16-350
Planning Techniques for Robotics

Search Algorithms:
A Search, Multi-goal A**

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Uninformed A* Search

- Computes g*-values for **relevant** (not all) states

Main function

$g(s_{start}) = 0;$ all other g-values are infinite; $OPEN = \{s_{start}\};$

ComputePath();

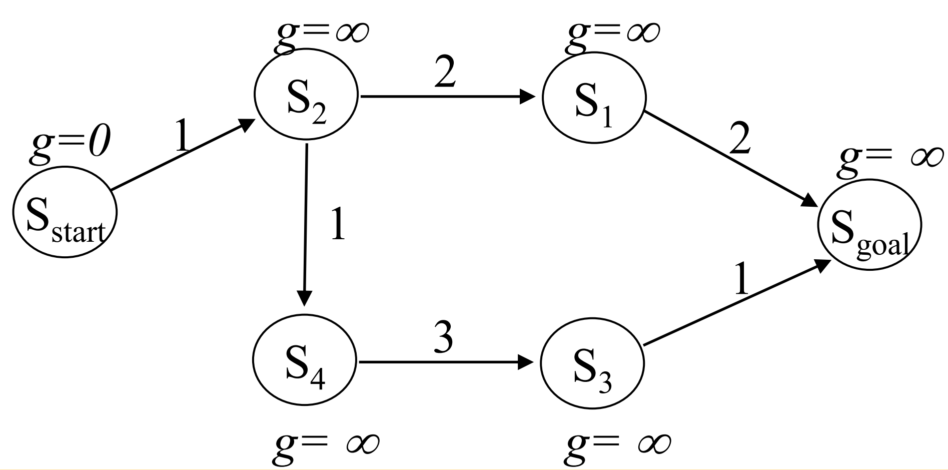
publish solution; //compute least-cost path using g-values

ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

  remove s with the smallest $g(s)$ from $OPEN$;

  expand s;



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for every successor *s'* of *s* such that *s'* not in *CLOSED*

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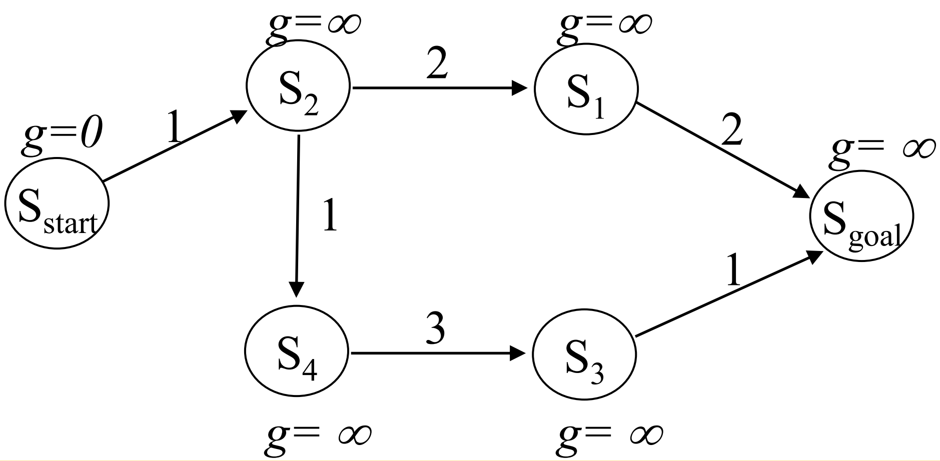
insert *s'* into *OPEN*;



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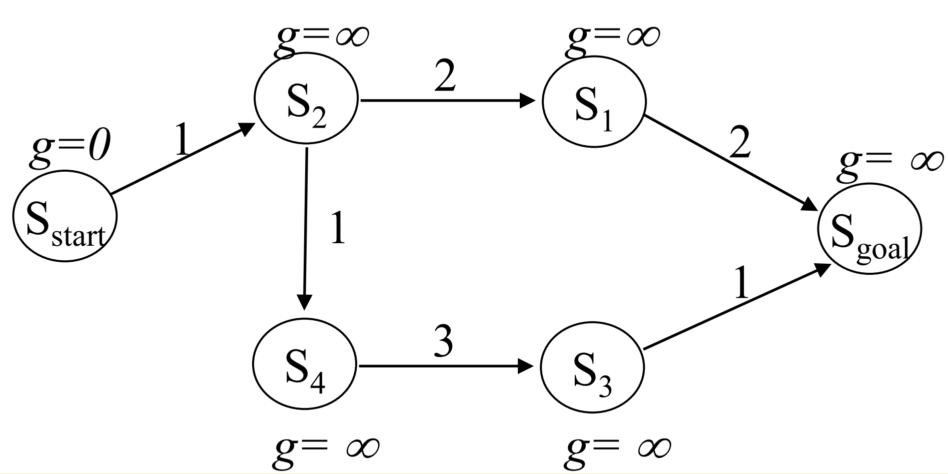
    if $g(s') > g(s) + c(s,s')$
      $g(s') = g(s) + c(s,s')$;
      insert s' into *OPEN*;

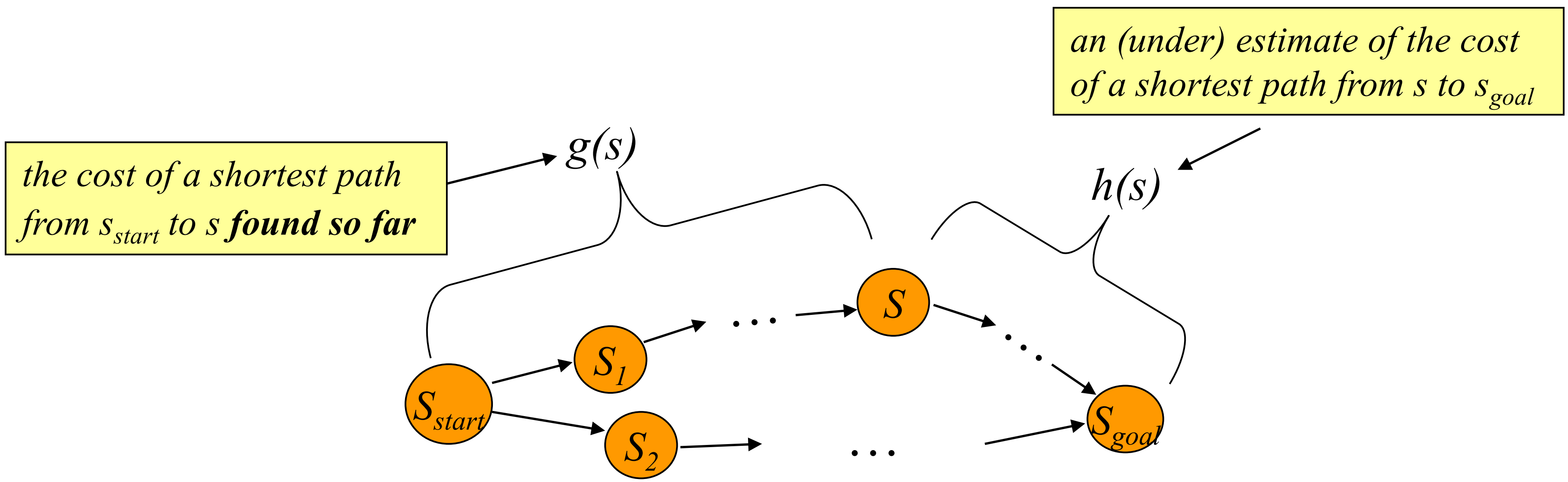
clarification: updates g(s') if s' is already in OPEN



A* Search [Hart, Nillson, Raphael, '68]

- Computes optimal g-values for relevant states

at any point of time:



