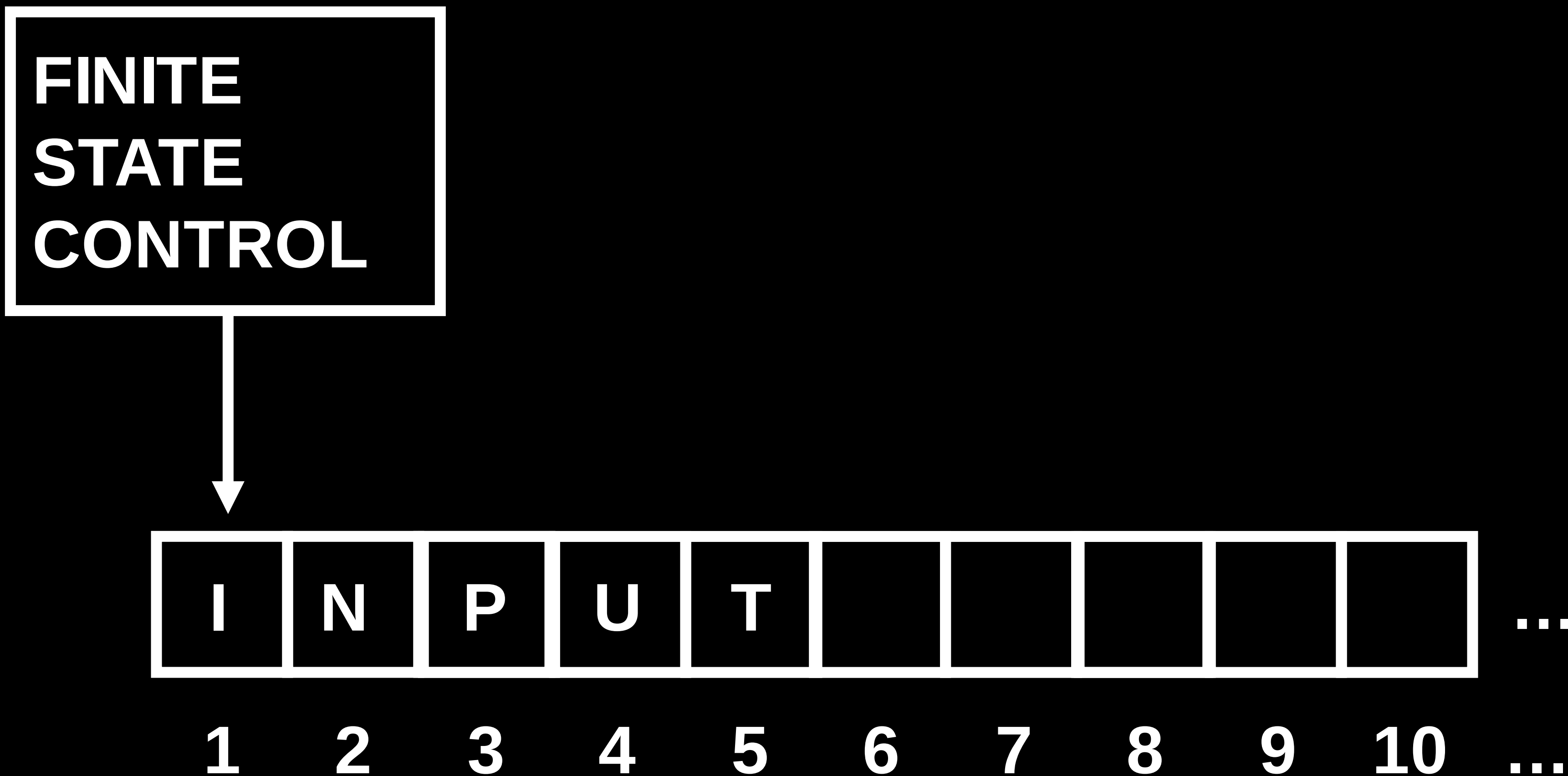
15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

Space Complexity: Savitch's Theorem and PSPACE-Completeness

Tuesday April 15

MEASURING SPACE COMPLEXITY



We measure space complexity by looking at the furthest tape cell reached during the computation

Let M = deterministic TM that halts on all inputs.

