Machine Learning Department
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Reinforcement Learning:
Value Iteration & Policy Iteration

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Lecture 21
Nov. 11, 2024

Reminders

- Homework 7: Deep Learning
  - Out: Fri, Nov. 8
  - Due: Sun, Nov. 17 at 11:59pm
- Homework 8: Deep RL
  - Out: Sun, Nov. 17
  - Due: Mon, Nov. 25 at 11:59pm

MARKOV DECISION PROCESSES

RL: Components

From the Environment (i.e. the MDP)

- State space, $\mathcal{S}$
- Action space, $\mathcal{A}$
- Reward function, $R(s, a)$, $R : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$
- Transition probabilities, $p(s' \mid s, a)$
  - Deterministic transitions:

$$p(s' \mid s, a) = \begin{cases} 1 \text{ if } \delta(s, a) = s' \\ 0 \text{ otherwise} \end{cases}$$

where $\delta(s, a)$ is a transition function

Markov Assumption

$$p(s_{t+1} \mid s_t, a_t, \ldots, s_1, a_1) = p(s_{t+1} \mid s_t, a_t)$$

From the Model

- Policy, $\pi : \mathcal{S} \rightarrow \mathcal{A}$
- Value function, $V^{\pi} : \mathcal{S} \rightarrow \mathbb{R}$
  - Measures the expected total payoff of starting in some state s and *executing* policy π

Markov Decision Process (MDP)

- For **supervised learning** the **PAC learning framework** provided assumptions about where our data came from:

$$\mathbf{x} \sim p^*(\cdot) \text{ and } y = c^*(\cdot)$$

- For **reinforcement learning** we assume our data comes from a **Markov decision process** (MDP)

Markov Decision Processes (MDP)

In RL, the source of our data is an MDP:

1. Start in some initial state $s_0 \in \mathcal{S}$
2. For time step t:
   1. Agent observes state $s_t \in \mathcal{S}$
   2. Agent takes action $a_t \in \mathcal{A}$ where $a_t = \pi(s_t)$
   3. Agent receives reward $r_t \in \mathbb{R}$ where $r_t = R(s_t, a_t)$ $R(s_t, a_t, s_{t+1})$
   4. Agent transitions to state $s_{t+1} \in \mathcal{S}$ where $s_{t+1} \sim p(s' \mid s_t, a_t)$
3. Total reward is $\sum_{t=0}^{\infty} \gamma^t r_t$
   – The value γ is the "discount factor", a hyperparameter $0 < \gamma < 1$

- Makes the same Markov assumption we used for n-gram ~~HMMs~~! The next state only depends on the current state and action.
- *Def.*: we **execute** a policy π by taking action $a = \pi(s)$ when in state s

RL: Objective Function

- Goal: Find a policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$ for choosing "good" actions that maximize:

$$\mathbb{E}[\text{total reward}] = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

randomness from trans. probs.

- The above is called the
  "infinite horizon expected future discounted reward"

$$V^{\pi}(s_0) = R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \cdots$$

$s_{t+1} \sim P(s_{t+1} \mid s_t, a_t)$ $\quad a_t = \pi(s_t)$

immediate reward	no discounting	finite horizon	median reward
r_0	$\sum_{t=0}^{\infty} r_t$	$\frac{1}{h} \sum_{t=0}^{h} r_t$	

RL: Optimal Value Function & Policy

- Bellman Equations: (recursive definition of $V^\pi(s)$)

$$V^\pi(s) = R(s,\pi(s)) + \gamma \sum_{s'\in S} p(s' \mid s,\pi(s)) \left[R(s',\pi(s')) + \gamma \sum_{s''\in S} p(s'' \mid s',\pi(s')) \left[R(s'',\pi(s'')) + \gamma \sum_{s'''\in S} p(s''' \mid s'',a'') [\ \ ] \right] \right]$$

- Optimal policy: $= R(s,\pi(s)) + \gamma \sum_{s'\in S} p(s' \mid s,\pi(s)) V^\pi(s')$
  - Given $V^*, \ R(s,a), \ p(s' \mid s,a), \gamma$ we can compute this!

