10-301/601: Introduction to Machine Learning Lecture 20: Markov Decision Processes

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4/1/24

Front Matter

- Announcements
  - HW7 released 3/28, due 4/8 at 11:59 PM
    - Please be mindful of your grace day usage (see the course syllabus for the policy)
  - **We will have lecture on 4/5 (Friday) and recitation on 4/8 (next Monday)**

Learning Paradigms

- Supervised learning - $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(n)}, y^{(n)}\right)\right\}_{n=1}^{N}$
  - Regression - $y^{(n)} \in \mathbb{R}$
  - Classification - $y^{(n)} \in \{1, \dots, C\}$
- Reinforcement learning - $\mathcal{D} = \left\{\left(\boldsymbol{s}^{(n)}, \boldsymbol{a}^{(n)}, r^{(n)}\right)\right\}_{n=1}^{N}$

Source: https://techobserver.net/2019/06/argo-ai-self-driving-car-research-center/

Source: https://www.wired.com/2012/02/high-speed-trading/

Reinforcement Learning: Examples

Source: https://www.cnet.com/news/boston-dynamics-robot-dog-spot-finally-goes-on-sale-for-74500/

Source: https://twitter.com/alphagomovie

AlphaGo

Source: https://www.youtube.com/watch?v=WXuK6gekU1Y&ab_channel=DeepMind

Outline

- Problem formulation
  - Time discounted cumulative reward
  - Markov decision processes (MDPs)
- Algorithms:
  - Value & policy iteration (dynamic programming)
  - (Deep) Q-learning (temporal difference learning)

Reinforcement Learning: Problem Formulation

- State space, $\mathcal{S}$
- Action space, $\mathcal{A}$
- Reward function
  - Stochastic, $p(r \mid s, a)$
  - Deterministic, $R\colon \mathcal{S} \times \mathcal{A} \to \mathbb{R}$
- Transition function
  - Stochastic, $p(s' \mid s, a)$
  - Deterministic, $\delta\colon \mathcal{S} \times \mathcal{A} \to \mathcal{S}$

Reinforcement Learning: Problem Formulation

- Policy, $\pi : \mathcal{S} \rightarrow \mathcal{A}$
  - Specifies an action to take in *every* state
- Value function, $V^{\pi}: \mathcal{S} \rightarrow \mathbb{R}$
  - Measures the expected total payoff of starting in some state s and executing policy π, i.e., in every state, taking the action that π returns

Toy Example

- $\mathcal{S}$ = all empty squares in the grid
- $\mathcal{A}$ = {up, down, left, right}
- Deterministic transitions
- Rewards of +1 and -1 for entering the labelled squares
- Terminate after receiving either reward

