

10-301/601: Introduction to Machine Learning

Lecture 3 – Decision Trees

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9/4/24

Front Matter

- Announcements:
 - HW1 released 8/26, due 9/4 (today!) at 11:59 PM
 - Reminder: we will grant (basically) any extension requests for this assignment!
 - HW2 released 9/4 (today!), due 9/16 at 11:59 PM
 - Unlike HW1, you will only have:
 - 1 (graded) submission for the written portion
 - 10 submissions of the programming portion to our autograder

Q & A:

How do these in-class polls work?

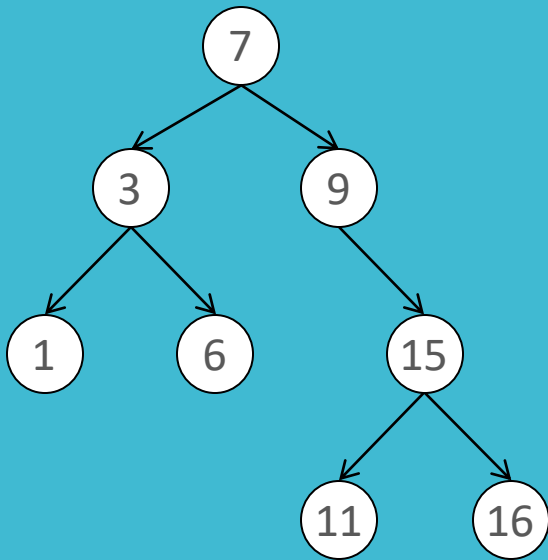
- Open the poll, either by clicking the [Poll] link on the schedule page of our course website or going to <http://poll.mlcourse.org>
- Sign into Google Forms using your **Andrew email**
- Answer all poll questions **during lecture for full credit** or **within 24 hours for half credit**
- Avoid the **toxic option** (will be clearly specified in lecture) which gives **negative poll points**
- You have 8 free “poll points” for the semester that will excuse you from all polls from a single lecture; you cannot use more than 3 poll points consecutively.

Poll Question 1:

Which of the following did you bring to class today?
Select all that apply

- A. A smartphone
- B. A flip phone
- C. A payphone
- D. No phone

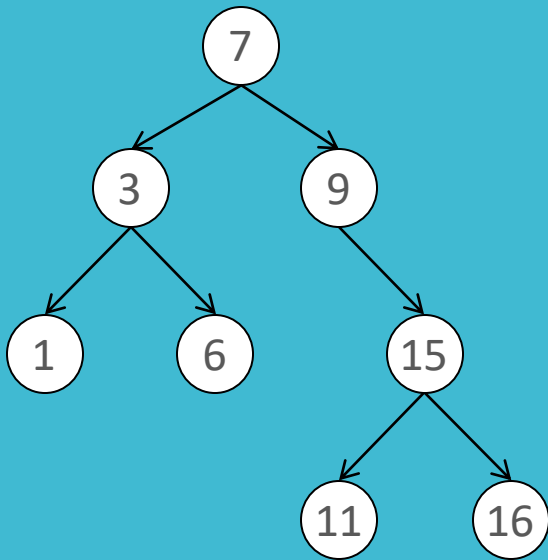
Background: Recursion



- A **binary search tree** (BST) consists of nodes, where each node:
 - has a value, v
 - up to 2 children, a left descendant and a right descendant
 - all its left descendants have values less than v and its right descendants have values greater than v
- We like BSTs because they permit search in $O(\log(n))$ time, assuming n nodes in the tree

```
def contains_iterative(node, key):  
    cur = node  
    while true:  
        if key < cur.value & cur.left != null:  
            cur = cur.left  
        else if cur.value < key & cur.right != null:  
            cur = cur.right  
        else:  
            break  
    return key == cur.value
```

Background: Recursion

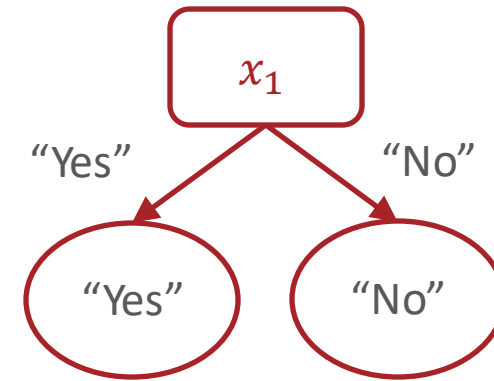


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 - all its left descendants have values less than v and its right descendants have values greater than v
- We like BSTs because they permit search in $O(\log(n))$ time, assuming n nodes in the tree

```
def contains_recursive(node, key):  
    if key < node.value & node.left != null:  
        return contains(node.left, key)  
    else if node.value < key & node.right != null:  
        return contains(node.right, key)  
    else:  
        return key == node.value
```