Definition: The space complexity of M is the function s : $N \to N$, where s(n) is the furthest tape cell reached by M on any input of length n.

Let N be a non-deterministic TM that halts on all inputs in all of its possible branches.

Definition: The space complexity of N is the function s : $N \to N$, where s(n) is the furthest tape cell reached by M, on any branch if its computation, on any input of length n.

Definition: SPACE(s(n)) =
{ L | L is a language decided by a O(s(n)) space deterministic Turing Machine }

Definition: NSPACE(t(n)) =
{ L | L is a language decided by a O(s(n)) space non-deterministic Turing Machine }

$$\mathbf{PSPACE} = \bigcup_{k \in \mathbb{N}} \mathbf{SPACE}(n^k)$$

$$\mathbf{NPSPACE} = \bigcup_{k \in \mathbb{N}} \mathbf{NSPACE}(n^k)$$

3SAT $\in$ SPACE(n) $\subset$ PSPACE

(	x	$\vee$	$\neg$	y	$\vee$	x	)		(	y	$\vee$	x	$\vee$	y	)						
(	x	$\vee$	$\neg$	y	$\vee$	x	)		(	y	$\vee$	x	$\vee$	y	)		#	x		y	
(	x	$\vee$	$\neg$	y	$\vee$	x	)		(	y	$\vee$	x	$\vee$	y	)		#	x	0	y	0
(	x	$\vee$	$\neg$	y	$\vee$	x	)		(	y	$\vee$	x	$\vee$	y	)		#	x	0	y	1
(	x	$\vee$	$\neg$	y	$\vee$	x	)		(	y	$\vee$	x	$\vee$	y	)		#	x	1	y	0

Assume a deterministic Turing machine that halts on all inputs runs in space s(n)

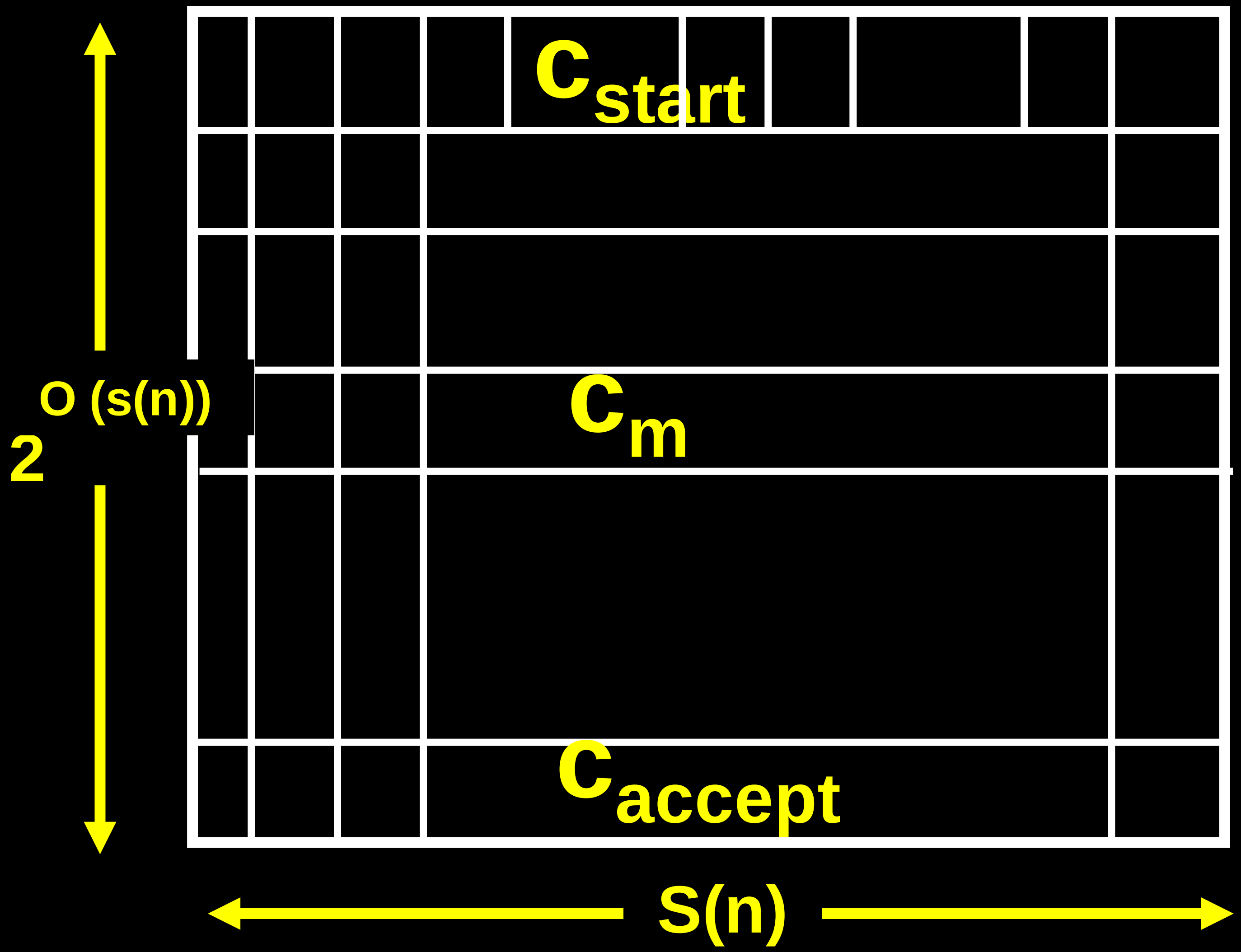
Question: What's an upper bound on the number of time steps for this machine?