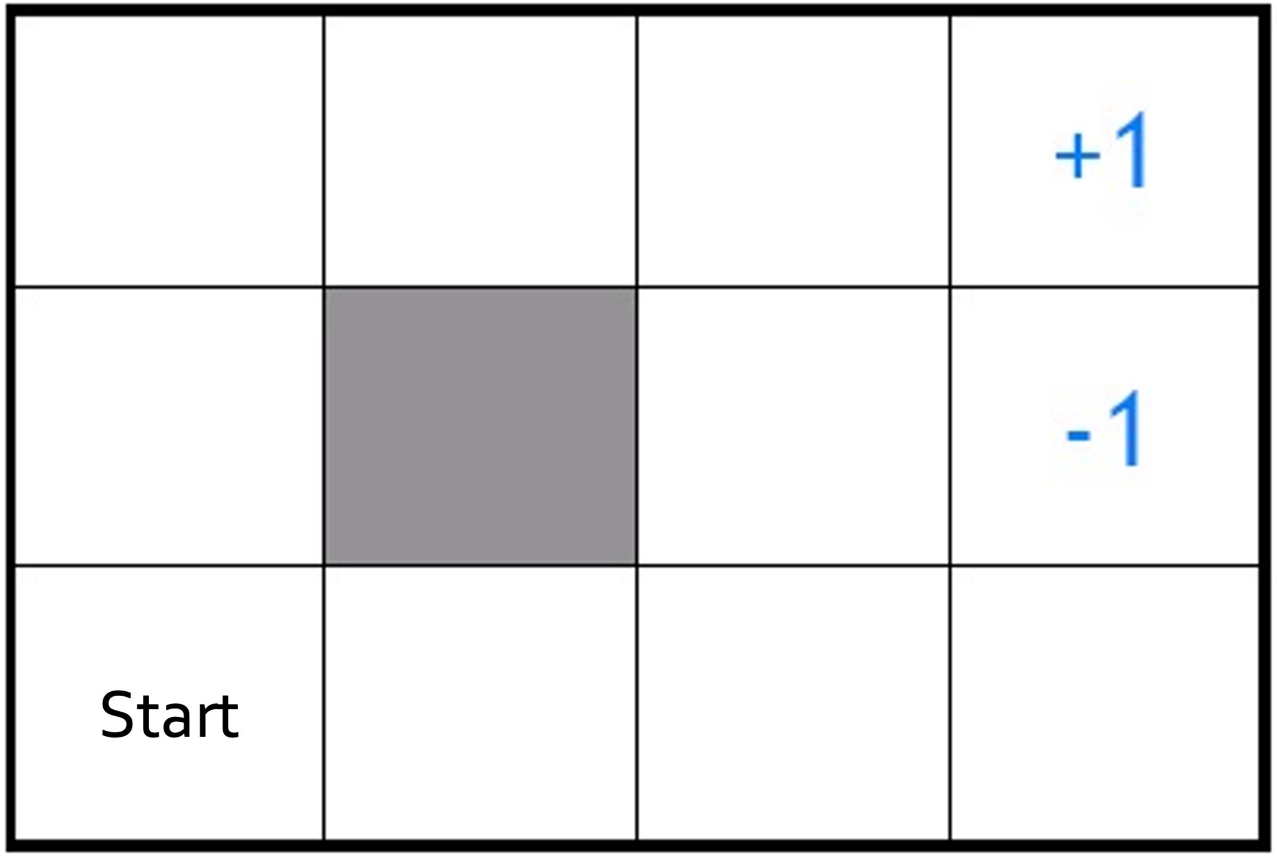


Figure courtesy of Eric Xing

Toy Example

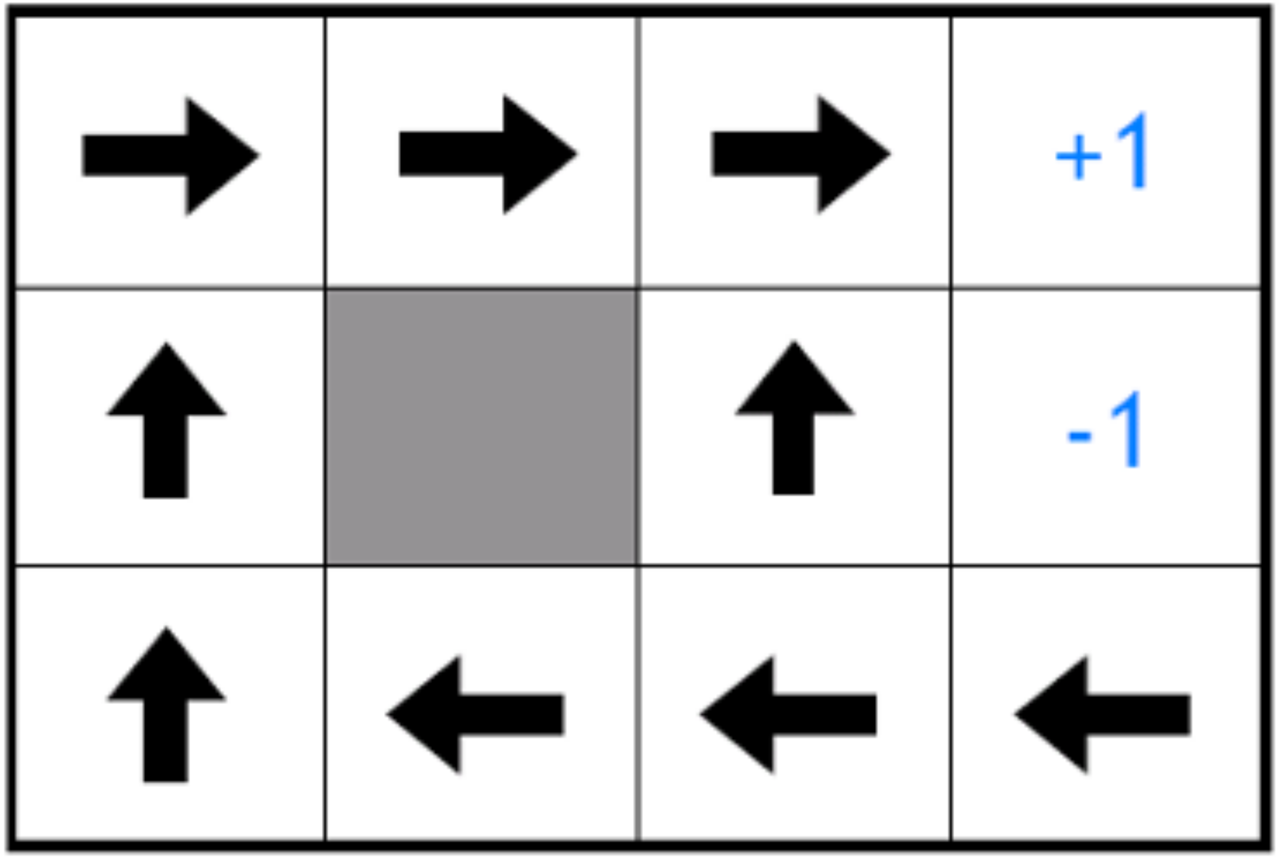


Figure courtesy of Eric Xing

Poll Question 1:

Is this policy optimal?

A. Yes

B. TOXIC

C. No

Poll Question 2:

Justify your answer to the previous question

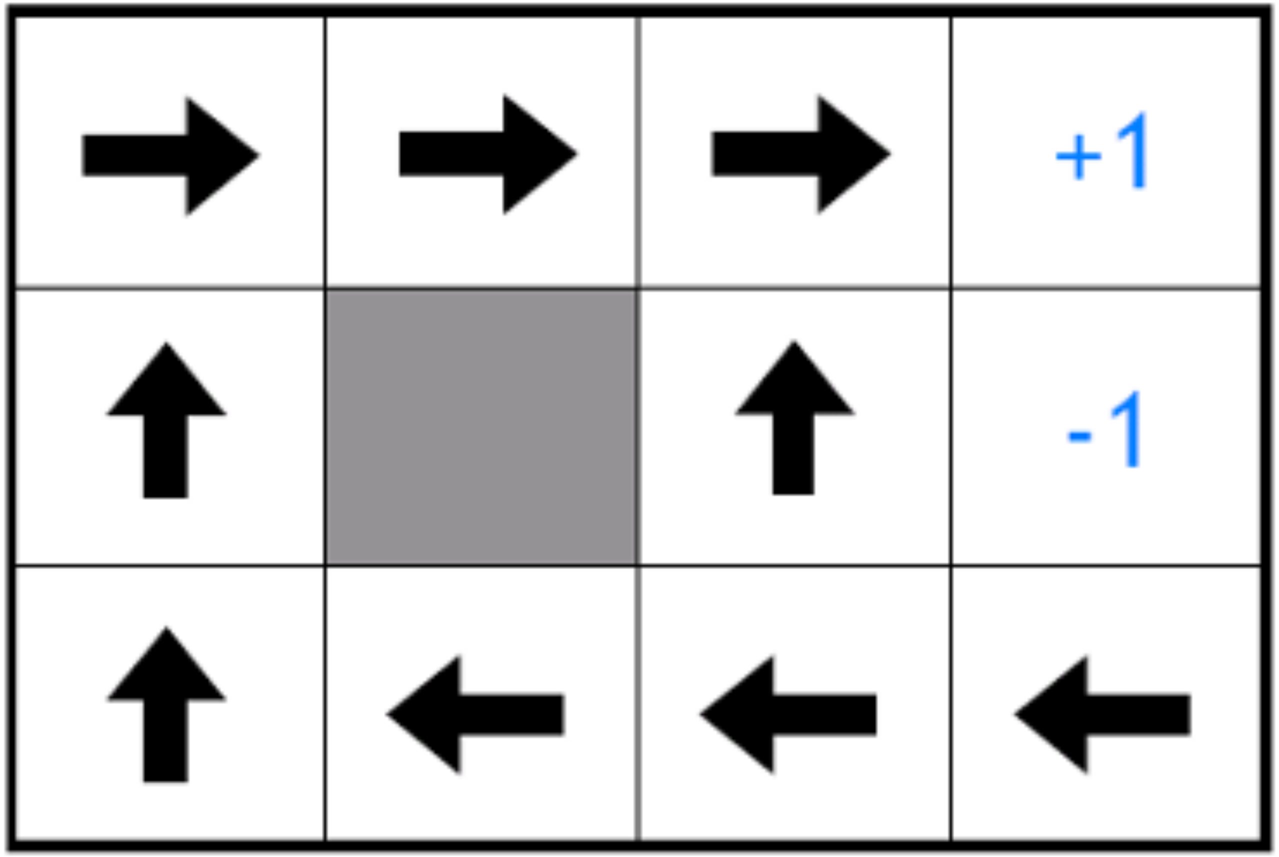


Figure courtesy of Eric Xing

Toy Example

Optimal policy given a reward of -2 per step

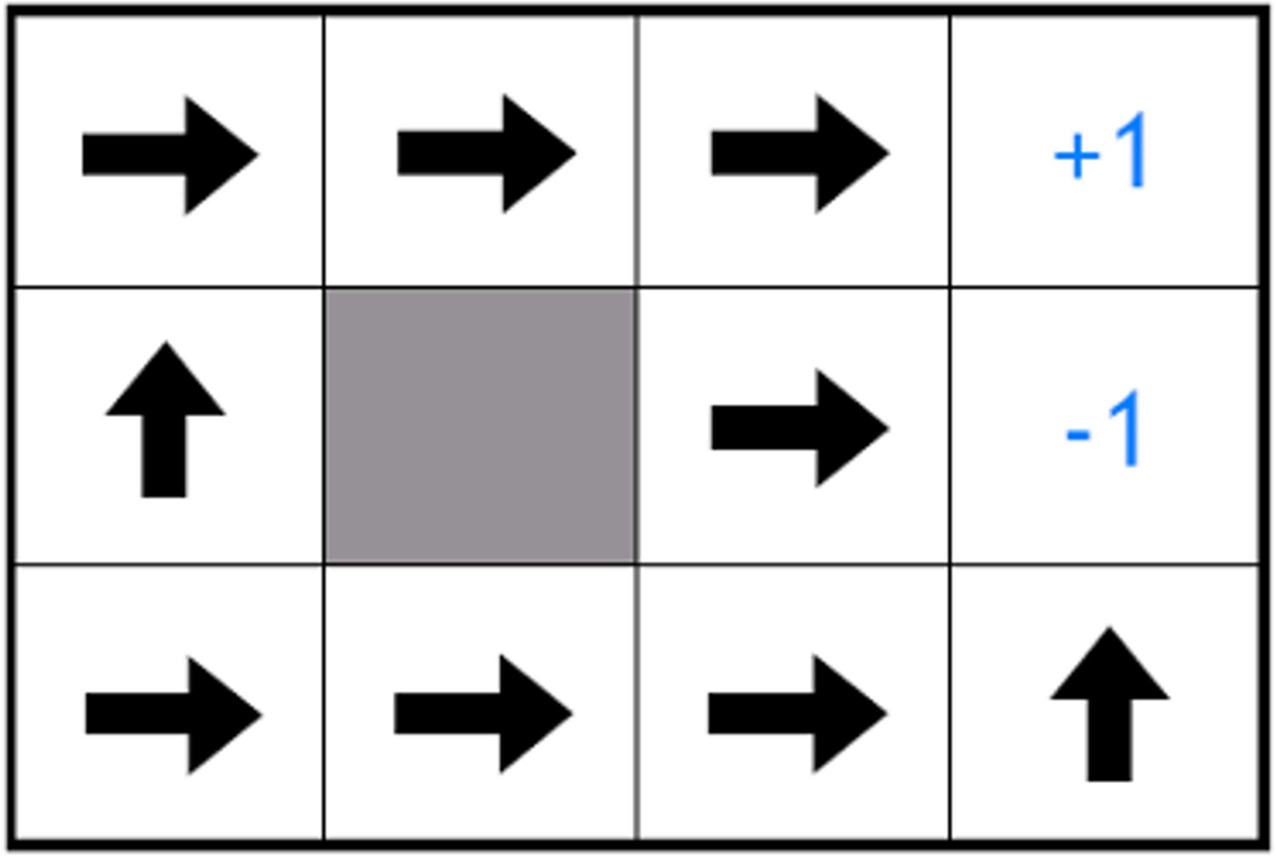