Recall: Decision Stump

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
Yes	Low	Normal	No
No	Medium	Normal	No
No	Low	Abnormal	Yes
Yes	Medium	Normal	Yes
Yes	High	Abnormal	Yes



Recall: Decision Stump Questions

1. How can we pick which feature to split on?
2. Why stop at just one feature? **Don't!**
 - a) If we split on more than one feature, how do we decide the order to split on?

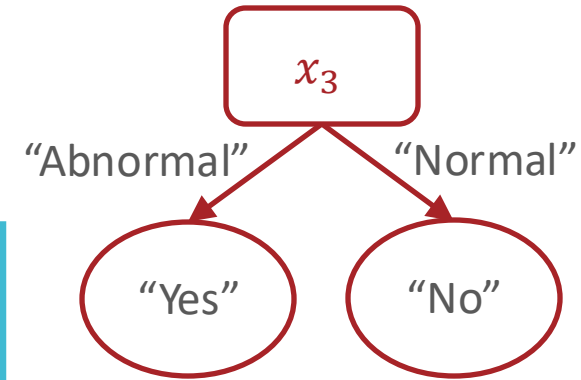
Decision Tree: In-class Activity

1. Group 1: Answer the questions to determine which leaf node corresponds to your feature values
2. Group 2:
 - a) Take a blue sticky note if you prefer dogs to cats; otherwise, take a red sticky note
 - b) Answer the questions to determine which leaf node corresponds to your feature values and place your sticky note there
 - c) Answer the new question to determine which new leaf node to move your sticky note to
3. Group 3: Answer the questions to determine which leaf node corresponds to your feature values

From Decision Stump

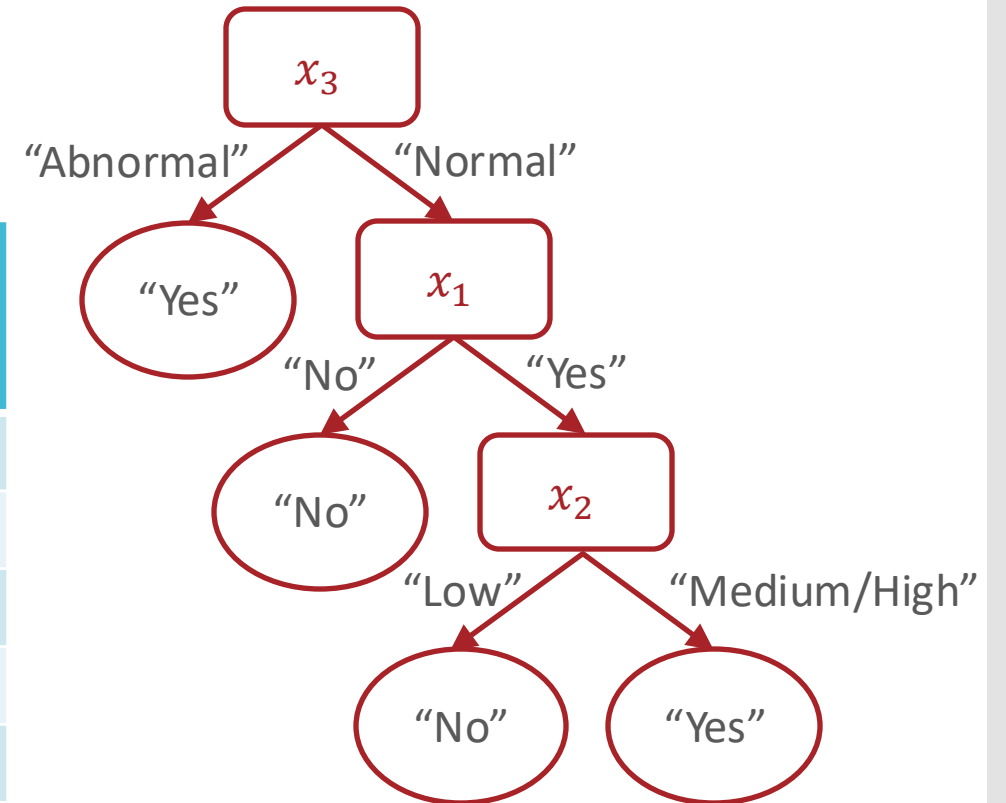
...