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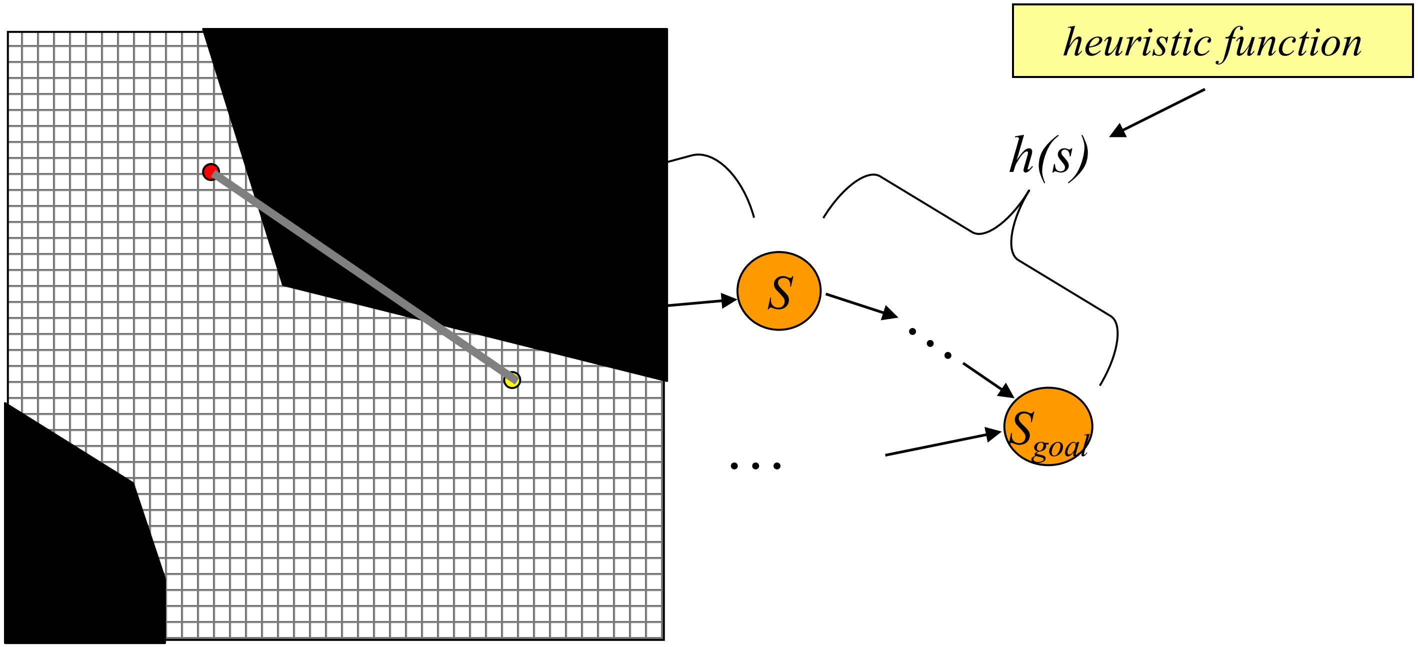


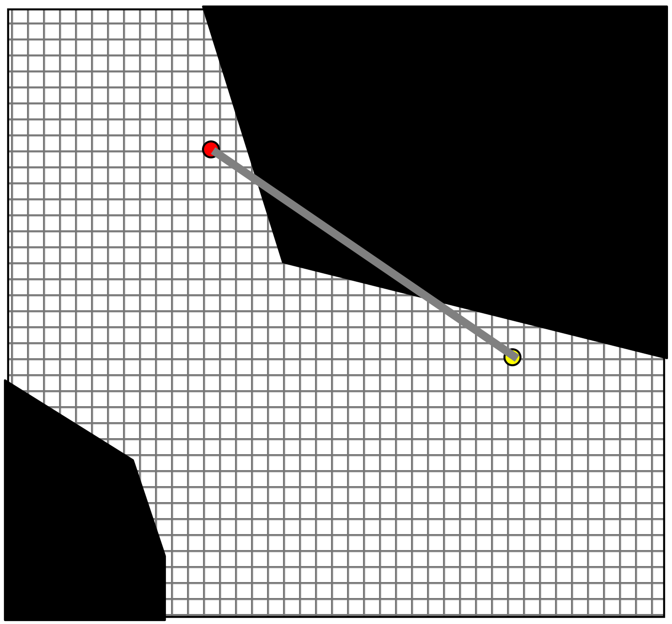
one popular heuristic function – Euclidean distance

A* Search [Hart, Nillson, Raphael, '68]

minimal cost from s to s_{goal}

- Heuristic function must be:
  - admissible: for every state s, $h(s) \leq c^*(s, s_{goal})$
  - consistent (satisfy triangle inequality):

    $h(s_{goal}, s_{goal}) = 0$ *and for every* $s \neq s_{goal}$, $h(s) \leq c(s, succ(s)) + h(succ(s))$
  - admissibility provably follows from consistency and often (not always) consistency follows from admissibility



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  - consistent (satisfy triangle inequality). *Why triangle inequality?*

    $h(s_{goal},s_{goal}) = 0$ and for every $s \neq s_{goal}$, $h(s) \leq c(s,succ(s)) + h(succ(s))$
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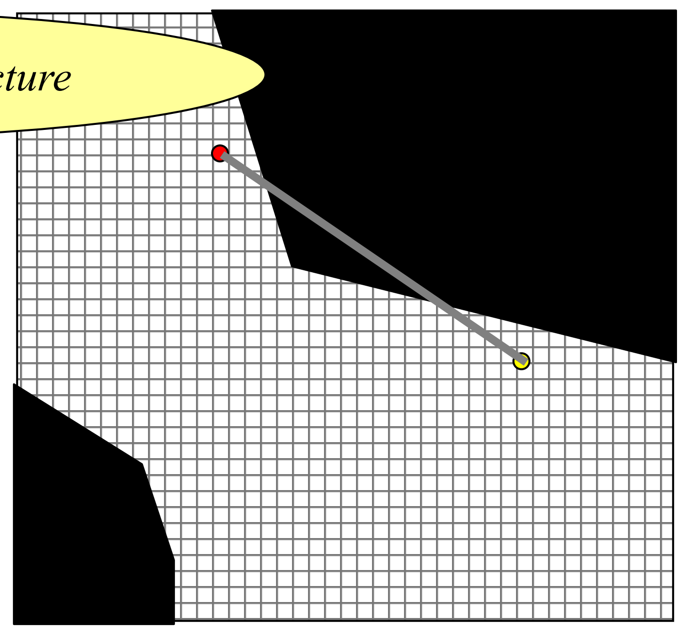
More on this in the later lecture



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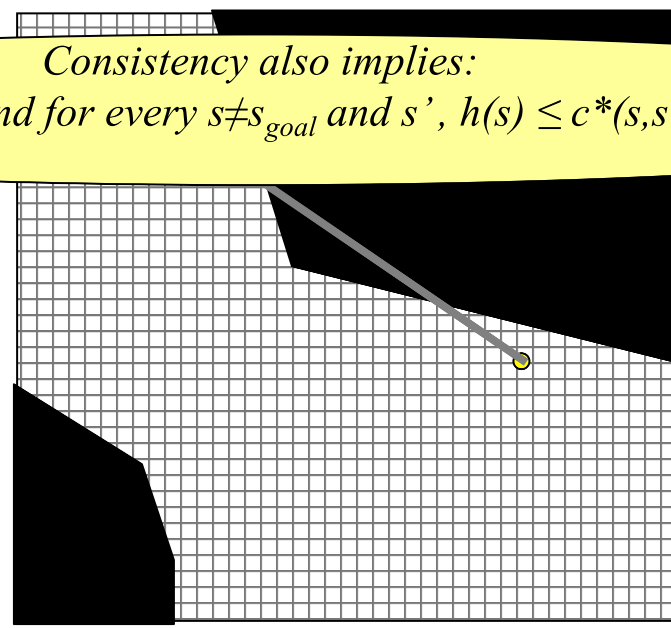

Consistency also implies:
$h(s_{goal}, s_{goal}) = 0$ *and for every* $s \neq s_{goal}$ *and* s', $h(s) \leq c^*(s, s') + h(s')$

A*: Uninformed vs. Informed Search

- A*: expands states in the order of $\boldsymbol{f} = \boldsymbol{g}+\boldsymbol{h}$ values
- Uninformed A*: expands states in the order of $\boldsymbol{g}$ values

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- Computes optimal g-values for relevant states

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ComputePath();

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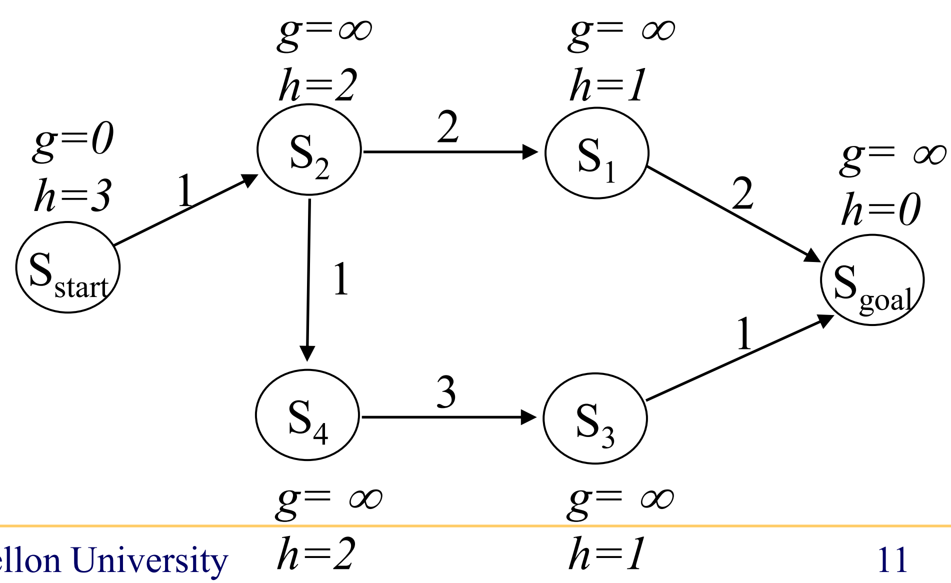