Figure courtesy of Eric Xing

Toy Example

Optimal policy given a reward of -0.1 per step

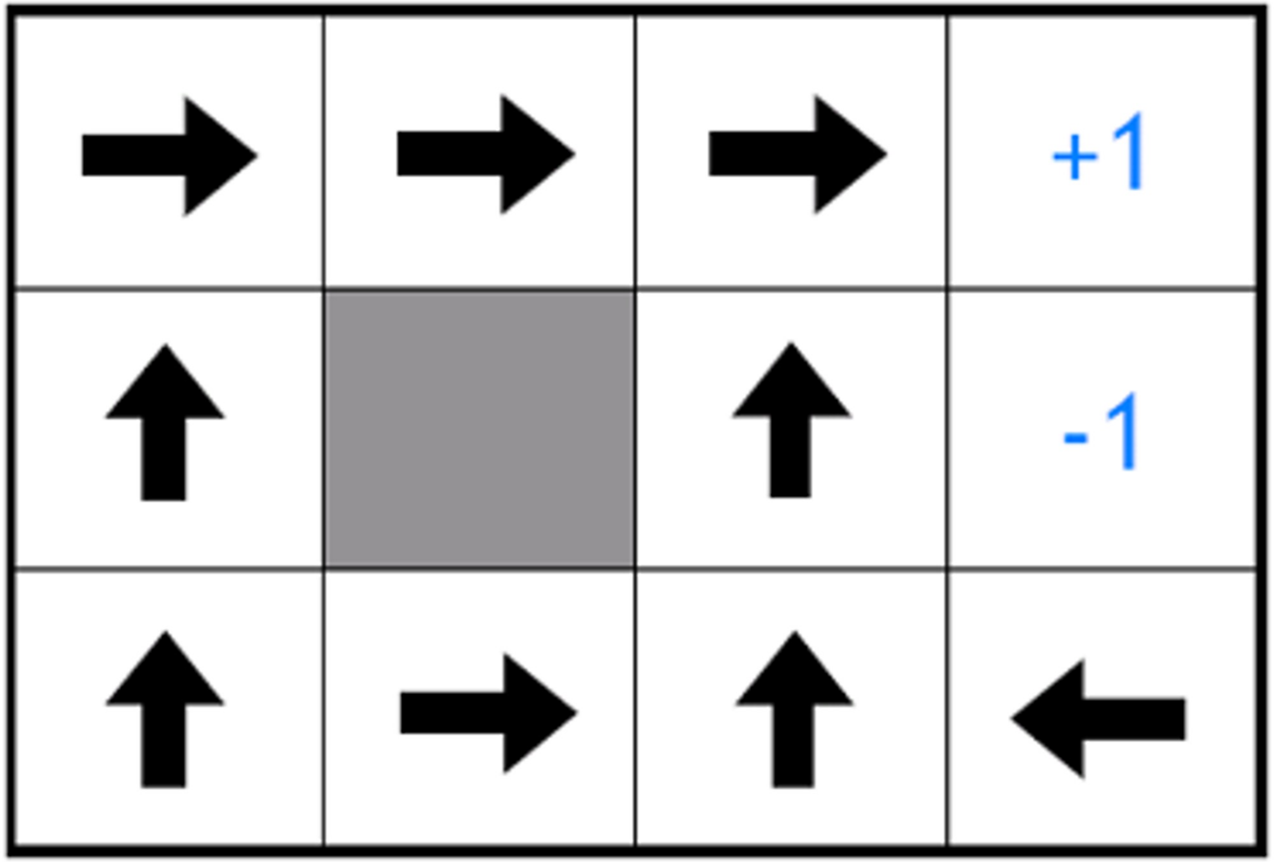


Figure courtesy of Eric Xing

Markov Decision Process (MDP)