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
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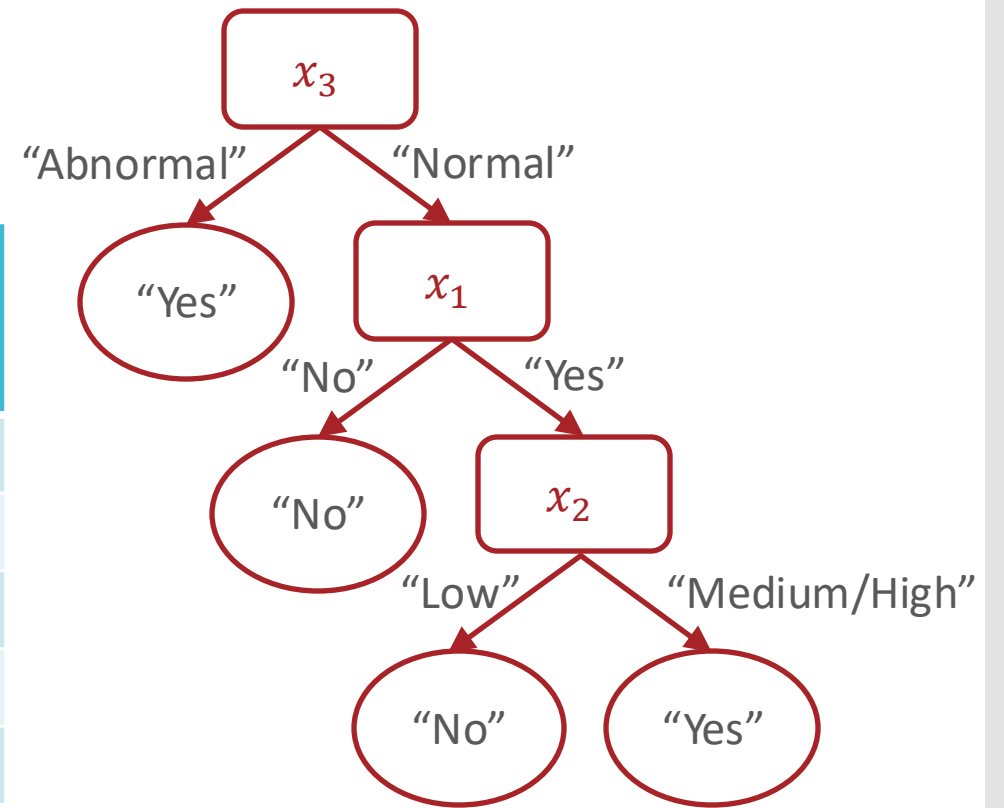
From Decision Stump to Decision Tree