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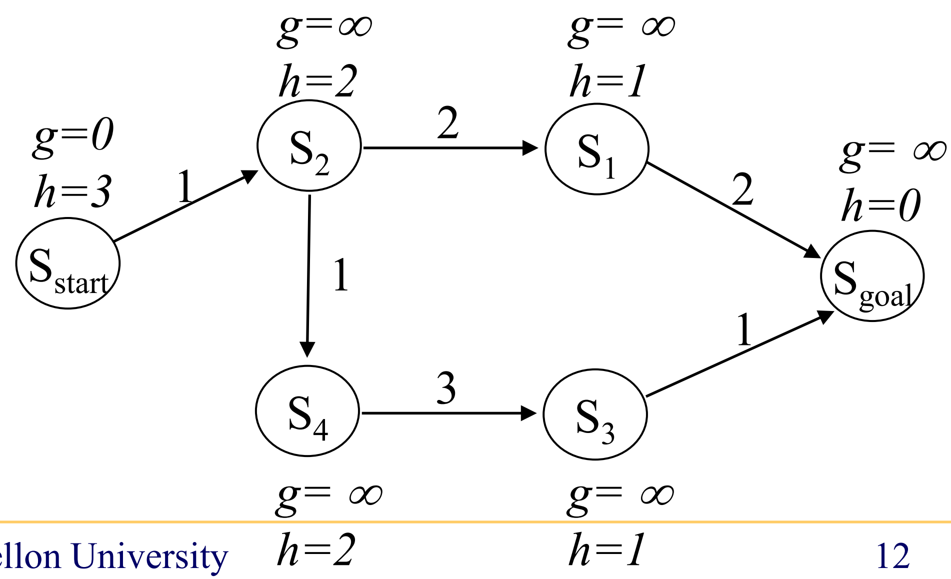

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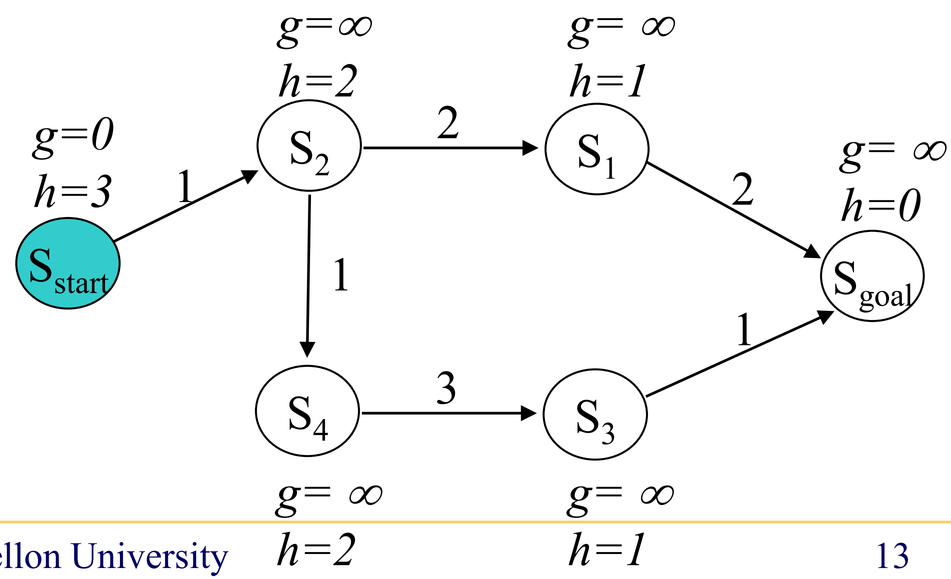
$OPEN = \{s_{start}\}$

next state to expand: s_{start}



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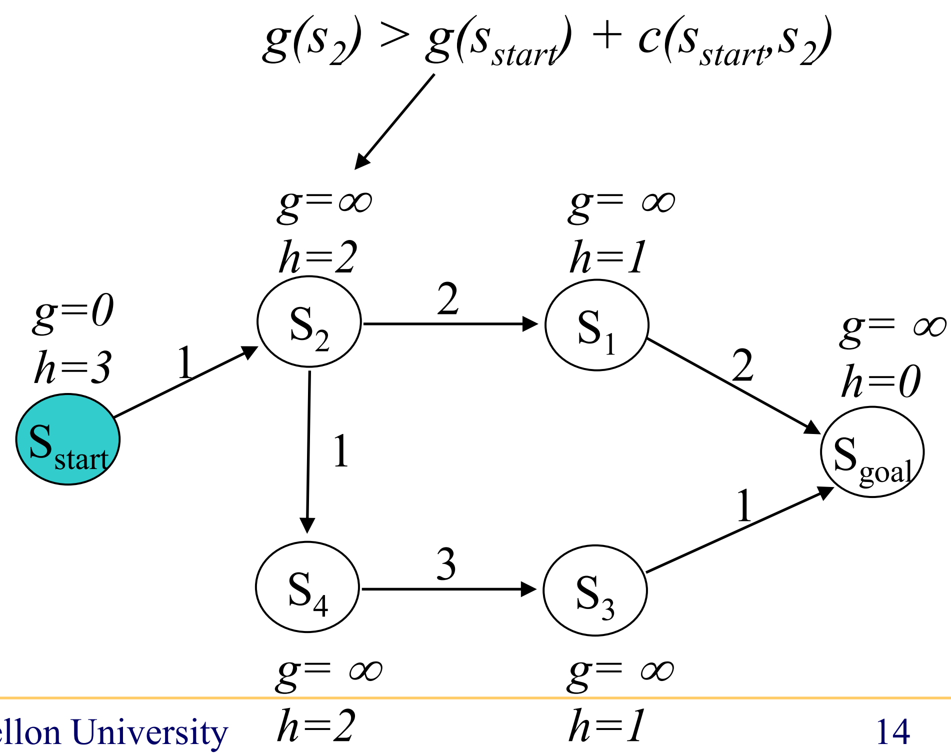

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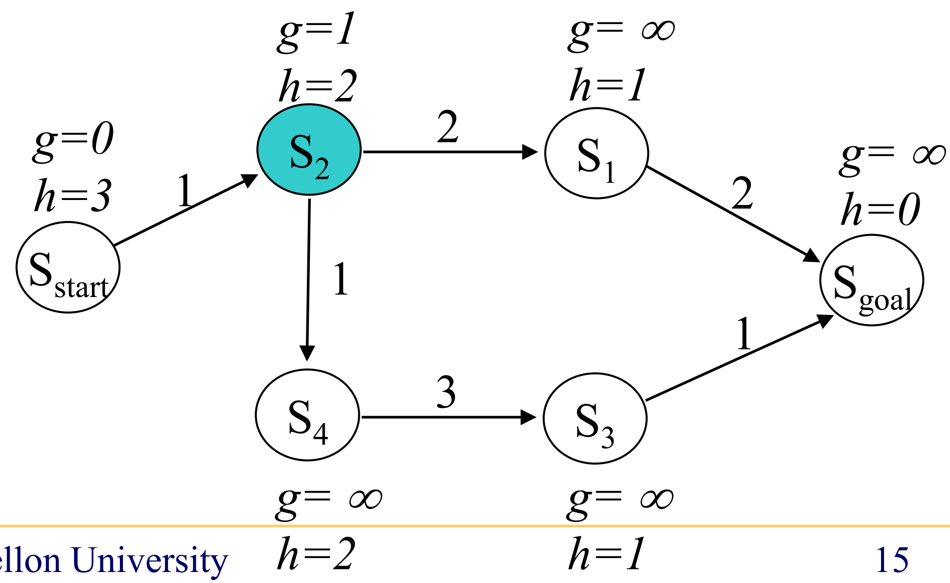