$\pi^*(s) =$ the action that maximizes expected future discounted reward

$$= \underset{a\in A}{\operatorname{argmax}} \ \underbrace{R(s,a)}_{\text{immediate reward}} + \underbrace{\gamma \sum_{s'\in S} p(s' \mid s,a) V^{\pi^*}(s')}_{\text{expected future reward}}$$

s	$V^*(s)$
s_{11}	+90
s_{12}	+81
s_{13}	+72
⋮	
s_{34}	−5
⋮	
s_{54}	7
s_{55}	8

- Optimal value function:

$$= R(s,\pi^*(s)) + \gamma \sum_{s'\in S} p(s' \mid s, \pi^*(s)) V^*(s')$$

$$V^*(s) = V^{\pi^*}(s) = \max_{a\in A} R(s,a) + \gamma \sum_{s'\in S} p(s' \mid s,a) V^*(s')$$

  - System of $|\mathcal{S}|$ equations and $|\mathcal{S}|$ variables (each variable is some $V^*(s)$ for some state s)
  - Can be written without π^*

EXPLORATION VS. EXPLOITATION

MDP Example: Multi-armed bandit

Single state: $|\mathcal{S}| = 1$

Three actions: $\mathcal{A} = \{1, 2, 3\}$

Deterministic transitions

Rewards are stochastic

Bandit 1	Bandit 2	Bandit 3
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???
???	???	???

Exploration vs. Exploitation Tradeoff

- In RL, there is a **tension** between two strategies an agent can follow when interacting with its environment:
  - **Exploration**: the agent takes actions to visit (state, action) pairs it has not seen before, with the hope of uncovering previously unseen high reward states
  - **Exploitation**: the agent takes actions to visit (state, action) pairs it knows to have high reward, with the goal of maximizing reward given its current (possibly limited) knowledge of the environment
- Balancing these two is critical to success in RL!
  - If the agent **only explores**, it performs no better than a random policy
  - If the agent **only exploits**, it will likely never discover an optimal policy
- One approach for trading off between these:
  the ϵ-greedy policy

FIXED POINT ITERATION

Fixed Point Iteration

$$f_1(x_1, \ldots, x_n) = 0$$
$$\vdots$$
$$f_n(x_1, \ldots, x_n) = 0$$

$$x_1 = g_1(x_1, \ldots, x_n)$$
$$\vdots$$
$$x_n = g_n(x_1, \ldots, x_n)$$

$$x_1^{(t+1)} = g_1(x_1^{(t)}, \ldots, x_n^{(t)})$$
$$\vdots$$
$$x_n^{(t+1)} = g_n(x_1^{(t)}, \ldots, x_n^{(t)})$$

- Fixed point iteration is a general tool for solving systems of equations
- Under the right conditions, it will converge

1. Assume we have n equations and n variables, written f(**x**) = 0 where **x** is a vector
2. Rearrange the equations s.t. each variable x_i has one equation where it is isolated on the LHS
3. Initialize the parameters. variable values
4. For i in {1,…,n}, update each parameter and increment *t*:
5. Repeat #5 until convergence

Fixed Point Iteration

$+\cancel{x} \quad +\cancel{x}$

$$\cos(y) - x = 0$$
$$\sin(x) - y = 0$$

$+y \quad +y$

$$x = \cos(y)$$
$$y = \sin(x)$$

$$x^{(t+1)} = \cos(y^{(t)})$$
$$y^{(t+1)} = \sin(x^{(t)})$$

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Fixed Point Iteration

$$\cos(y) - x = 0$$
$$\sin(x) - y = 0$$

$$x = \cos(y)$$
$$y = \sin(x)$$

$$x^{(t+1)} = \cos(y^{(t)})$$
$$y^{(t+1)} = \sin(x^{(t)})$$

We can implement our example in a few lines of code

```
from math import *

def f(x, y):
    eq1 = cos(y) - x
    eq2 = sin(x) - y
    return (eq1, eq2)

def g(x, y):
    x = cos(y)
    y = sin(x)
    return (x, y)

def fpi(x0, y0, n):
    '''Solves the system of equations by fixed point iteration
    starting at x0 and stopping after n iterations. Also
    includes an auxiliary function f to test at each value.'''
    x = x0
    y = y0
    for i in range(n):
        ox, oy = f(x,y)
        print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
        x,y = g(x,y)
    i += 1
    print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
    return x,y

if __name__ == "__main__":
    x,y = fpi(-1, -1, 20)
```