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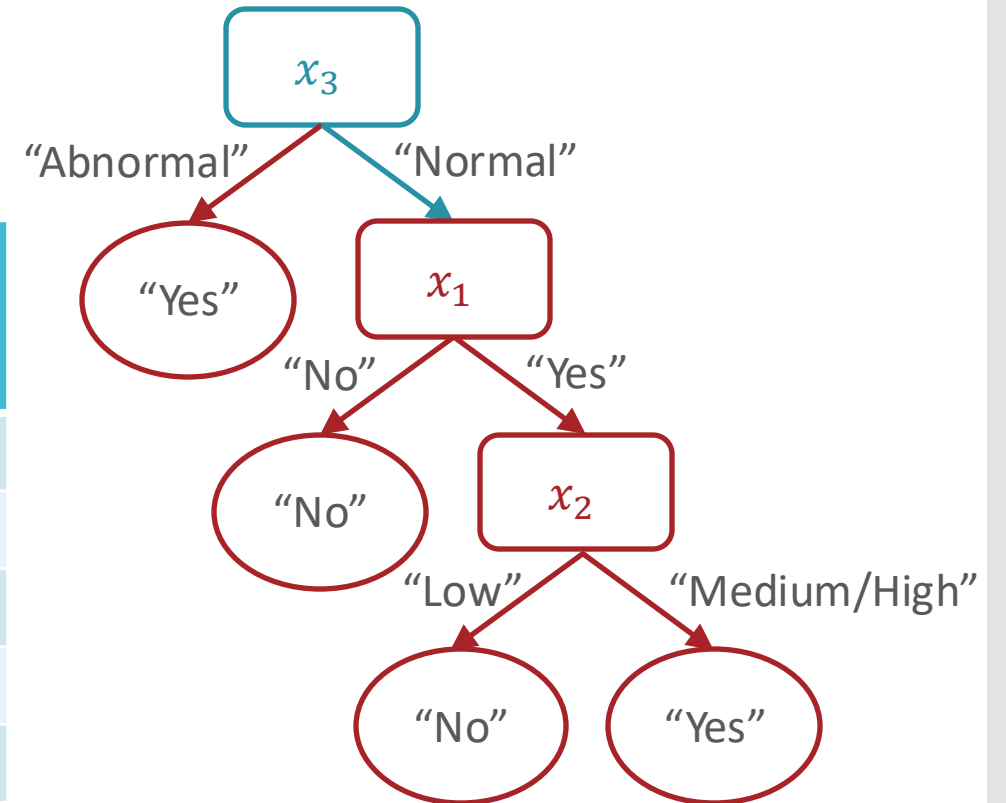
From Decision Stump to Decision Tree

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No	Low	Abnormal	Yes
Yes	Medium	Normal	Yes
Yes	High	Abnormal	Yes
No	High	Normal	No