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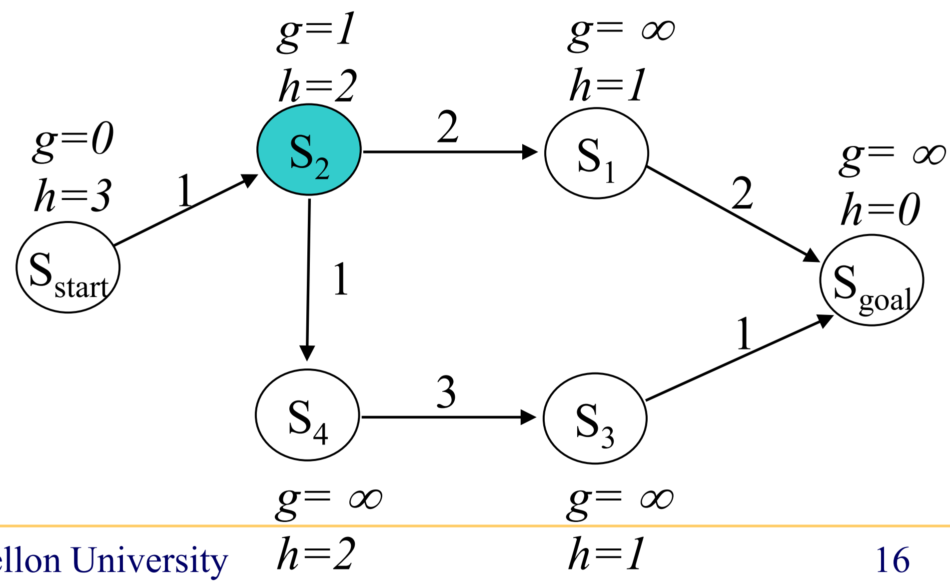
$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

$CLOSED = \{s_{start}\}$

$OPEN = \{s_2\}$

next state to expand: s_2



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if $g(s') > g(s) + c(s,s')$

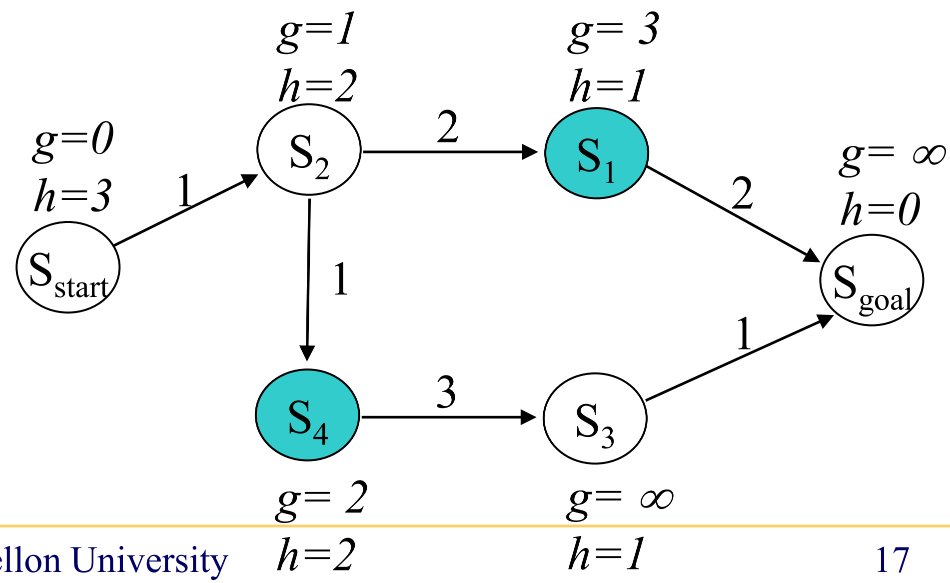
$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2\}$

$OPEN = \{s_1, s_4\}$

next state to expand: s_1



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if $g(s') > g(s) + c(s,s')$

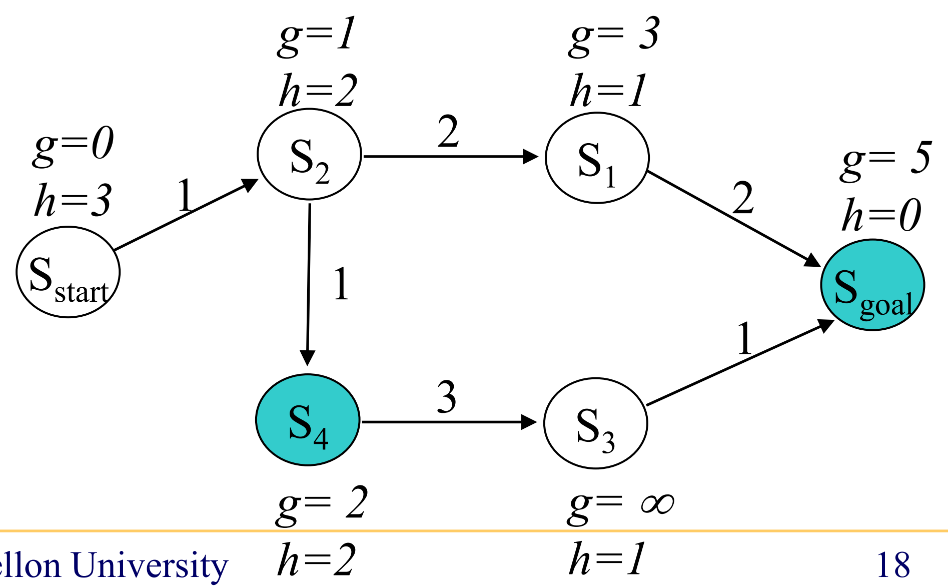
$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2, s_1\}$

$OPEN = \{s_4, s_{goal}\}$

next state to expand: s_4



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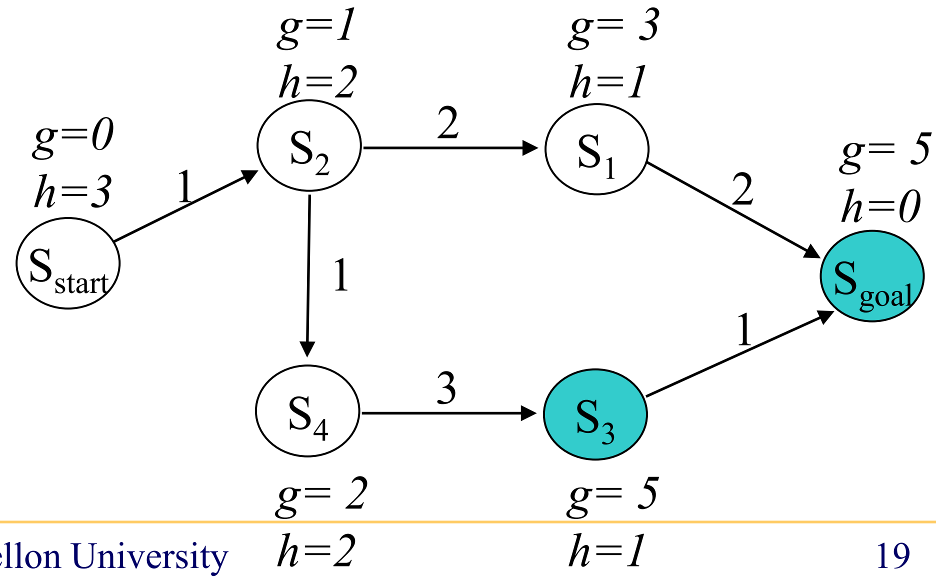
insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2, s_1, s_4\}$

$OPEN = \{s_3, s_{goal}\}$

next state to expand: s_{goal}



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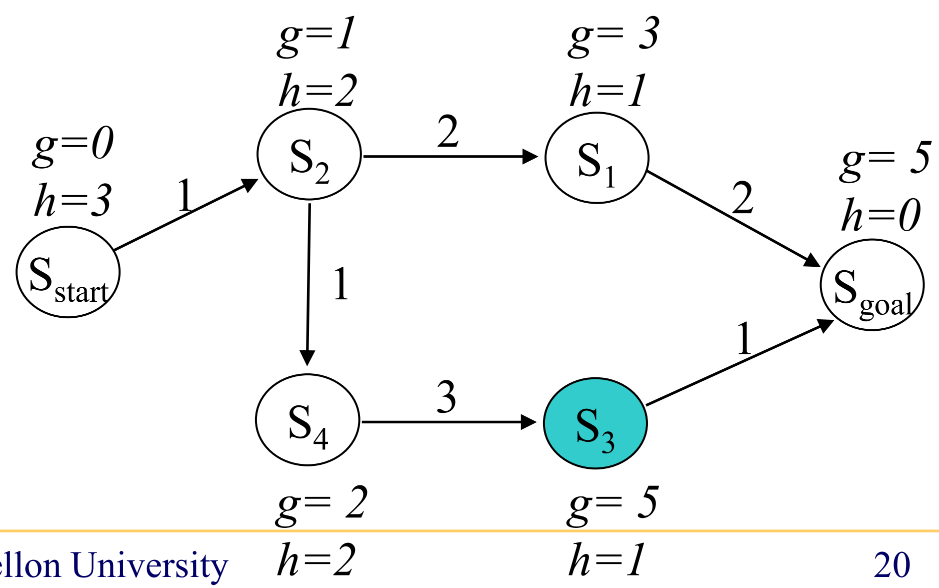
$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