A configuration gives a head position, state, and tape contents. Number of configurations is at most:

$$s(n)\ |Q|\ |\Gamma|^{s(n)} = 2^{O(s(n))}$$

Number of Configurations

$$s(n)\,|Q|\,|\Gamma|^{s(n)} = 2^{O(s(n))}$$



MORAL:

Space S computations can be simulated in at most $2^{O(S)}$ time steps

PSPACE $\subseteq$ EXPTIME

$$\text{EXPTIME} = \bigcup_{k \in N} \text{TIME}(2^{n^k})$$

SAVITCH'S THEOREM

Is NTIME(t(n)) $\subseteq$ TIME(t(n))?

Is NTIME(t(n)) $\subseteq$ TIME(t(n)^k) for some k > 1?

We don't know in general!

If the answer is yes, then P = NP…
What about the space-bounded setting?

$$\mathbf{NSPACE(s(n)) \subseteq SPACE(s(n)^2)}$$

$$\mathbf{s(n) \geq n}$$

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If the answer is yes, then P = NP…
What about the space-bounded setting?

therefore PSPACE = NPSPACE

SAVITCH'S THEOREM

Theorem: For functions s(n) where s(n) $\geq$ n

$$\mathrm{NSPACE}(s(n)) \subseteq \mathrm{SPACE}(s(n)^2)$$

Proof Try:

Let N be a non-deterministic TM with space complexity s(n)

Construct a deterministic machine M that tries every possible branch of N

Since each branch of N uses space at most s(n), then M uses space at most s(n) for each branch ..

SAVITCH'S THEOREM

Theorem: For functions s(n) where s(n) ≥ n

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Construct a deterministic machine M that tries every possible branch of N

Since each branch of N uses space at most s(n), then M uses space at most s(n) for each branch ..

There are 2^(O(2^O(s))) branches to keep track of!

We need to simulate a non-deterministic computation and save as much space as possible

IDEA: Given two configurations C_1 and C_2 of an $s(n)$ space machine N, and a number t, determine if N can get from C_1 to C_2 within t steps

Procedure CANYIELD(C_1, C_2, t):

If $t = 0$ then *accept* iff $C_1 = C_2$
If $t = 1$ then *accept* iff C_1 yields C_2 within one step.