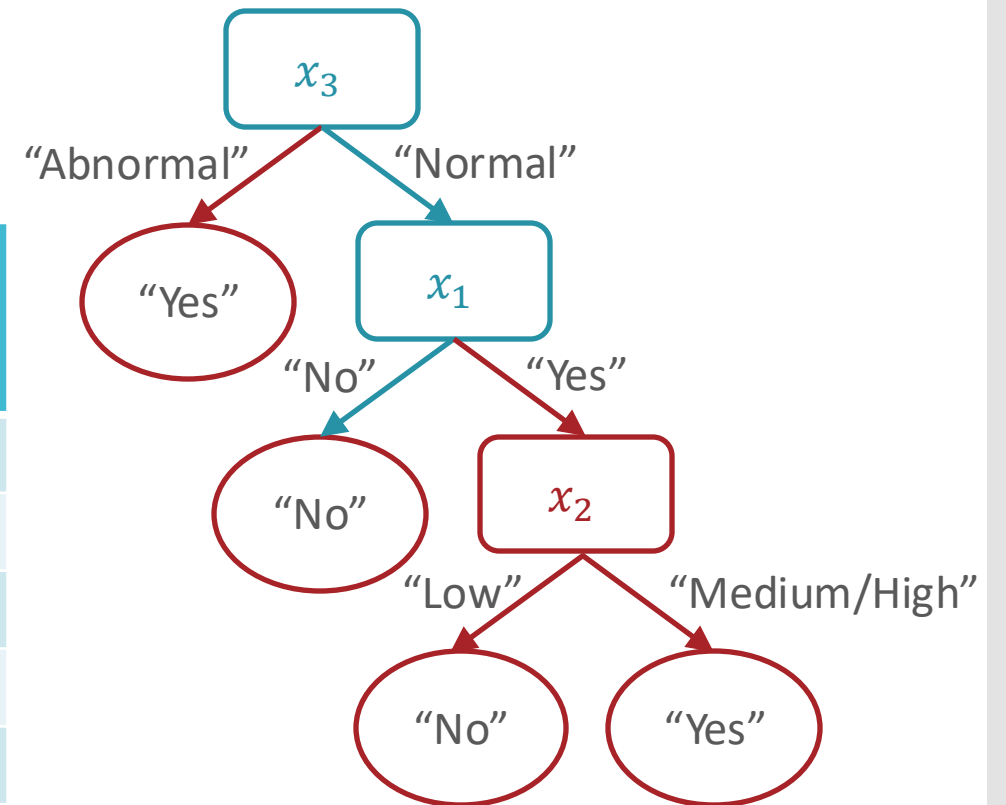
From Decision Stump to Decision Tree

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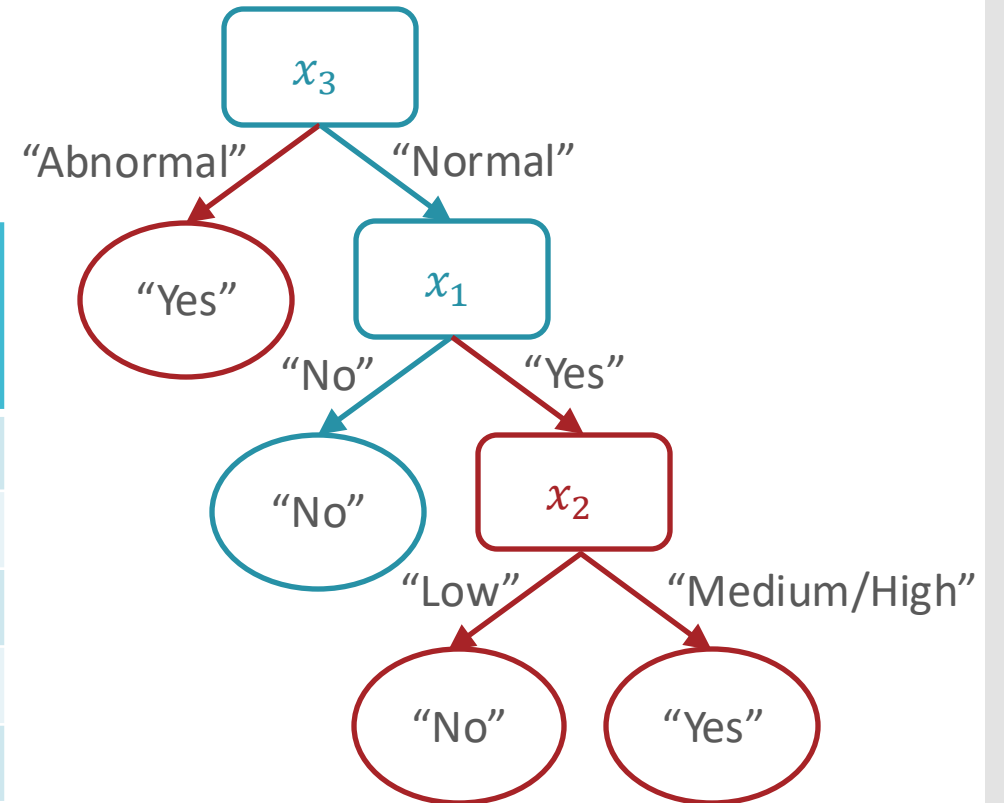
From Decision Stump to Decision Tree

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From Decision Stump to Decision Tree

x_1 Family History	x_2 Resting Blood Pressure	x_3 Cholesterol	y Heart Disease?
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Yes	High	Abnormal	Yes
No	High	Normal	No



Decision Tree: Pseudocode

```
def  $h(x')$ :
```

```
- walk from root node to a leaf node
```

```
while(true):
```

```
    if current node is internal (non-leaf):
```

```
        check the associated attribute,  $x_d$ 
```

```
        go down branch according to  $x'_d$ 
```

```
    if current node is a leaf node:
```

```
        return label stored at that leaf
```


Decision Tree: Example

Learned from medical records of 1000 women

Negative examples are C-sections

```
[833+,167-] .83+ .17-  
Fetal_Presentation = 1: [822+,116-] .88+ .12-  
| Previous_Csection = 0: [767+,81-] .90+ .10-  
| | Primiparous = 0: [399+,13-] .97+ .03-  
| | Primiparous = 1: [368+,68-] .84+ .16-  
| | | Fetal_Distress = 0: [334+,47-] .88+ .12-  
| | | Fetal_Distress = 1: [34+,21-] .62+ .38-  
| Previous_Csection = 1: [55+,35-] .61+ .39-  
Fetal_Presentation = 2: [3+,29-] .11+ .89-  
Fetal_Presentation = 3: [8+,22-] .27+ .73-
```

Decision Tree Questions

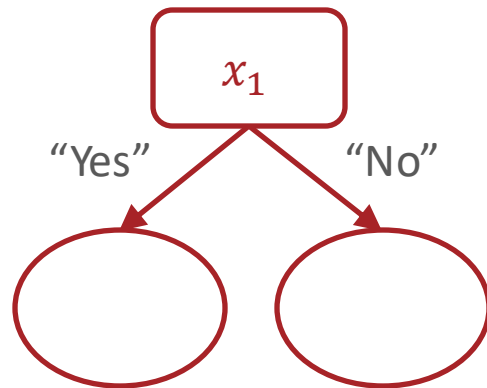
1. How can we pick which feature to split on?
2. Why stop at just one feature?
 - a) If we split on more than one feature, how do we decide the order to split on?