$CLOSED = \{s_{start}, s_2, s_1, s_4, s_{goal}\}$

$OPEN = \{s_3\}$

done



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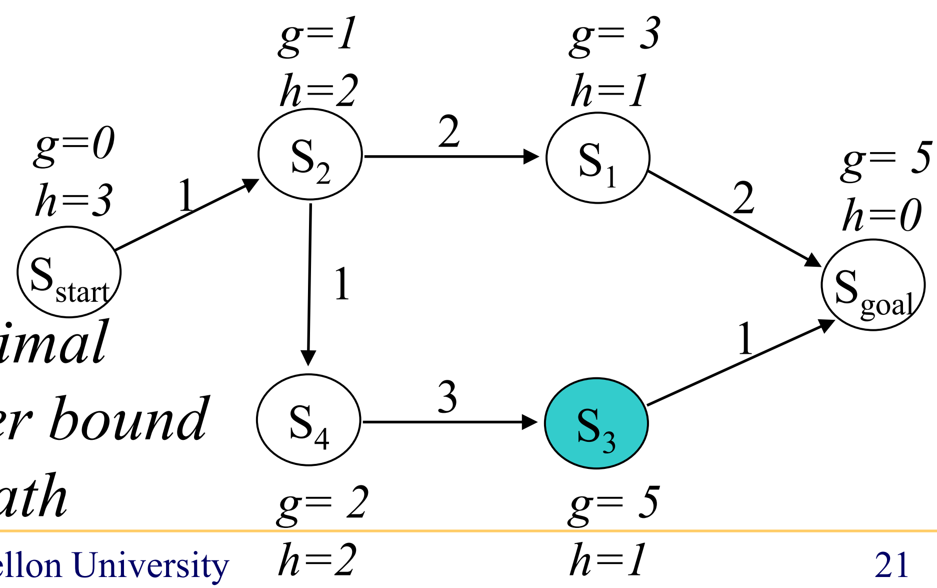

for every expanded state g(s) is optimal

for every other state g(s) is an upper bound

we can now compute a least-cost path

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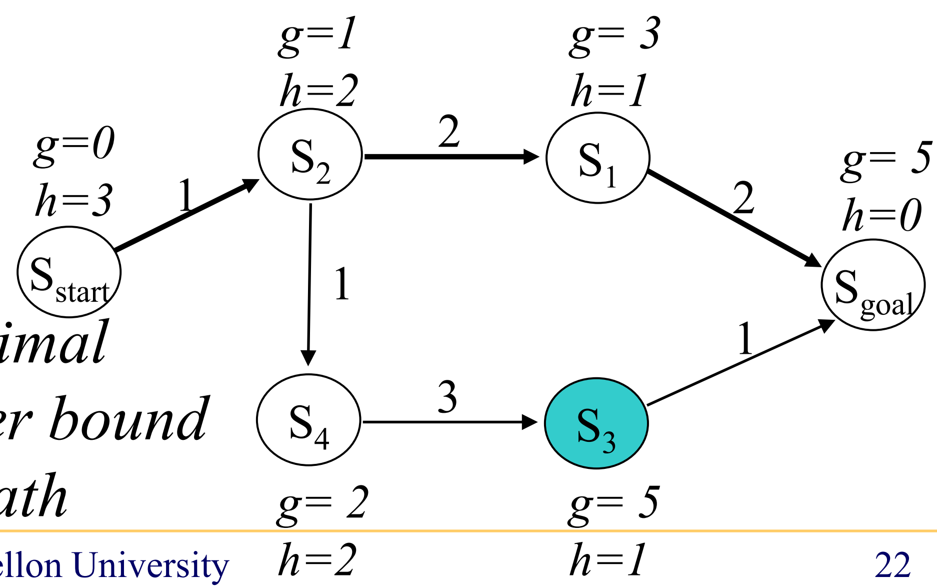


for every expanded state g(s) is optimal

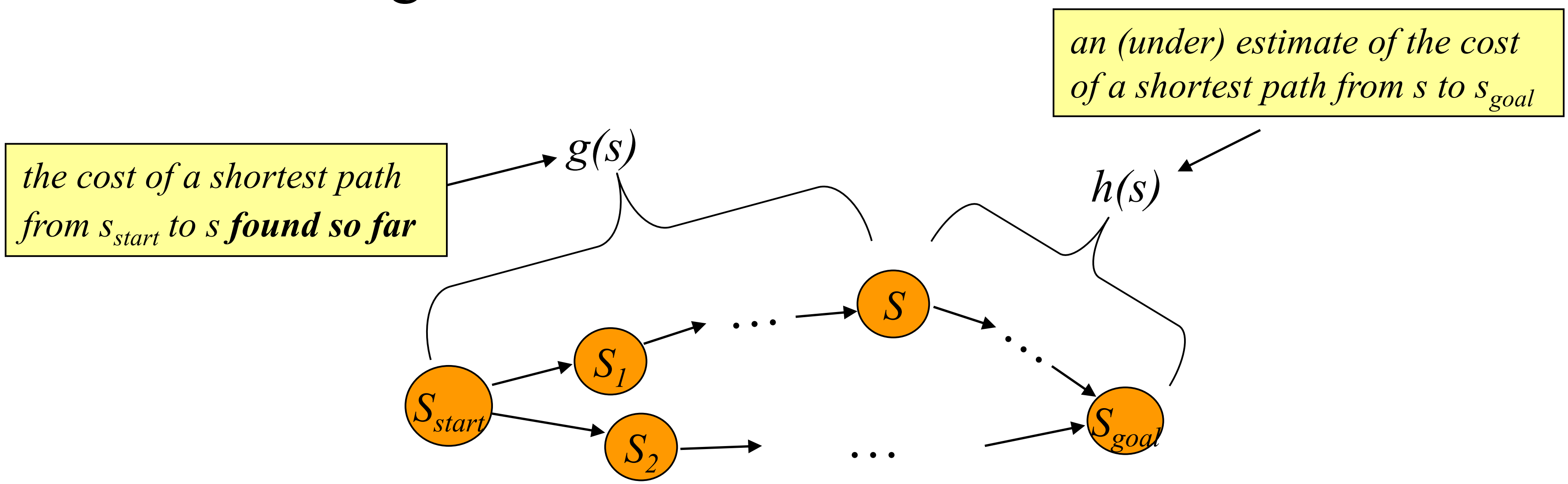
for every other state g(s) is an upper bound

we can now compute a least-cost path

A*: Uninformed vs. Informed Search

- A*: expands states in the order of $\boldsymbol{f} = \boldsymbol{g}+\boldsymbol{h}$ values
- Uninformed A*: expands states in the order of $\boldsymbol{g}$ values

- Intuitively: *f(s)* – estimate of the cost of a least cost path from start to goal via state *s*