- Assume the following model for our data:

1. Start in some initial state s_0
2. For time step t:
    1. Agent observes state s_t
    2. Agent takes action $a_t = \pi(s_t)$
    3. Agent receives reward $r_t \sim p(r \mid s_t, a_t)$
    4. Agent transitions to state $s_{t+1} \sim p(s' \mid s_t, a_t)$
3. Total reward is $\sum_{t=0}^{\infty} \gamma^t r_t$

- MDPs make the *Markov assumption*: the reward and next state only depend on the current state and action.

Reinforcement Learning: Key Challenges

- The algorithm has to gather its own training data
- The outcome of taking some action is often stochastic or unknown until after the fact
- Decisions can have a delayed effect on future outcomes (exploration-exploitation tradeoff)

MDP Example: Multi-armed bandit

- Single state: $|\mathcal{S}| = 1$
- Three actions: $\mathcal{A} = \{1, 2, 3\}$
- Deterministic transitions
- Rewards are stochastic

Reinforcement Learning: Objective Function

- Find a policy $\pi^* = \underset{\pi}{\operatorname{argmax}}\ V^{\pi}(s)\ \forall\, s \in \mathcal{S}$
- Assume deterministic transitions and deterministic rewards
- $V^{\pi}(s) =$ *discounted* total reward of starting in state s and executing policy π forever

$$= R(s_0, \pi(s_0)) + \gamma R\left(s_1 = \delta(s_0, \pi(s_0)), \pi(s_1)\right) + \gamma^2 R\left(s_2 = \delta(s_1, \pi(s_1)), \pi(s_2)\right) + \cdots$$

$$= R(s_0, \pi(s_0)) + \sum_{t=1}^{\infty} \gamma^t R\left(\delta(s_{t-1}, \pi(s_{t-1})), \pi(s_t)\right)$$

where $0 < \gamma < 1$ is some discount factor for future rewards

Reinforcement Learning: Objective Function

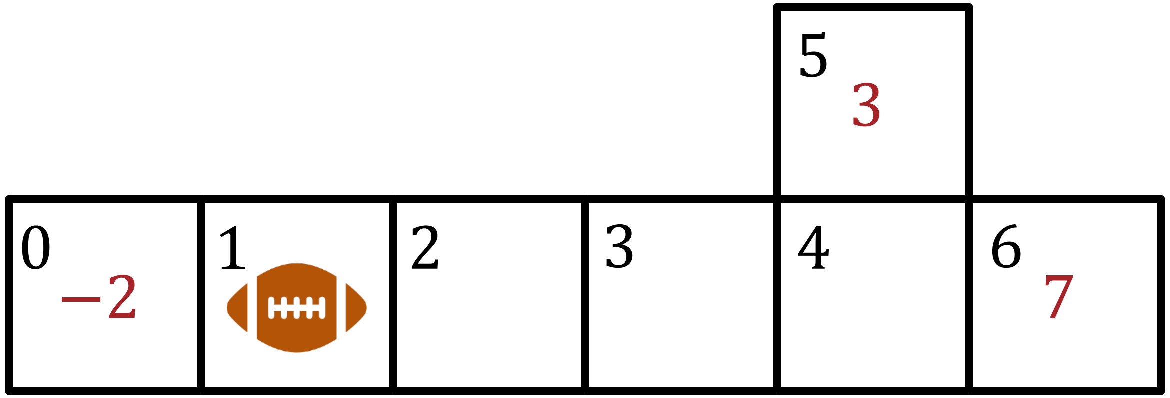
- Find a policy $\pi^* = \underset{\pi}{\operatorname{argmax}}\ V^\pi(s)\ \forall\, s \in \mathcal{S}$
- Assume stochastic transitions and deterministic rewards
- $V^\pi(s) = \mathbb{E}[$*discounted* total reward of starting in state s and executing policy π forever$]$