Splitting Criterion

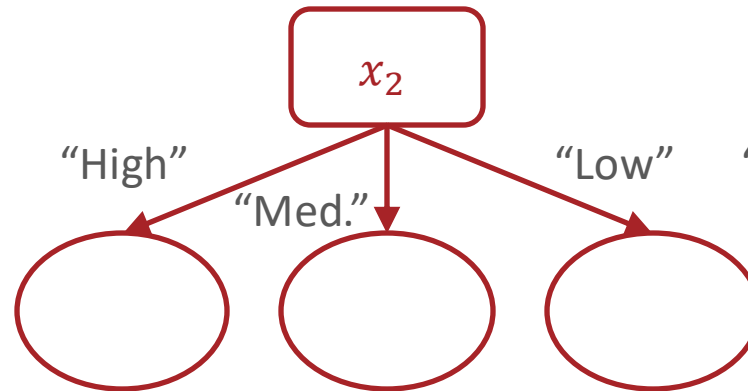
- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion

Training Error Rate as a Splitting Criterion

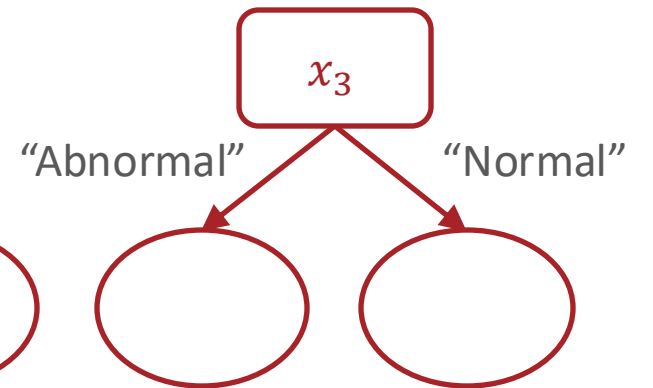
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Training error rate:



Training error rate:



Training error rate:

Poll Question 2:

Which feature would you split on using training error rate as the splitting criterion?

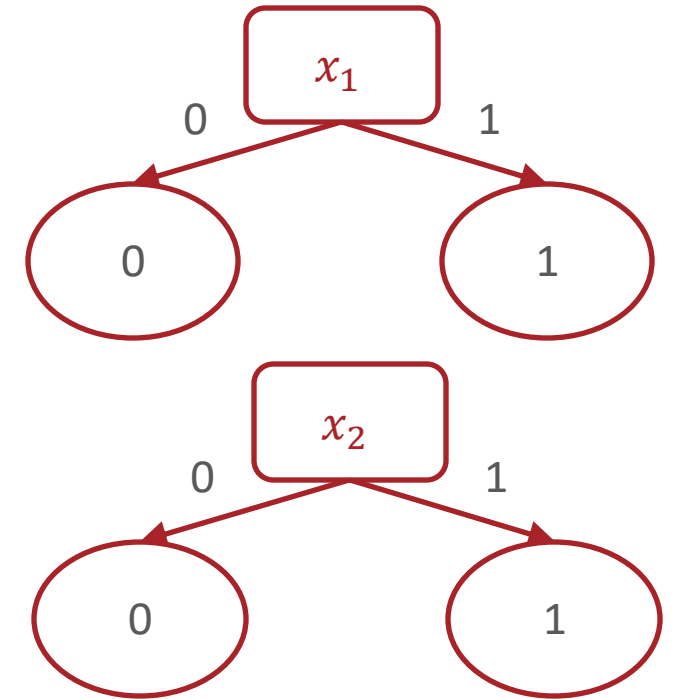
x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1

- A. x_1
- B. x_2
- C. Either x_1 or x_2
- D. Neither x_1 nor x_2

Poll Question 2:

Which feature would you split on using training error rate as the splitting criterion?

x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1



Training error rate: 2/8

Splitting Criterion

- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion
- Potential splitting criteria:
 - Training error rate (minimize)
 - Gini impurity (minimize) → CART algorithm
 - Mutual information (maximize) → ID3 algorithm