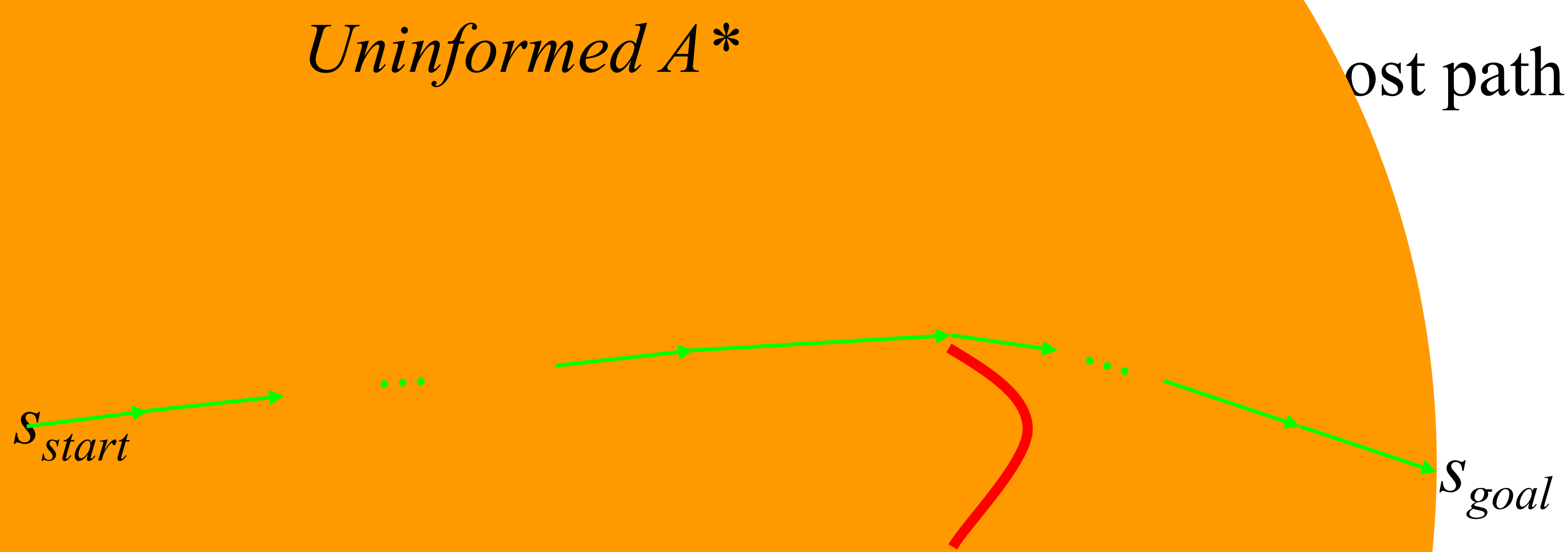


Informed Search

+***h*** values

of ***g*** values

ost path



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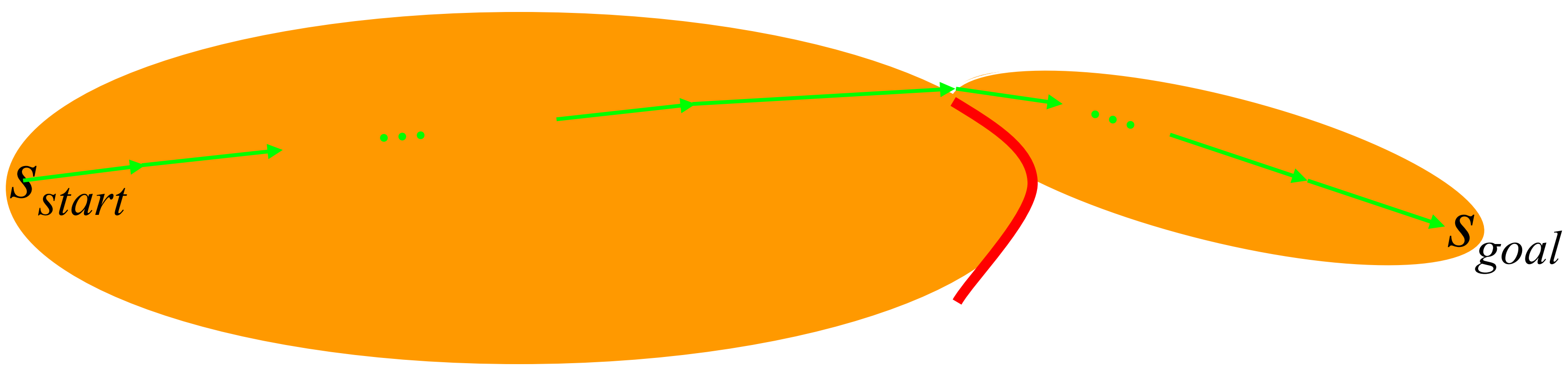


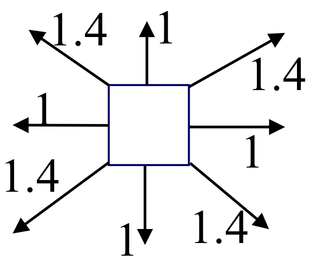
A with Heuristics=Euclidean Distance*

A* Search

- Example on a Grid-based Graph:

$$h(cell\ <x,y>) = max(|x-x_{goal}|, |y-y_{goal}|)$$

8-connected grid



	A	B	C	D	E	F
1	h=5	h=4	h=3	h=2	h=1	h=1
2	h=5	h=4	h=3	h=2	h=1	h=0
3	h=5	h=4			h=1	h=1
4	h=5	h=4	**h=3**	h=2	h=2	h=2

goal

robot

A* Search: Proofs

Theorem 1. For every expanded state *s*, it is guaranteed that $g(s)=g^*(s)$

Sketch of proof by induction:

- *assume all previously expanded states have optimal g-values*
- *next state to expand is s: $f(s) = g(s)+h(s)$ – min among states in OPEN*
- *assume $g(s)$ is suboptimal (we will prove that it is impossible by contradiction)*
- *then there must be at least one state s' on an optimal path from start to s such that it is in OPEN but wasn't expanded*
- $g(s') + h(s') \geq g(s)+h(s)$
- *but* $g(s') + c^*(s',s) < g(s)$ =>
- $g(s') + c^*(s',s) + h(s) < g(s) + h(s)$ => *(from consistency of h-values)*
- $g(s') + h(s') < g(s) + h(s)$ => *CONTRADICTION*
- *thus it must be the case that $g(s)$ is optimal*

A* Search: Proofs

Theorem 2. Once the search terminates, it is guaranteed that $g(s_{goal})=g^*(s_{goal})$

Sketch of proof:

Proof?

A* Search: Proofs

Theorem 3. Once the search terminates, the least-cost path from s_{start} to s_{goal} can be re-constructed by backtracking *(start with s_{goal} and from any state s backtrack to the predecessor state s' such that $s' = \arg\min_{s'' \in pred(s)} (g(s'') + c(s'', s))$)*

Sketch of proof:

- *every backtracking step from state s moves to a predecessor state s' that continues to be on a least-cost path (because all predecessors u not on a least-cost path will have have g(u)+cost(u,s) that are strictly larger than g(s')+cost(s',s))*

A* Search: Proofs

Theorem 4 (complexity). No state is expanded more than once by A*

Sketch of proof:

Proof?

A* Search: Proofs

Theorem 5. Given a graph and a heuristic function, **A* performs a minimal number of expansions to find a <u>provably</u> optimal path** *(provided goal state is always expanded first among the states with the same f-values in OPEN)*

Implementation Details of A* Search

- Computes optimal g-values for relevant states

ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

insert s into $CLOSED$;

for every successor s' of s such that s' not in $CLOSED$

if $g(s') > g(s) + c(s,s')$

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How to implement OPEN?

How to implement CLOSED?

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$g(s') = g(s) + c(s,s')$;

insert s' into $OPEN$;

How to implement OPEN?

Typically, a priority queue built using a binary heap

How to implement CLOSED?

Typically, each state has a Boolean flag indicating if it was already closed

A* Search with Backpointers

- After search terminates, least-cost path is given by backtracking backpointers from s_{goal} to s_{start}

Main function

$g(s_{start}) = 0;$ all other g-values are infinite; $OPEN = \{s_{start}\};$

set all backpointers bp to NULL;

ComputePath();

publish solution; //**backtrack least-cost path using backpointers *bp***

ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

  remove s with the smallest $[f(s) = g(s)+h(s)]$ from $OPEN$;

  insert s into $CLOSED$;

  for every successor s' of s such that s' not in $CLOSED$

    if $g(s') > g(s) + c(s,s')$

      $g(s') = g(s) + c(s,s');$ **bp(s') = s;**

      insert s' into $OPEN$;

Support for Multiple Goal Candidates

- How to compute a least-cost path to any one of the possible goals?
  - Example 1: Computing a least-cost path to a parking spot given multiple parking spaces (some are better, some are worse, some are closer, some are further)
  - Example 2: Catching a moving target whose future trajectory is known (i.e., multiple potential intercept points)
  - Example 3: Mapping/exploration (covered in future lectures)

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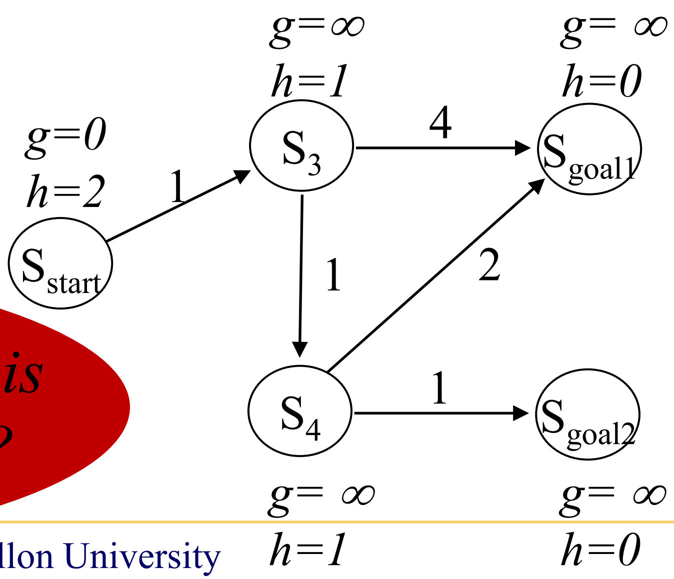

How to find a least-cost path that is lowest across all possible goals?

