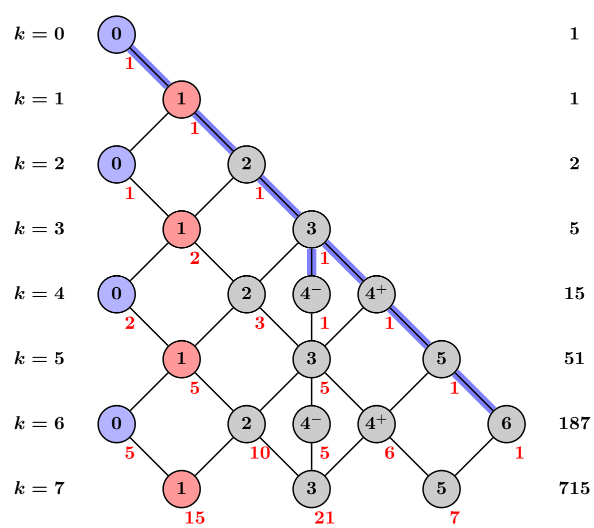
$\mathbf{D}_\infty$:



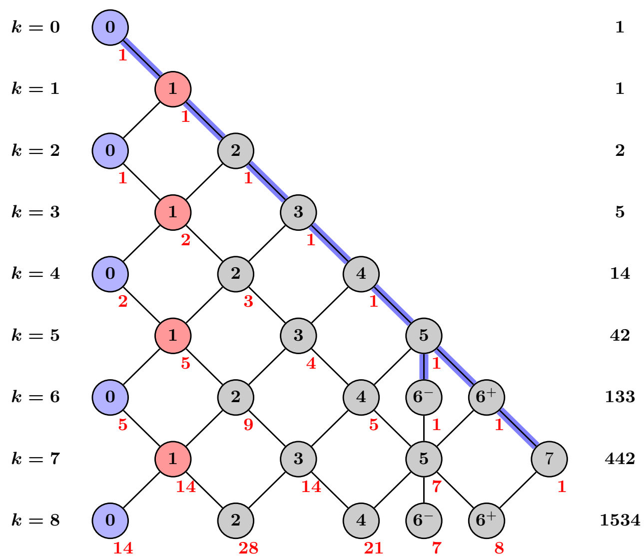
$\mathbf{T}$:



$\mathbf{O}$:



$\mathbf{I}$:

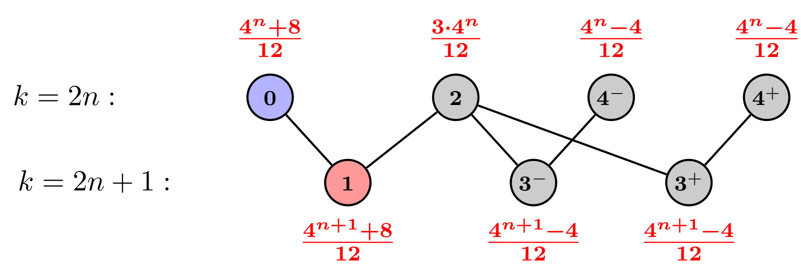


4.3 Dimensions

Using an inductive proof on the structure of the Bratteli diagram, we can compute the dimensions of the irreducible $\mathsf{Z}_k(\mathsf{G})$-modules. The dimension of the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$ is the multiplicity of the trivial G-module (the blue node) at level $2k$. That dimension is also the sum of the squares of the irreducible G-modules occurring in $\mathsf{V}^{\otimes k}$. We record $\dim \mathsf{Z}_k(\mathsf{G})$ for $\mathsf{G} = \mathbf{T}, \mathbf{O}, \mathbf{I}$ in the next result. The cyclic and dihedral cases can be found in Sections 3 and 4.

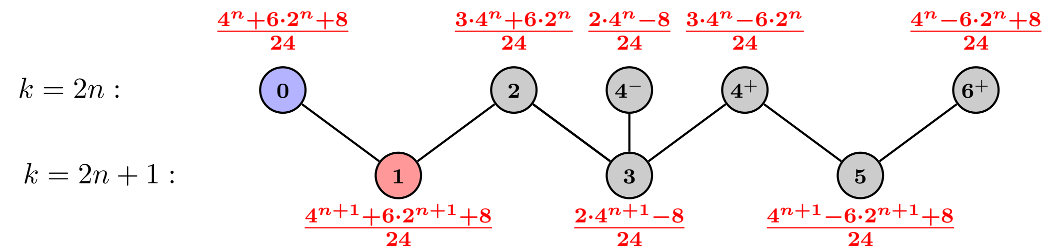
Theorem 4.1. *For $k \geq 1$,*

(a) $\dim \mathsf{Z}_k(\mathbf{T}) = \dfrac{4^k + 8}{12}$ ([OEIS] Sequence A047849). *For $k \geq 2$, the dimensions of the irreducible $\mathsf{Z}_k(\mathbf{T})$-modules, and thus also the multiplicities of the irreducible $\mathbf{T}$-modules in $\mathsf{V}^{\otimes k}$, are as shown in the following diagram:*

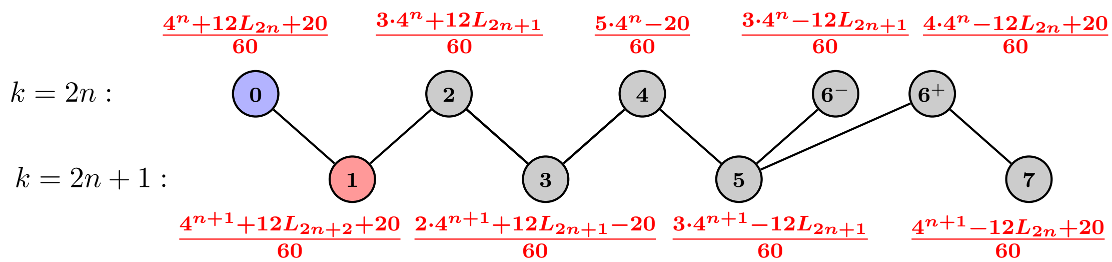


(b) $\dim \mathsf{Z}_k(\mathbf{O}) = \dfrac{4^k + 6 \cdot 2^k + 8}{24}$ ([OEIS] Sequence A007581). *For $k \geq 2$, the dimensions of the irreducible $\mathsf{Z}_k(\mathbf{O})$-modules, and thus also the multiplicities of the irreducible $\mathbf{O}$-modules in*

$\mathsf{V}^{\otimes k}$, *are as shown in the following diagram:*



(c) $\dim \mathsf{Z}_k(\mathbf{I}) = \dfrac{4^k + 12L_{2k} + 20}{60}$, *where* L_n *is the Lucas number defined by* $L_0 = 2, L_1 = 1$, *and* $L_{n+2} = L_{n+1} + L_n$. *For* $k \geq 6$, *the dimensions of the irreducible* $\mathsf{Z}_k(\mathbf{I})$*-modules, and thus also the multiplicities of the irreducible* $\mathbf{I}$*-modules in* $\mathsf{V}^{\otimes k}$, *are as shown in the following diagram:*



Proof. The proofs of the dimension formulas for the irreducible modules are by induction on k. The base cases are given in the Bratteli diagrams in the previous section. The inductive step is given by verifying that each dimension formula at level k (for $k = 2n$ and $k = 2n+1$) equals the sum of the dimension formulas on level $k-1$ connected to the given formula by an edge. Each of these is a straightforward calculation. The fact that $\dim \mathsf{Z}_k(\mathsf{G}) = \dim \mathsf{Z}_{2k}^{(0)}$ follows from (1.17). □

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