$$= \mathbb{E}_{p(s' \mid s,\, a)}[R\big(s_0 = s, \pi(s_0)\big) + \gamma R\big(s_1, \pi(s_1)\big) + \gamma^2 R\big(s_2, \pi(s_2)\big) + \cdots]$$

$$= \sum_{t=0}^{\infty} \gamma^t \mathbb{E}_{p(s' \mid s,\, a)}\big[R\big(s_t, \pi(s_t)\big)\big]$$

where $0 < \gamma < 1$ is some discount factor for future rewards

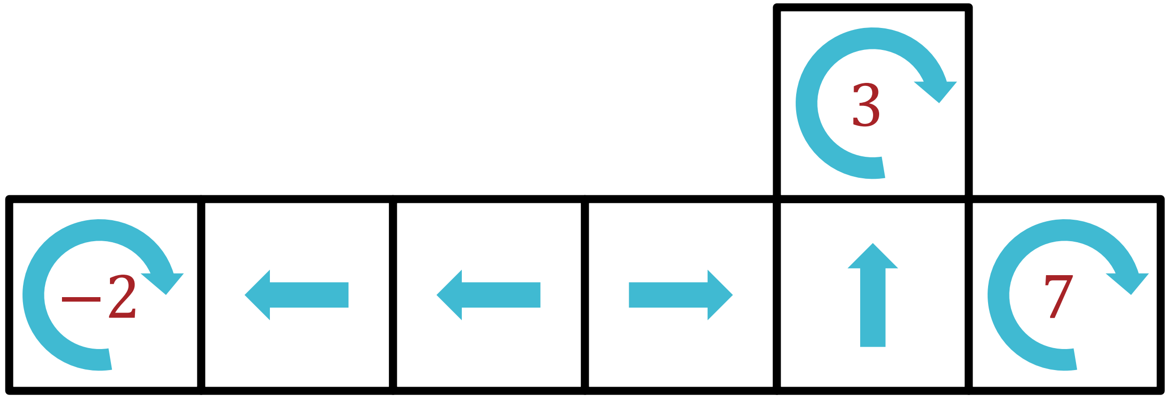
Value Function: Example



$$R(s,a) = \begin{cases} -2 \text{ if entering state 0 (safety)} \\ 3 \text{ if entering state 5 (field goal)} \\ 7 \text{ if entering state 6 (touch down)} \\ 0 \text{ otherwise} \end{cases}$$

$$\gamma = 0.9$$

Value Function: Example



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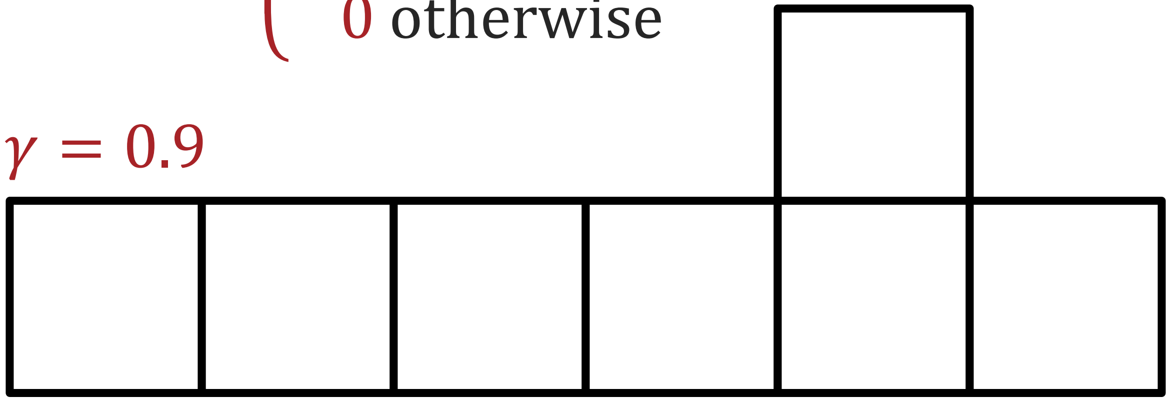