Introducing "**imaginary**" goal

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ComputePath function

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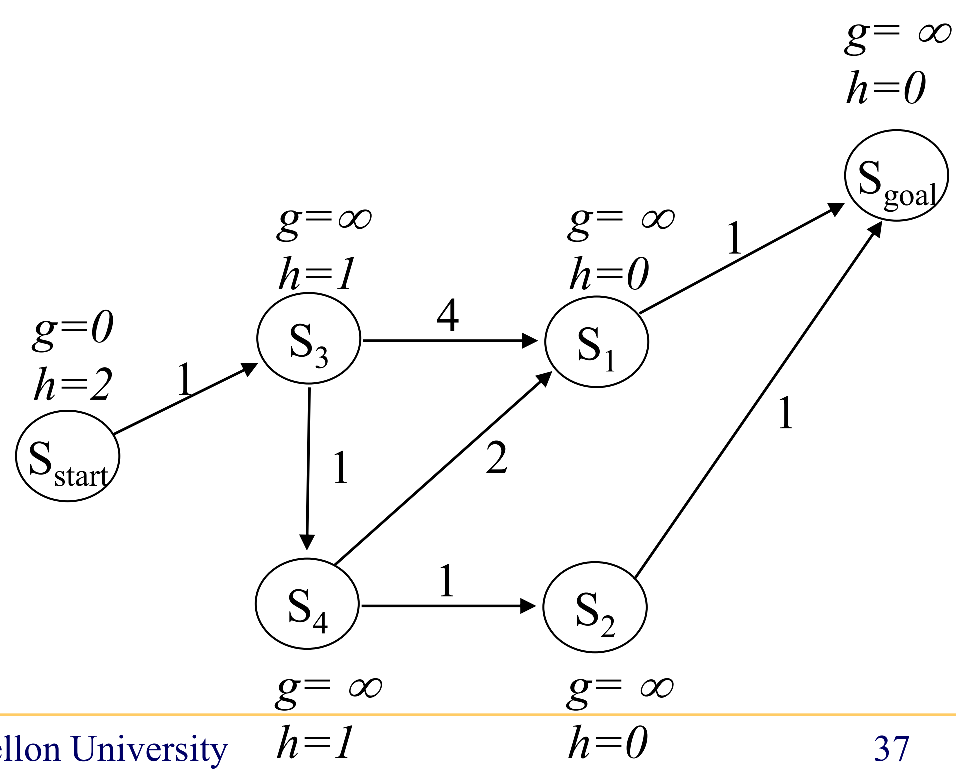
      insert s' into $OPEN$;

Equivalent problem but with a single goal!



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for every successor s' of s such that s' not in $CLOSED$

if $g(s') > g(s) + c(s,s')$

$g(s') = g(s) + c(s,s')$;

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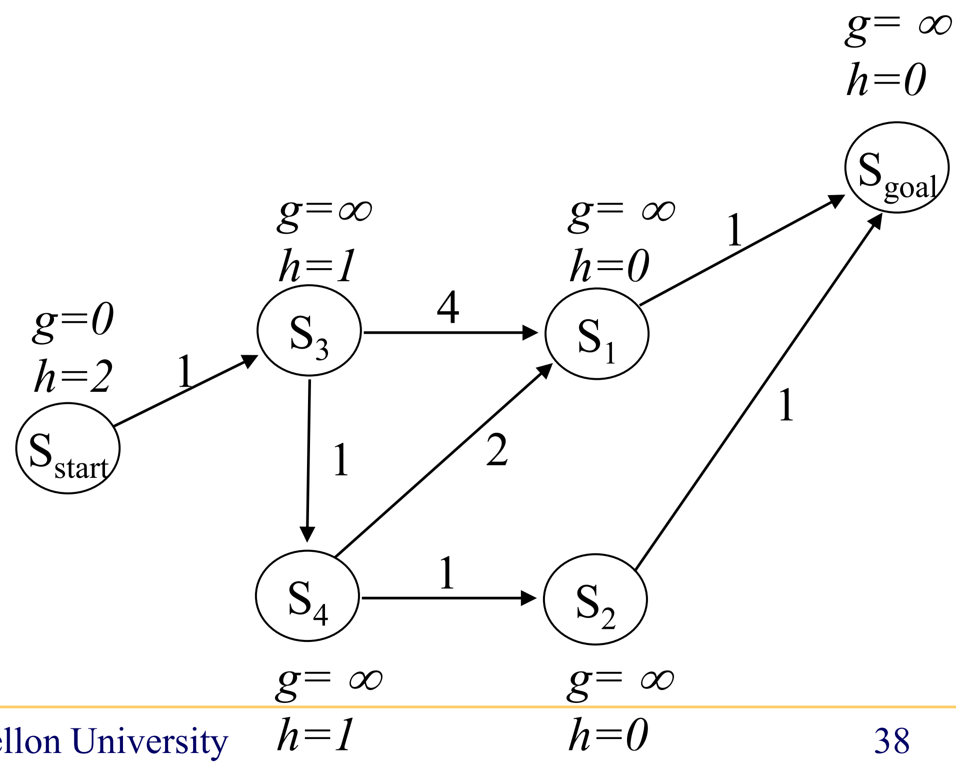
Equivalent problem but with a single goal!

How to prove it?



Support for “unequal” goals

Main function

$g(s_{start}) = 0$; all other g-values are infinite; $OPEN = \{s_{start}\}$;

ComputePath();

publish solution;

ComputePath function

while(s_{goal} is not expanded and $OPEN \neq 0$)

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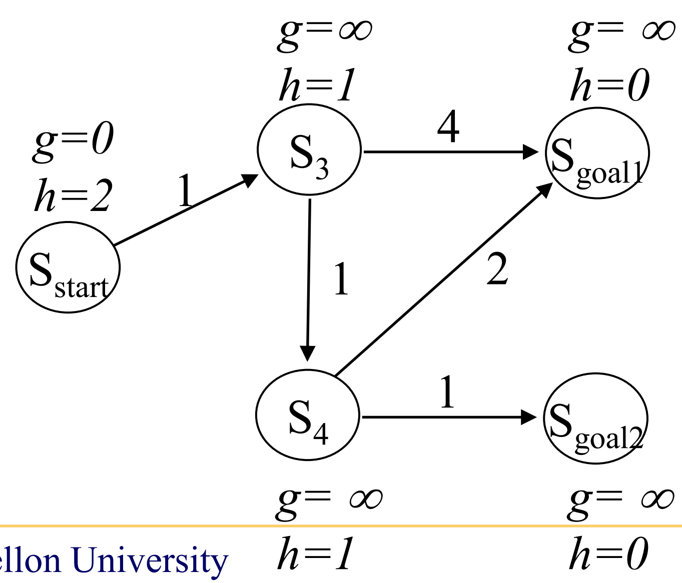
What if some goals are better than others?