Use transition map of N to check [uses space $O(s(n))$]

If $t > 1$, then Accept if and only if

for some configuration C_m of size $s(n)$, both
CANYIELD(C_1,C_m ,$t/2$) and CANYIELD(C_m,C_2, $t/2$) accept

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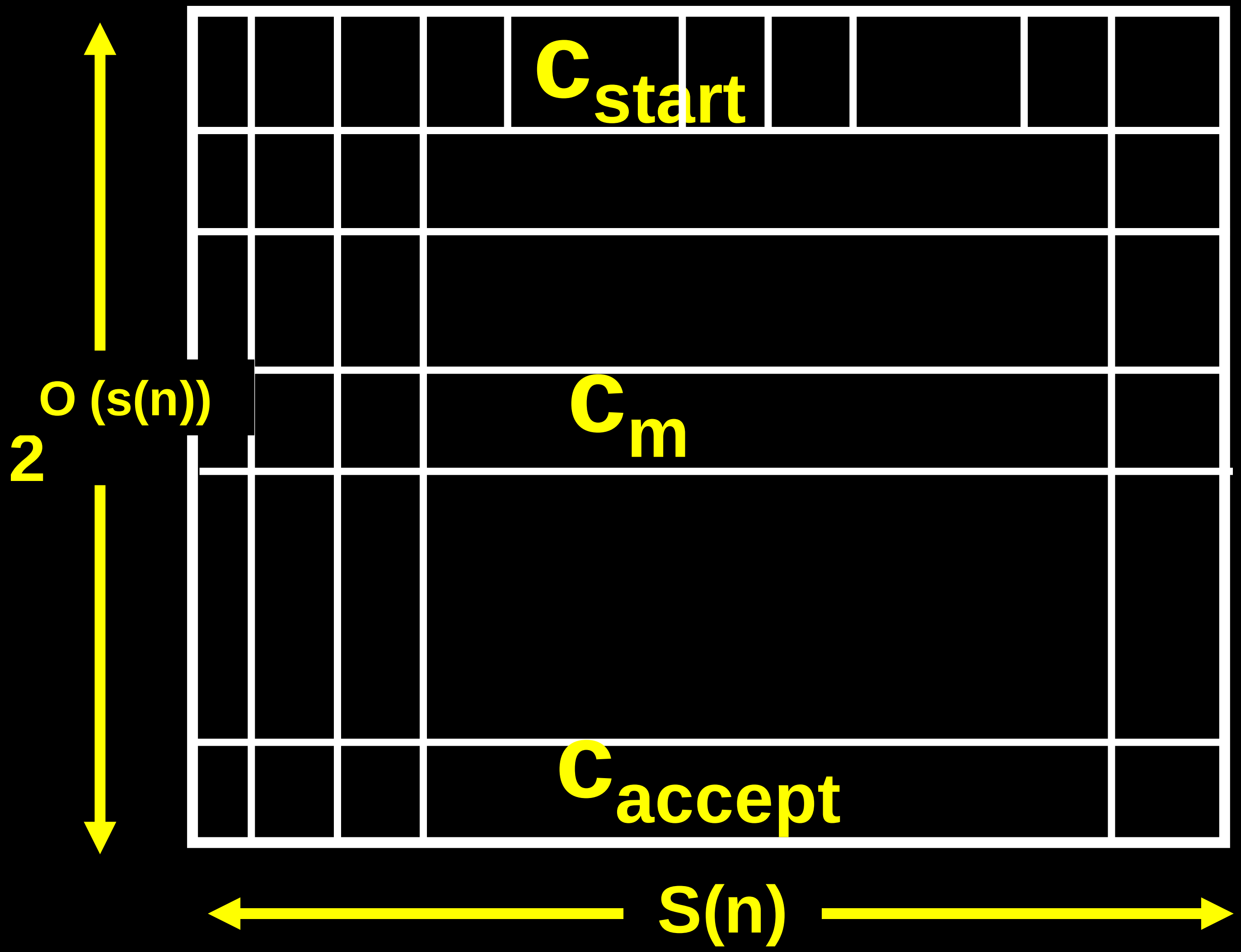
CANYIELD(C_1, C_2, t) has log(t) levels of recursion.

Each level of recursion uses O(s(n)) additional space to store C_m.

