Recitation 1
Background

10-301/10-601: Introduction to Machine Learning

02/04/2021

1 Probability and Statistics

You should be familiar with event notations for probabilities, i.e. $P(A \cup B)$ and $P(A \cap B)$, where A and B are binary events.

In this class, however, we will mainly be dealing with random variable notations, where A and B are random variables that can take on different states, i.e. a_1, a_2, and b_1, b_2, respectively. Below are some notation equivalents, as well as basic probability rules to keep in mind.

- $P(A = a_1 \cap B = b_1) = P(A = a_1, B = b_1) = p(a_1, b_1)$
- $P(A = a_1 \cup B = b_1) = \sum_{b \in B} p(a_1, b) + \sum_{a \in A} p(a, b_1) - p(a_1, b_1)$
- $p(a_1 \mid b_1) = \frac{p(a_1, b_1)}{p(b_1)}$
- $p(a_1) = \sum_{b \in B} p(a_1, b)$

1. Two random variables, A and B, each can take on two values, a_1, a_2, and b_1, b_2, respectively. a_1 and b_2 are considered disjoint (mutually exclusive). $P(A = a_1) = 0.5$, $P(B = b_2) = 0.5$.
    - What is $p(a_1, b_2)$?
    - What is $p(a_1, b_1)$?
    - What is $p(a_1 \mid b_2)$?

    - $P(A = a_1, B = b_2) = 0$
    - $P(A = a_1, B = b_1) = p(b_1 \mid a_1)p(a_1) = 0.5$ since $p(b_1 \mid a_1) = 1$
    - $P(A = a_1 \mid B = b_2) = 0$
2. Now, instead, a_1 and b_2 are not disjoint, but the two random variables A and B are independent.
    - What is $p(a_1, b_2)$?
    - What is $p(a_1, b_1)$?

- What is $p(a_1 \mid b_2)$?

- $p(a_1, b_2) = 0.25$
- $p(a_1, b_1) = 0.25$ since now $p(b_1 \mid a_1) = 0.5$
- $p(a_1 \mid b_2) = 0.5$

3. A student is looking at her activity tracker (Fitbit/Apple Watch) data and she notices that she seems to sleep better on days that she exercises. They observe the following:

Exercise	Good Sleep	Probability
yes	yes	0.3
yes	no	0.2
no	no	0.4
no	yes	0.1

- What is the $P(GoodSleep = yes \mid Exercise = yes)$?
- Why doesn't $P(GoodSleep = yes, Exercise = yes) = P(GoodSleep = yes) \cdot P(Exercise = yes)$?
- The student merges her activity tracker data with her food logs and finds that the $P(Eatwell = yes \mid Exercise = yes, GoodSleep = yes)$ is 0.25. What is the probability of all three happening on the same day?

- $P(GoodSleep = yes \mid Exercise = yes) = \frac{0.3}{0.3+0.2} = 0.6$
- Good Sleep and Exercise are not independent.
- $P(Eatwell = yes, Exercise = yes, GoodSleep = yes) = P(Eatwell = yes \mid Exercise = yes, GoodSleep = yes) * P(Exercise = yes, GoodSleep = yes) = 0.075$

4. What is the expectation of X where X is a single roll of a fair 6-sided dice ($S = \{1, 2, 3, 4, 5, 6\}$)? What is the variance of X?
E[X] = 3.5
Var[X] = 2.917

For variance, we can do $E[(X - E[X])^2]$ or use the equivalent formulation $E[X^2] - E[X]^2$. In the first method, this gives $\frac{1}{6}\sum_{x \in \{1,2,3,4,5,6\}} (x - 3.5)^2$

5. Imagine that we had a new dice where the sides were $S = \{3, 4, 5, 6, 7, 8\}$. How do the expectation and the variance compare to our original dice?
E[X] = 5.5

Var[X] = 2.917, note $Var[X + a] = Var[X]$ for scalar a

2 Calculus

1. If $f(x) = x^3 e^x$, find $f'(x)$.

   $f'(x) = 3x^2 e^x + x^3 e^x$ by product rule

2. If $f(x) = e^x$, $g(x) = 4x^2 + 2$, find $h'(x)$, where $h(x) = f(g(x))$.

   $h'(x) = 8xe^{4x^2+2}$ by chain rule

3. If $f(x, y) = y \log(1 - x) + (1 - y) \log(x)$, $x \in (0, 1)$, evaluate $\frac{\partial f(x,y)}{\partial x}$ at the point $(\frac{1}{2}, \frac{1}{2})$.

   $\frac{\partial f(x,y)}{\partial x} = -\frac{y}{1-x} + \frac{1-y}{x}$. Therefore, $\frac{\partial f(x,y)}{\partial x}|_{x=\frac{1}{2},y=\frac{1}{2}} = 0$.

4. Find $\frac{\partial}{\partial w_j}\mathbf{x}^T\mathbf{w}$, where $\mathbf{x}$ and $\mathbf{w}$ are M-dimensional real-valued vectors and $1 \leq j \leq M$.

   $\mathbf{x}^T\mathbf{w} = \sum_{i=1}^{M} x_i w_i$. Therefore, $\frac{\partial}{\partial w_j}\mathbf{x}^T\mathbf{w} = x_j$.

3 Vectors, Matrices, and Geometry

1. **Inner Product:** $\mathbf{u} = \begin{bmatrix}6 & 1 & 2\end{bmatrix}^T, \mathbf{v} = \begin{bmatrix}3 & -10 & -2\end{bmatrix}^T$, what is the inner product of $\mathbf{u}$ and $\mathbf{v}$? What is the geometric interpretation?