Fixed Point Iteration

```
$ python fixed-point-iteration.py
i= 0 x=-1.0000 y=-1.000 f(x,y)=(1.5403, 0.1585)
i= 1 x=0.5403 y=0.5144 f(x,y)=(0.3303, 0.0000)
i= 2 x=0.8706 y=0.7647 f(x,y)=(-0.1490, 0.0000)
i= 3 x=0.7216 y=0.6606 f(x,y)=(0.0681, 0.0000)
i= 4 x=0.7896 y=0.7101 f(x,y)=(-0.0313, 0.0000)
i= 5 x=0.7583 y=0.6877 f(x,y)=(0.0144, 0.0000)
i= 6 x=0.7727 y=0.6981 f(x,y)=(-0.0066, 0.0000)
i= 7 x=0.7661 y=0.6933 f(x,y)=(0.0031, 0.0000)
i= 8 x=0.7691 y=0.6955 f(x,y)=(-0.0014, 0.0000)
i= 9 x=0.7677 y=0.6945 f(x,y)=(0.0006, 0.0000)
i=10 x=0.7684 y=0.6950 f(x,y)=(-0.0003, 0.0000)
i=11 x=0.7681 y=0.6948 f(x,y)=(0.0001, 0.0000)
i=12 x=0.7682 y=0.6949 f(x,y)=(-0.0001, 0.0000)
i=13 x=0.7681 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=14 x=0.7682 y=0.6948 f(x,y)=(-0.0000, 0.0000)
i=15 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=16 x=0.7682 y=0.6948 f(x,y)=(-0.0000, 0.0000)
i=17 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=18 x=0.7682 y=0.6948 f(x,y)=(-0.0000, 0.0000)
i=19 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=20 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
```

We can implement our example in a few lines of code

```python
from math import *

def f(x, y):
    eq1 = cos(y) - x
    eq2 = sin(x) - y
    return (eq1, eq2)

def g(x, y):
    x = cos(y)
    y = sin(x)
    return (x, y)

def fpi(x0, y0, n):
    '''Solves the system of equations by fixed point iteration
    starting at x0 and stopping after n iterations. Also
    includes an auxiliary function f to test at each value.'''
    x = x0
    y = y0
    for i in range(n):
        ox, oy = f(x,y)
        print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
        x,y = g(x,y)
    i += 1
    print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
    return x,y

if __name__ == "__main__":
    x,y = fpi(-1, -1, 20)
```

VALUE ITERATION

Q1

RL Terminology

Question: Match each term (on the left) to the corresponding statement or definition (on the right)

Terms:

A. a reward function 3.
B. a transition probability 5
C. a policy 2
D. state/action/reward triples 7
E. a value function 1
F. transition function 4
G. an optimal policy 6
H. Matt's favorite statement

↳ #1

Statements:

1. gives the expected future discounted reward of a state
2. maps from states to actions
3. quantifies immediate success of agent
4. is a deterministic map from state/action pairs to states
5. quantifies the likelihood of landing a new state, given a state/action pair
6. is the desired output of an RL algorithm
7. can be influenced by trading off between exploitation/exploration

RL: Optimal Value Function & Policy

- Bellman Equations:

$$V^\pi(s) = \quad R(s,\pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,\pi(s)) V^\pi(s')$$

- Optimal policy:
  - Given V^*, $R(s,a)$, $p(s' \mid s,a), \gamma$ we can compute this!

$$\pi^*(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}} \underbrace{R(s,a)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^*(s')}_{\text{(Discounted) Future reward}}$$

- Optimal value function:

$$V^*(s) = \max_{a \in \mathcal{A}} R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^*(s')$$

  - System of $|\mathcal{S}|$ equations and $|\mathcal{S}|$ variables (each variable is some $V^*(s)$ for some state s)
  - Can be written without π^*

Example: Path Planning

Value Iteration

Algorithm:

① Initialize $V(s)=0 \ \forall s$ (or random)

② while not converged:

For $s \in S$:

For stochastic trans. $\rightarrow \left[V(s) = \max_{a \in A} R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a)\, V(s') \right.$

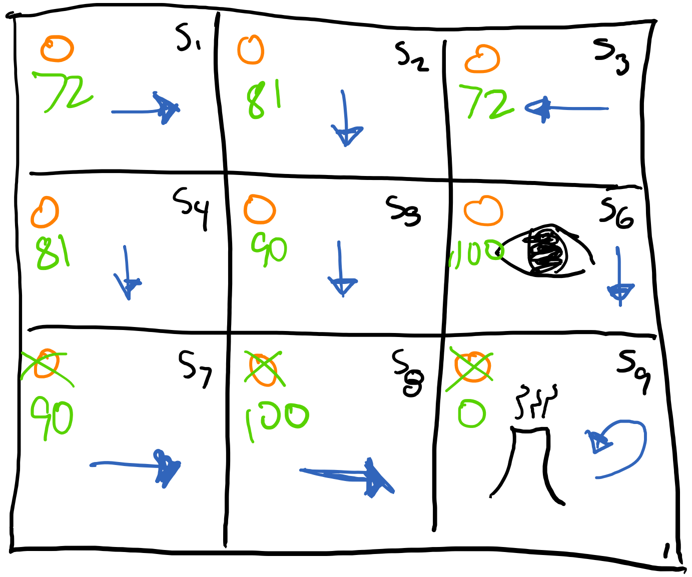
For deterministic trans $\rightarrow \left[V(s) = \max_{a \in A} R(s,a) + \gamma\, V(\underbrace{\delta(s,a)}_{s'}) \right.$