So CANYIELD(C_1, C_2, t) uses O(s(n) log(t)) space.

Number of Configurations

$$s(n)\ |Q|\ |\Gamma|^{s(n)} = 2^{O(s(n))}$$



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M: On input w,
Output the result of CANYIELD(c_{start}, c_{accept}, $2^{ds(n)}$)
CANYIELD(C_1, C_2, $2^{ds(n)}$) uses $O(s(n) \log(2^{ds(n)}))$ space.

IDEA: Given two configurations C_1 and C_2 of an s(n) space machine N, and a number t, determine if N can get from C_1 to C_2 within t steps

Procedure CANYIELD(C_1, C_2, t):

If t = 0 then *accept* iff $C_1 = C_2$
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Use transition map of N to check [uses space O(s(n))]

If t > 1, then Accept **if and only if**

for some configuration C_m of size s(n), both CANYIELD(C_1,C_m ,t/2) and CANYIELD(C_m,C_2, t/2) accept

M: On input w,
Output the result of CANYIELD(c_{start}, c_{accept}, $2^{ds(n)}$)
Here d > 0 is chosen so that $2^{ds(|w|)}$ upper bounds the number of configurations of N(w)

Theorem: For a function s where $s(n) \geq n$

$$\boxed{NSPACE(s(n)) \subseteq SPACE(s(n)^2)}$$

Proof:

Let N be a nondeterministic TM using s(n) space

Modify N so that when it accepts, it goes to a special state q_s, clears its tape, and moves its head to the leftmost cell

N has a UNIQUE accepting configuration: $C_{acc} = q_s\square\ldots\square$

Theorem: For a function s where $s(n) \geq n$

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Construct a deterministic M that on input w, runs CANYIELD(C_0, C_{acc}, $2^{ds(|w|)}$)

Here $d > 0$ is chosen so that $2^{d\,s(|w|)}$ upper bounds the number of configurations of N(w)

=> $2^{ds(|w|)}$ is an upper bound on the running time of N(w).

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Why does it take only $s(n)^2$ space?

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Let N be a nondeterministic TM using s(n) space

Modify N so that when it accepts, it goes to a special state q_s, clears its tape, and moves its head to the leftmost cell

