



# 10-301/10-601 Introduction to Machine Learning

Machine Learning Department  
School of Computer Science  
Carnegie Mellon University

## Reinforcement Learning: Value Iteration & Policy Iteration

Matt Gormley, Hoda Heidari, Henry Chai

Lecture 21

Apr. 3, 2024

# Reminders

- **Homework 7: Deep Learning**
  - Out: Thu, Mar. 28
  - Due: Mon, Apr. 8 at 11:59pm
- **Schedule Notes**
  - Lecture 22: Fri, Apr. 5
  - HW8 Recitation: Mon, Apr. 8
- **Homework 8: Deep RL**
  - Out: Mon, Apr. 8
  - Due: Fri, Apr. 19 at 11:59pm

# **MARKOV DECISION PROCESSES**

# RL: Components

## From the Environment (i.e. the MDP)

- State space,  $\mathcal{S}$
- Action space,  $\mathcal{A}$
- Reward function,  $R(s, a)$ ,  $R : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$
- Transition probabilities,  $p(s' | s, a)$ 
  - Deterministic transitions:

$$p(s' | s, a) = \begin{cases} 1 & \text{if } \delta(s, a) = s' \\ 0 & \text{otherwise} \end{cases}$$

where  $\delta(s, a)$  is a transition function

Markov Assumption

$$p(s_{t+1} | s_t, a_t, \dots, s_1, a_1) \\ = p(s_{t+1} | s_t, a_t)$$

## From the Model

- Policy,  $\pi : \mathcal{S} \rightarrow \mathcal{A}$
- Value function,  $V^\pi : \mathcal{S} \rightarrow \mathbb{R}$ 
  - Measures the expected total payoff of starting in some state  $s$  and executing policy  $\pi$

# Markov Decision Process (MDP)

- For **supervised learning** the **PAC learning framework** provided assumptions about where our data came from:

$$\mathbf{x} \sim p^*(\cdot) \text{ and } y = c^*(\cdot)$$

- For **reinforcement learning** we assume our data comes from a **Markov decision process (MDP)**

# Markov Decision Processes (MDP)

In RL, the source of our data is an MDP:

1. Start in some initial state  $s_0 \in \mathcal{S}$
2. For time step  $t$ :
  1. Agent observes state  $s_t \in \mathcal{S}$
  2. Agent takes action  $a_t \in \mathcal{A}$  where  $a_t = \pi(s_t)$
  3. Agent receives reward  $r_t \in \mathbb{R}$  where  $r_t = R(s_t, a_t)$
  4. Agent transitions to state  $s_{t+1} \in \mathcal{S}$  where  $s_{t+1} \sim p(s' | s_t, a_t)$
3. Total reward is  $\sum_{t=0}^{\infty} \gamma^t r_t$ 
  - The value  $\gamma$  is the “discount factor”, a hyperparameter  $0 < \gamma < 1$

- Makes the same Markov assumption we used for HMMs! The next state only depends on the current state and action.
- *Def.:* we **execute** a policy  $\pi$  by taking action  $a = \pi(s)$  when in state  $s$

# RL: Objective Function

- Goal: Find a policy  $\pi : \mathcal{S} \rightarrow \mathcal{A}$  for choosing “good” actions that maximize:

$$\mathbb{E}[\text{total reward}] = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r_t \right]$$

- The above is called the  
“infinite horizon expected future discounted reward”

# RL: Optimal Value Function & Policy

- Bellman Equations:
- Optimal policy:
  - Given  $V^*$ ,  $R(s, a)$ ,  $p(s' | s, a)$ ,  $\gamma$  we can compute this!
- Optimal value function:
  - System of  $|\mathcal{S}|$  equations and  $|\mathcal{S}|$  variables (each variable is some  $V^*(s)$  for some state  $s$ )
  - Can be written without  $\pi^*$