   The inner product (aka dot product) of the two vectors $\mathbf{u} \cdot \mathbf{v} = 4$. Geometrically, this value is proportional to the projection of $\mathbf{u}$ on $\mathbf{v}$.

2. **Cauchy-Schwarz inequality** (Optional): Given $\mathbf{u} = \begin{bmatrix}3 & 1 & 2\end{bmatrix}^T, \mathbf{v} = \begin{bmatrix}3 & -1 & 4\end{bmatrix}^T$, what is $||\mathbf{u}||_2$ and $||\mathbf{v}||_2$? What is $\mathbf{u} \cdot \mathbf{v}$? How do $\mathbf{u} \cdot \mathbf{v}$ and $||\mathbf{u}||_2||\mathbf{v}||_2$ compare? Is this always true?

   $||\mathbf{u}||_2 = \sqrt{3^2 + 1^2 + 2^2} = 3.74$ and $||\mathbf{v}||_2 = \sqrt{3^2 + (-1)^2 + 4^2} = 5.10$
   $\mathbf{u} \cdot \mathbf{v} = 16$. Since $||\mathbf{u}||_2||\mathbf{v}||_2 = 19.074$, $||\mathbf{u}||_2||\mathbf{v}||_2 > \mathbf{u} \cdot \mathbf{v}$.

In the general case, the Cauchy-Schwarz inequality states that $\forall \mathbf{u}, \mathbf{v} : (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) \geq (\mathbf{u} \cdot \mathbf{v})^2$ where $\cdot$ denotes a valid inner product operation.

3. **Matrix algebra.** Generally, if $\mathbf{A} \in \mathbb{R}^{M \times N}$ and $\mathbf{B} \in \mathbb{R}^{N \times P}$, then $\mathbf{AB} \in \mathbb{R}^{M \times P}$ and $(AB)_{ij} = \sum_k A_{ik} B_{kj}$.

   Given $\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 1 & -1 \\ 3 & -2 & 2 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

   - What is $\mathbf{AB}$? Does $\mathbf{BA} = \mathbf{AB}$? What is $\mathbf{Bu}$?
   - What is rank of $\mathbf{A}$?
   - What is $\mathbf{A}^T$?
   - Calculate $\mathbf{uv}^T$.
   - What are the eigenvalues of $\mathbf{A}$?

   - $\mathbf{AB} = \begin{bmatrix} 21 & -11 & 10 \\ 8 & -2 & 2 \\ 12 & -8 & 8 \end{bmatrix}$, $\mathbf{AB} \neq \mathbf{BA}$, $\mathbf{Bu} = \begin{bmatrix} 8 \\ -2 \\ 9 \end{bmatrix}$
   - Rank of $\mathbf{A} = 3$
   - $\mathbf{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 2 & 4 \end{bmatrix}$
   - $\mathbf{uv}^T = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 15 & 10 & 5 \end{bmatrix}$
   - The eigenvalues of $\mathbf{A}$ are 1, 2 and 4. In general, we find the eigenvalues for square matrices by finding the roots of the matrix's characteristic polynomial.

4. **Geometry:** Given a line $2x + y = 2$ in the two-dimensional plane,

   - If a given point (α, β) satisfies $2\alpha + \beta > 2$, where does it lie relative to the line?
   - What is the relationship of vector $\mathbf{v} = [2, 1]^T$ to this line?
   - What is the distance from origin to this line?

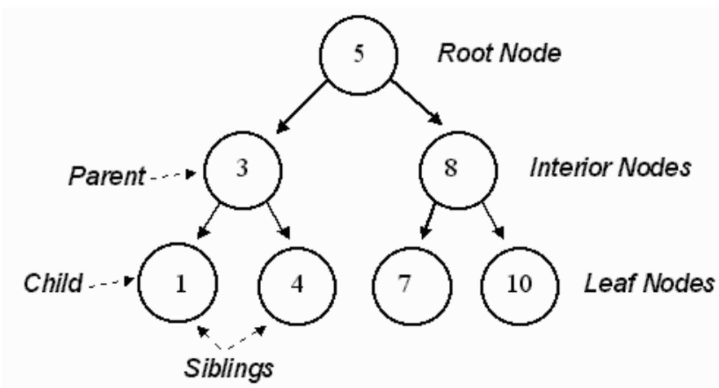
- Above the line.
- This vector is orthogonal to the line.
- The distance is $\frac{2}{\sqrt{5}}$. Generally the distance from a point (α, β) to a line $Ax + By + C = 0$ is given by $\frac{|A\alpha+B\beta+C|}{\sqrt{A^2+B^2}}$.

4 CS Fundamentals

1. For each (f, g) functions below, is $f(n) \in \mathcal{O}(g(n))$ or $g(n) \in \mathcal{O}(f(n))$ or both?
   - $f(n) = \log_2(n)$, $g(n) = \log_3(n)$
   - $f(n) = 2^n$, $g(n) = 3^n$
   - $f(n) = \frac{n}{50}$, $g(n) = \log_{10}(n)$

   - both
   - $f(n) \in \mathcal{O}(g(n))$
   - $g(n) \in \mathcal{O}(f(n))$

2. Find the DFS traversal and BFS traversal of the following binary tree. What are the time complexities of the traversals?



DFS (pre-order): 5, 3, 1, 4, 8, 7, 10
DFS (in-order): 1, 3, 4, 5, 7, 8, 10
DFS (post-order): 1, 4, 3, 7, 10, 8, 5
BFS: 5, 3, 8, 1, 4, 7, 10

Time complexities are all $\mathcal{O}(n)$ where n is the number of nodes in the tree.