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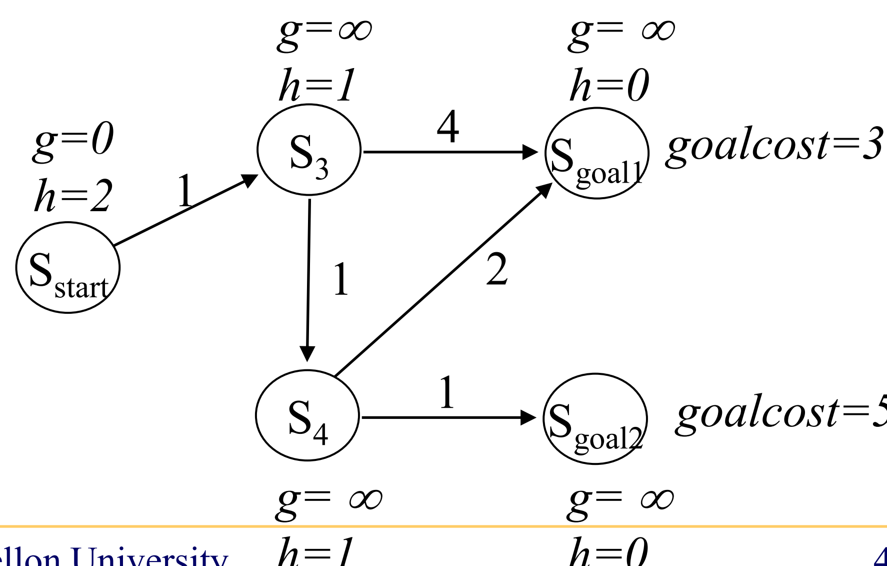

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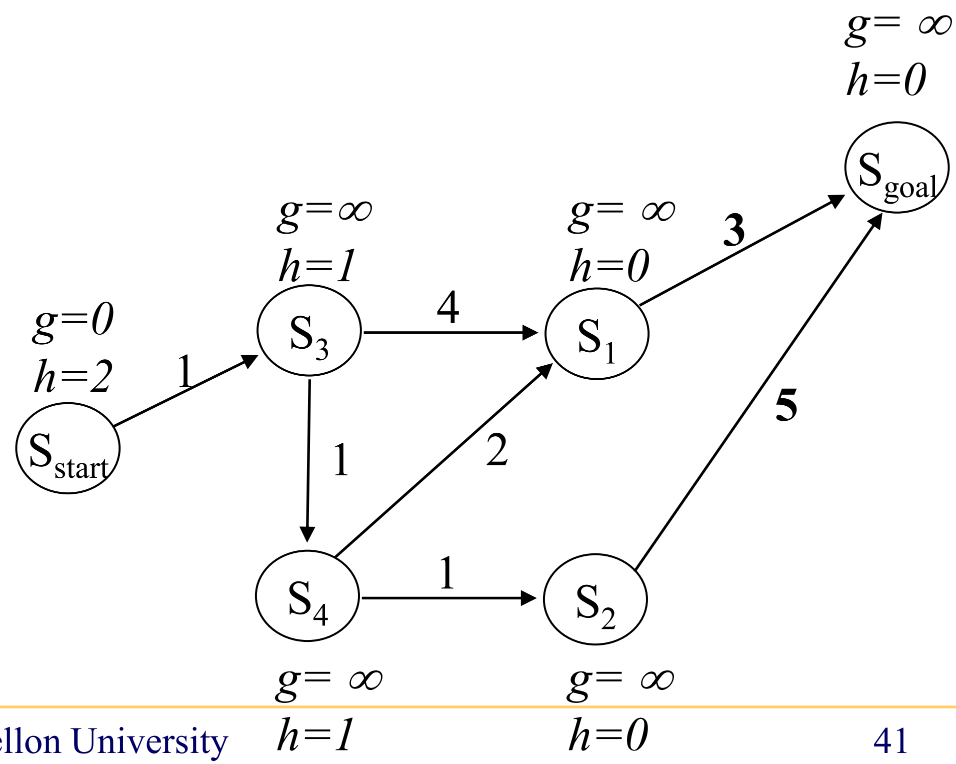

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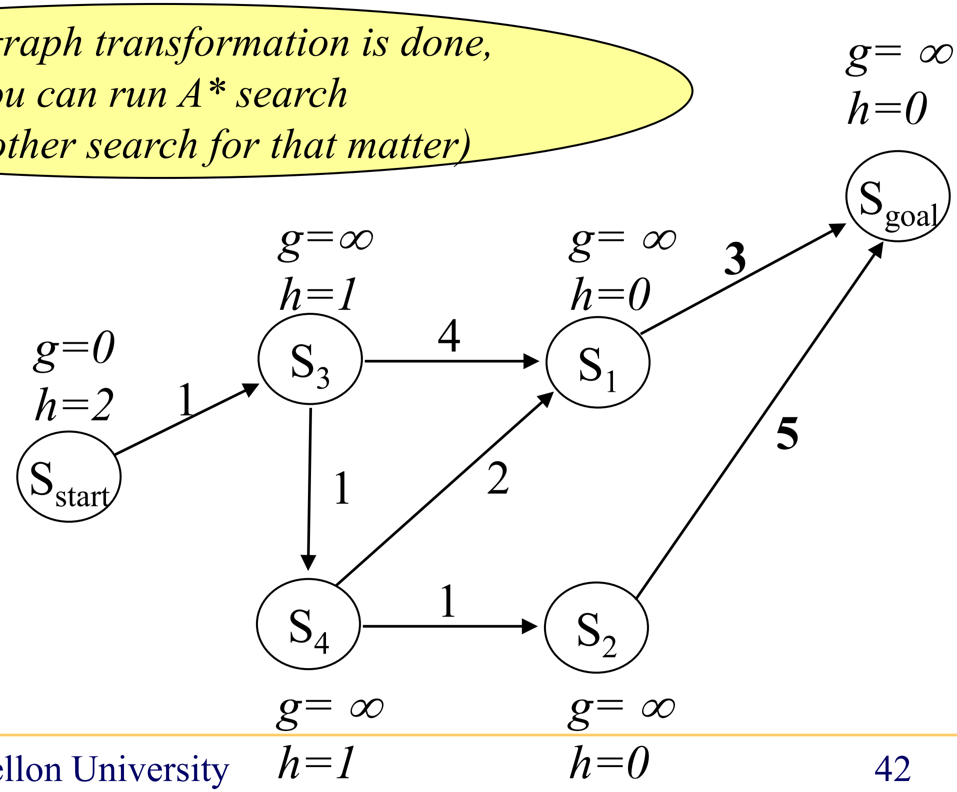
Once the graph transformation is done, you can run A search (or any other search for that matter)*



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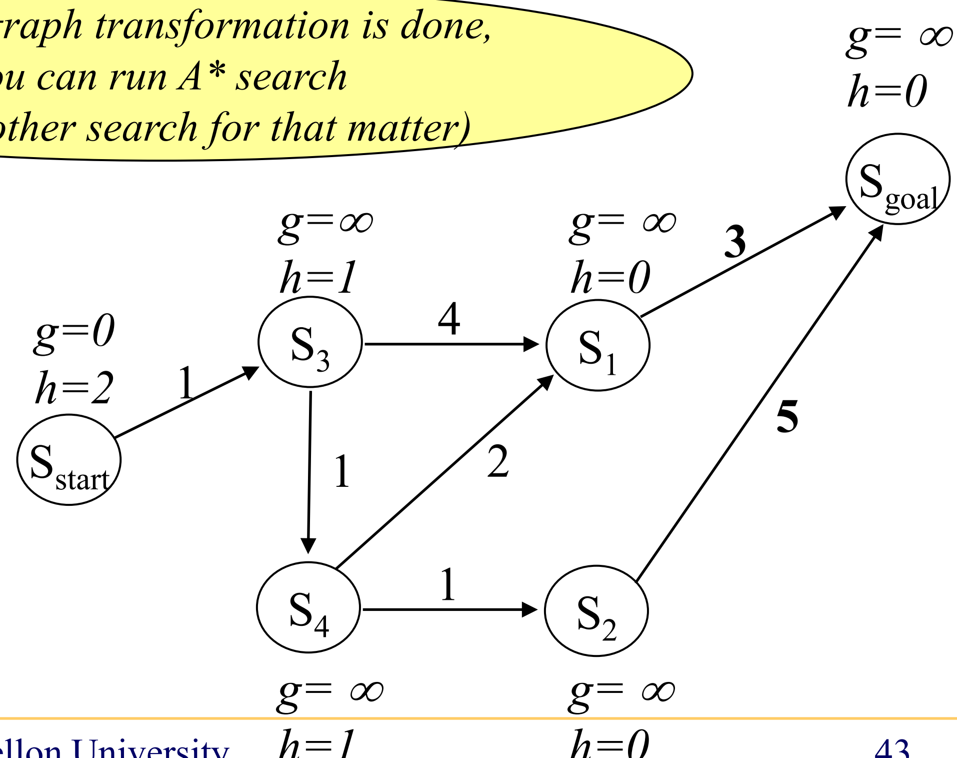
Any impact on how heuristics is computed?

Once the graph transformation is done, you can run A search (or any other search for that matter)*



What You Should Know…

- Operation of A*
- Understand why A* returns an optimal solution (e.g., understand the sketch of proof)
- Theoretical properties of A*
- Properties of heuristics (e.g., admissibility, consistency)
- Multi-goal A*