Splitting Criterion

- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion
- Potential splitting criteria:
 - Training error rate (minimize)
 - Gini impurity (minimize) → CART algorithm
 - **Mutual information** (maximize) → ID3 algorithm

Entropy

- The **entropy** of a *random variable* describes the uncertainty of its outcome: the higher the entropy, the less certain we are about what the outcome will be.

$$H(X) = - \sum_{v \in V(X)} P(X = v) \log_2(P(X = v))$$

where X is a (discrete) random variable

$V(X)$ is the set of possible values X can take on

Entropy

- The **entropy** of a *set* describes how uniform or pure it is: the higher the entropy, the more impure or “mixed-up” the set is

$$H(S) = - \sum_{v \in V(S)} \frac{|S_v|}{|S|} \log_2 \left(\frac{|S_v|}{|S|} \right)$$

where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If all the elements in S are the same, then

$$H(S) = -1 \log_2(1) = 0$$

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where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If S is split fifty-fifty between two values, then

$$H(S) = -\frac{1}{2} \log_2 \left(\frac{1}{2} \right) - \frac{1}{2} \log_2 \left(\frac{1}{2} \right) = -\log_2 \left(\frac{1}{2} \right) = 1$$

Mutual Information

- The **mutual information** between *two random variables* describes how much clarity knowing the value of one random variables provides about the other

$$I(Y; X) = H(Y) - H(Y|X)$$

$$= H(Y) - \sum_{v \in V(X)} P(X = v)H(Y|X = v)$$

where X and Y are random variables

$V(X)$ is the set of possible values X can take on

$H(Y|X = v)$ is the conditional entropy of Y given $X = v$

Mutual Information

- The **mutual information** between *a feature and the label* describes how much clarity knowing the feature provides about the label

$$\begin{aligned} I(y; x_d) &= H(y) - H(y|x_d) \\ &= H(y) - \sum_{v \in V(x_d)} f_v \left(H(Y_{x_d=v}) \right) \end{aligned}$$

where x_d is a feature and y is the set of all labels

$V(x_d)$ is the set of possible values x_d can take on

f_v is the fraction of data points where $x_d = v$

$Y_{x_d=v}$ is the set of all labels where $x_d = v$

Mutual Information: Example

x_d	y
1	1
1	1
0	0
0	0

$$\begin{aligned} I(x_d, Y) &= H(Y) - \sum_{v \in V(x_d)} (f_v) \left(H(Y_{x_d=v}) \right) \\ &= 1 - \frac{1}{2} H(Y_{x_d=0}) - \frac{1}{2} H(Y_{x_d=1}) \\ &= 1 - \frac{1}{2} (0) - \frac{1}{2} (0) = 1 \end{aligned}$$

Mutual Information: Example

x_d	y
1	1
0	1
1	0
0	0

$$\begin{aligned} I(x_d, Y) &= H(Y) - \sum_{v \in V(x_d)} (f_v) \left(H(Y_{x_d=v}) \right) \\ &= 1 - \frac{1}{2} H(Y_{x_d=0}) - \frac{1}{2} H(Y_{x_d=1}) \\ &= 1 - \frac{1}{2} (1) - \frac{1}{2} (1) = 0 \end{aligned}$$

Poll Question 3:

Which feature would you split on using mutual information as the splitting criterion?