③ Return

$$\pi^{greedy}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a)\, V(s')$$

$$= \arg\max_a R(s,a) + \gamma\, V(\delta(s,a))$$

Example:



Π s_9 is terminal

$A = [\uparrow, \downarrow, \leftarrow, \rightarrow]$

$R(s,a) = -100$ if entering

$R(s,a) = +100$ if entering Π

$R(s,a) = 0$

transitions are deterministic

$\gamma = 0.9$

$V(s) = 0$ for $t=1$

Value Iteration

Algorithm 1 Value Iteration (deterministic transitions)

1: **procedure** ValueIteration($R(s,a)$ reward function, $\delta(s,a)$ transition function)
2: Initialize value function $V(s) = 0$ or randomly
3: **while** not converged **do**
4: **for** $s \in \mathcal{S}$ **do**
5: $V(s) = \max_a R(s,a) + \gamma V(\delta(s,a))$
6: Let $\pi(s) = \text{argmax}_a\, R(s,a) + \gamma V(\delta(s,a)), \ \forall s$
7: **return** π

Variant 1: without Q(s,a) table

Value Iteration

Algorithm 1 Value Iteration (stochastic transitions)

procedure VALUEITERATION($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)

  Initialize value function $V(s) = 0$ or randomly

  while not converged **do**

    for $s \in \mathcal{S}$ **do**

      $V(s) = \max_a \overbrace{R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')}^{Q(s,a)}$

  Let $\pi(s) = \text{argmax}_a\, R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s'), \;\forall s$

  return π

Variant 1: without Q(s,a) table

Value Iteration

Algorithm 1 Value Iteration (stochastic transitions)

procedure VALUEITERATION($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
  Initialize value function $V(s) = 0$ or randomly
  while not converged **do**
    for $s \in \mathcal{S}$ **do**
      for $a \in \mathcal{A}$ **do**
        $Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a)V(s')$
      $V(s) = \max_a Q(s,a)$
  Let $\pi(s) = \text{argmax}_a Q(s,a), \ \forall s$
  return π

Variant 2: with Q(s,a) table

Synchronous vs. Asynchronous Value Iteration

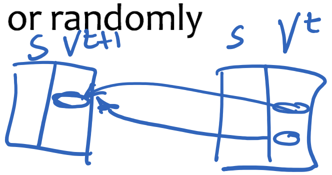
Algorithm 1 Asynchronous Value Iteration

procedure ASYNCHRONOUSVALUEITERATION($R(s,a), p(\cdot|s,a)$)
    Initialize value function $V(s) = 0$ or randomly
    while not converged **do**
        for $s \in \mathcal{S}$ **do**
            $V(s) = \max_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')$
    Let $\pi(s) = \text{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s'), \ \forall s$
    return π

asynchronous updates: compute and update V(s) for each state one at a time

Algorithm 1 Synchronous Value Iteration

procedure SYNCHRONOUSVALUEITERATION($R(s,a), p(\cdot|s,a)$)
    Initialize value function $V(s)^{(0)} = 0$ or randomly
    $t = 0$
    while not converged **do**
        for $s \in \mathcal{S}$ **do**
            $V(s)^{(t+1)} = \max_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')^{(t)}$
        $t = t + 1$
    Let $\pi(s) = \text{argmax}_a R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s'), \ \forall s$
    return π



synchronous updates: compute all the fresh values of V(s) from all the stale values of V(s), then update V(s) with fresh values

Value Iteration Convergence

very abridged

Theorem 1 (Bertsekas (1989))
V converges to V^*, if each state is visited infinitely often

Holds for both asynchronous and sychronous updates

Theorem 2 (Williams & Baird (1993))
if $max_s|V^{t+1}(s) - V^t(s)| < \epsilon$
then $max_s|V^{t+1}(s) - V^*(s)| < \frac{2\epsilon\gamma}{1-\gamma}, \forall s$

Provides reasonable stopping criterion for value iteration

Theorem 3 (Bertsekas (1987))
greedy policy will be optimal in a finite number of steps (even if not converged to optimal value function!)

Often greedy policy converges well before the value function