N has a UNIQUE accepting configuration: $C_{acc} = q_s\square\ldots\square$

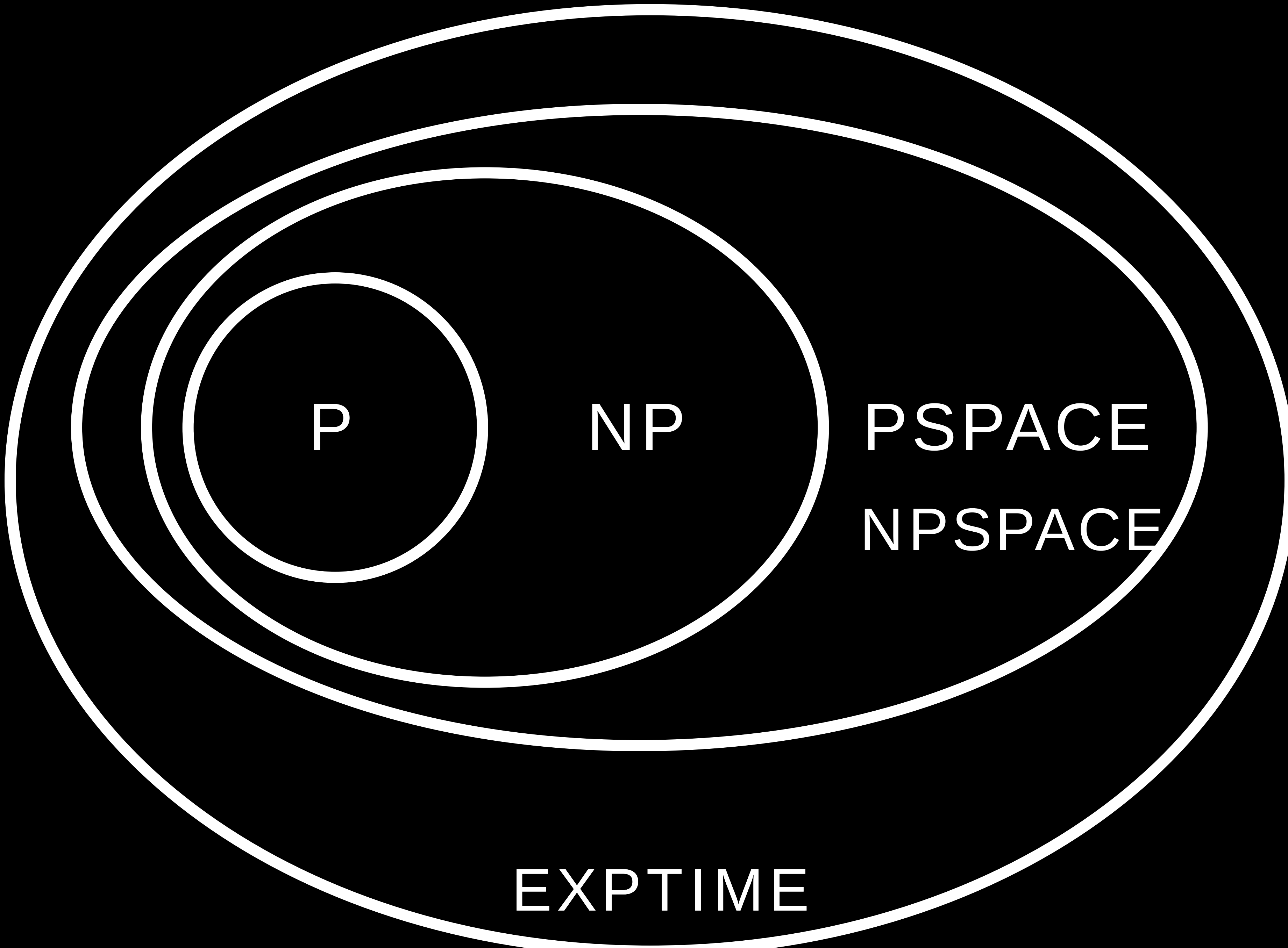
Construct a deterministic M that on input w, runs CANYIELD(C_0, C_{acc}, $2^{ds(|w|)}$)

Uses $\log(2^{d\,s(|w|)})$ recursions. Each level of recursion uses $O(s(n))$ extra space. Therefore uses $O(s(n)^2)$ space!

$$\mathrm{PSPACE} = \bigcup_{k \in N} \mathrm{SPACE}(n^k)$$

$$\mathrm{NPSPACE} = \bigcup_{k \in N} \mathrm{NSPACE}(n^k)$$

PSPACE = NPSPACE


P
NP
PSPACE
NPSPACE
EXPTIME

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXPTIME}$$

$$\mathbf{P} \neq \mathbf{EXPTIME}$$

TIME HIERARCHY THEOREM

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Intuition: **If you have more TIME to work with, then you can solve strictly more problems!**

Theorem: **For functions f, g where $g(n)/(f(n))^2 \to \infty$**

$$\mathrm{TIME}(g(n)) \not\subset \mathrm{TIME}(f(n))$$

So, for all k, since $2^n/n^{2k} \to \infty$,

$$\mathrm{TIME}(2^n) \not\subset \mathrm{TIME}(n^k)$$

Therefore, $\mathrm{TIME}(2^n) \not\subset \mathrm{P}$

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Proof IDEA: Diagonalization

Make a machine M that works in g(n) time and "does the opposite" of all f(n) time machines on at least one input

So L(M) is in TIME(g(n)) but not TIME(f(n))

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Proof IDEA: Diagonalization

Need $g(n) >> f(n)^2$ to ensure that you can simulate an arbitrary machine running in $f(n)$ time with a single machine that runs in $g(n)$ time.

So L(M) is in TIME(g(n)) but not TIME(f(n))

Definition: Language B is PSPACE-complete if:

1. $B \in$ PSPACE
2. Every A in PSPACE is poly-time reducible to B (i.e. B is PSPACE-hard)