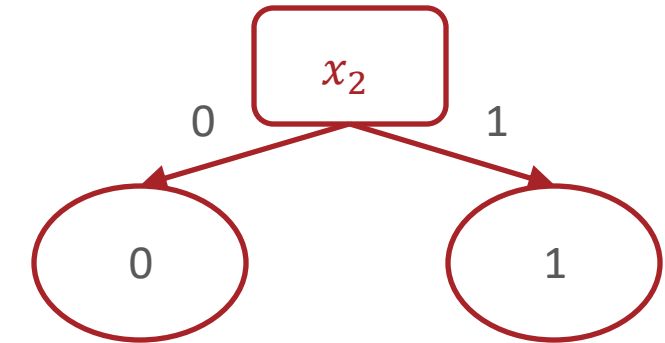
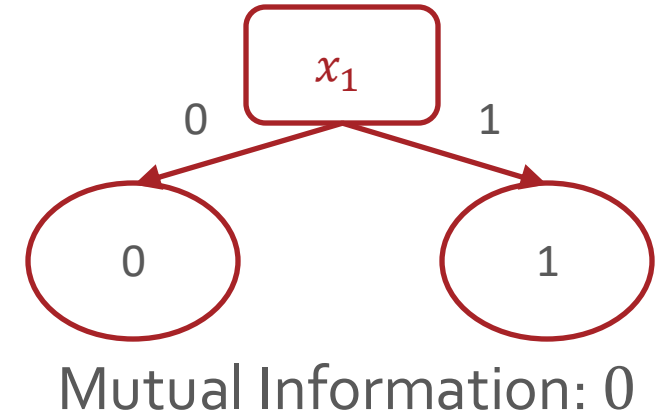
x_1	x_2	y
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- A. x_1
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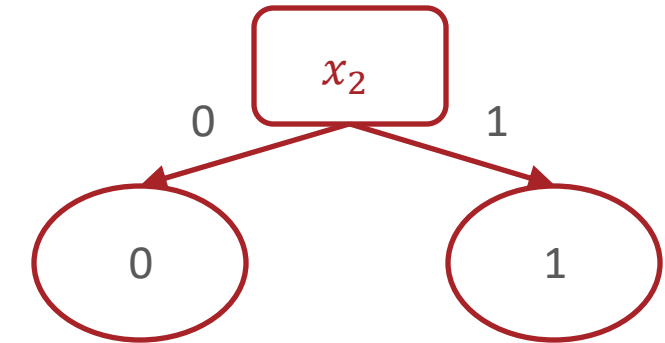
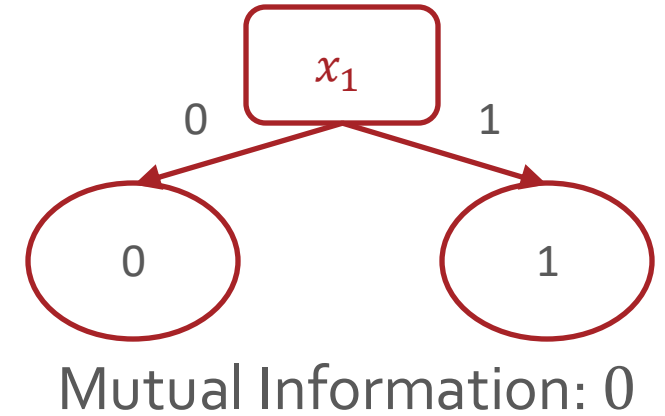


Mutual Information: $H(Y) - \frac{1}{2}H(Y_{x_2=0}) - \frac{1}{2}H(Y_{x_2=1})$

Poll Question 3:

Which feature would you split on using mutual information as the splitting criterion?

x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1



$$\text{Mutual Information: } -\frac{2}{8} \log_2 \frac{2}{8} - \frac{6}{8} \log_2 \frac{6}{8} - \frac{1}{2} (1) - \frac{1}{2} (0) \approx 0.31$$

Decision Tree Questions

1. How can we pick which feature to split on?
2. Why stop at just one feature?
 - a) If we split on more than one feature, how do we decide the order to split on?

Decision Tree: Pseudocode

```
def train( $\mathcal{D}$ ):  
    store root = tree_recurse( $\mathcal{D}$ )  
def tree_recurse( $\mathcal{D}'$ ):  
    q = new node()  
    base case - if (SOME CONDITION):  
    recursion - else:  
        find best attribute to split on,  $x_d$   
        q.split =  $x_d$   
        for  $v$  in  $V(x_d)$ , all possible values of  $x_d$ :  
             $\mathcal{D}_v = \{(x^{(n)}, y^{(n)}) \in \mathcal{D} \mid x_d^{(n)} = v\}$   
            q.children( $v$ ) = tree_recurse( $\mathcal{D}_v$ )  
    return q
```

Decision Tree: Pseudocode

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def train( $\mathcal{D}$ ):  
    store root = tree_recurse( $\mathcal{D}$ )  
def tree_recurse( $\mathcal{D}'$ ):  
    q = new node()  
    base case - if ( $\mathcal{D}'$  is empty OR  
        all labels in  $\mathcal{D}'$  are the same OR  
        all features in  $\mathcal{D}'$  are identical OR  
        some other stopping criterion):  
        q.label = majority_vote( $\mathcal{D}'$ )  
  
    recursion - else:  
        return q
```