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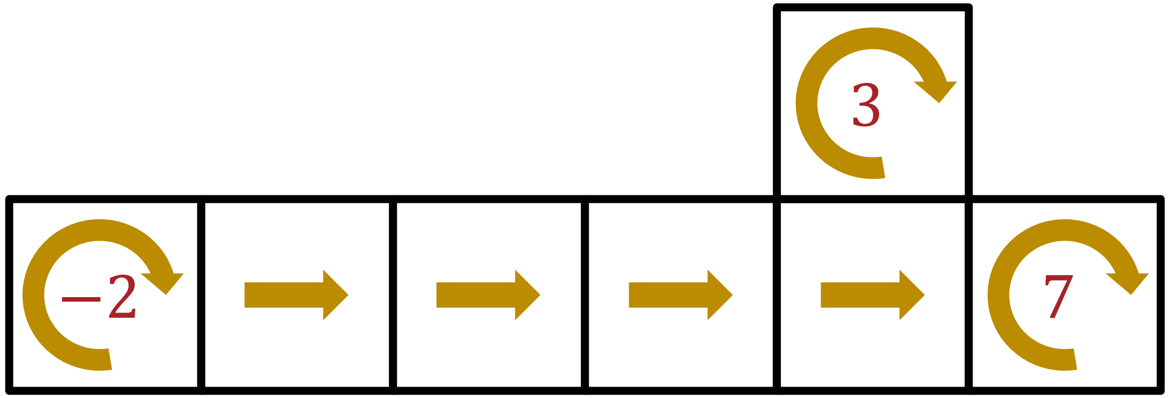

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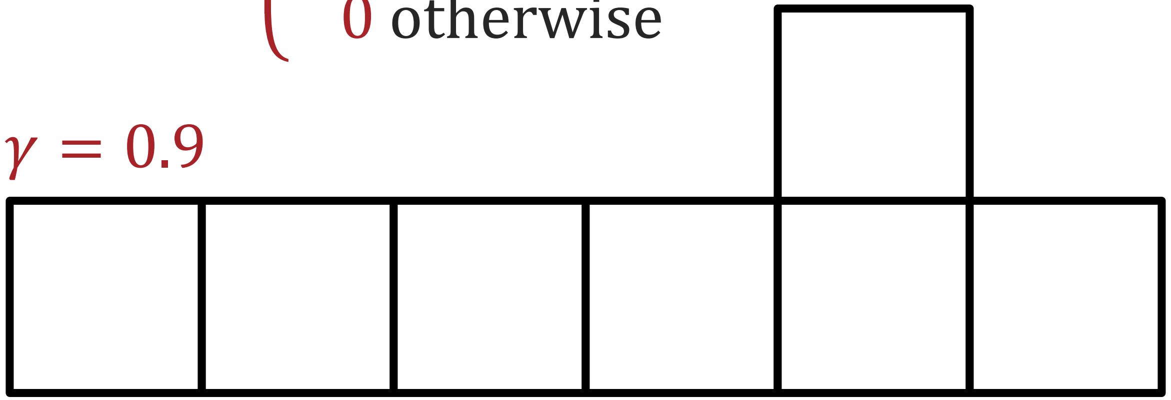

Value Function

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$$= \mathbb{E}[R(s_0, \pi(s_0)) + \gamma R(s_1, \pi(s_1)) + \gamma^2 R(s_2, \pi(s_2)) + \cdots \mid s_0 = s]$$

$$= R(s, \pi(s)) + \gamma \mathbb{E}[R(s_1, \pi(s_1)) + \gamma R(s_2, \pi(s_2)) + \ \ldots \mid s_0 = s]$$

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$$\underbrace{\mathrm{V}^{\pi}(\mathrm{s}) = R(s, \pi(s)) + \gamma \sum_{s_1 \in \mathcal{S}} p(s_1 \mid s, \pi(s)) V^{\pi}(s_1)}_{\text{Bellman equations}}$$