# **FIXED POINT ITERATION**

# Fixed Point Iteration

$$f_1(x_1, \dots, x_n) = 0$$

⋮

$$f_n(x_1, \dots, x_n) = 0$$

$$x_1 = g_1(x_1, \dots, x_n)$$

⋮

$$x_n = g_n(x_1, \dots, x_n)$$

$$x_1^{(t+1)} = g_1(x_1^{(t)}, \dots, x_n^{(t)})$$

⋮

$$x_n^{(t+1)} = g_n(x_1^{(t)}, \dots, x_n^{(t)})$$

- Fixed point iteration is a general tool for solving systems of equations
  - Under the right conditions, it will converge
1. Assume we have  $n$  equations and  $n$  variables, written  $f(\mathbf{x}) = 0$  where  $\mathbf{x}$  is a vector
  2. Rearrange the equations s.t. each variable  $x_i$  has one equation where it is isolated on the LHS
  3. Initialize the parameters.
  4. For  $i$  in  $\{1, \dots, n\}$ , update each parameter and increment  $t$ :
  5. Repeat #5 until convergence

# Fixed Point Iteration

$$\begin{aligned}\cos(y) - x &= 0 \\ \sin(x) - y &= 0\end{aligned}$$

$$\begin{aligned}x &= \cos(y) \\ y &= \sin(x)\end{aligned}$$

$$\begin{aligned}x^{(t+1)} &= \cos(y^{(t)}) \\ y^{(t+1)} &= \sin(x^{(t)})\end{aligned}$$

- Fixed point iteration is a general tool for solving systems of equations
  - Under the right conditions, it will converge
1. Assume we have  $n$  equations and  $n$  variables, written  $f(\mathbf{x}) = 0$  where  $\mathbf{x}$  is a vector
  2. Rearrange the equations s.t. each variable  $x_i$  has one equation where it is isolated on the LHS
  3. Initialize the parameters.
  4. For  $i$  in  $\{1, \dots, n\}$ , update each parameter and increment  $t$ :
  5. Repeat #5 until convergence

# Fixed Point Iteration

We can implement our example in a few lines of code

$$\cos(y) - x = 0$$

$$\sin(x) - y = 0$$

$$x = \cos(y)$$

$$y = \sin(x)$$

$$x^{(t+1)} = \cos(y^{(t)})$$

$$y^{(t+1)} = \sin(x^{(t)})$$

```
from math import *

def f(x, y):
    eq1 = cos(y) - x
    eq2 = sin(x) - y
    return (eq1, eq2)

def g(x, y):
    x = cos(y)
    y = sin(x)
    return (x, y)

def fpi(x0, y0, n):
    '''Solves the system of equations by fixed point iteration
    starting at x0 and stopping after n iterations. Also
    includes an auxiliary function f to test at each value.'''
    x = x0
    y = y0
    for i in range(n):
        ox, oy = f(x,y)
        print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
        x,y = g(x,y)
        i += 1
    print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
    return x,y

if __name__ == "__main__":
    x,y = fpi(-1, -1, 20)
```

# Fixed Point Iteration

```
$ python fixed-point-iteration.py
i= 0 x=-1.0000 y=-1.000 f(x,y)=(1.5403, 0.1585)
i= 1 x=0.5403 y=0.5144 f(x,y)=(0.3303, 0.0000)
i= 2 x=0.8706 y=0.7647 f(x,y)=(-0.1490, 0.0000)
i= 3 x=0.7216 y=0.6606 f(x,y)=(0.0681, 0.0000)
i= 4 x=0.7896 y=0.7101 f(x,y)=(-0.0313, 0.0000)
i= 5 x=0.7583 y=0.6877 f(x,y)=(0.0144, 0.0000)
i= 6 x=0.7727 y=0.6981 f(x,y)=(-0.0066, 0.0000)
i= 7 x=0.7661 y=0.6933 f(x,y)=(0.0031, 0.0000)
i= 8 x=0.7691 y=0.6955 f(x,y)=(-0.0014, 0.0000)
i= 9 x=0.7677 y=0.6945 f(x,y)=(0.0006, 0.0000)
i=10 x=0.7684 y=0.6950 f(x,y)=(-0.0003, 0.0000)
i=11 x=0.7681 y=0.6948 f(x,y)=(0.0001, 0.0000)
i=12 x=0.7682 y=0.6949 f(x,y)=(-0.0001, 0.0000)
i=13 x=0.7681 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=14 x=0.7682 y=0.6948 f(x,y)=(-0.0000, 0.0000)
i=15 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=16 x=0.7682 y=0.6948 f(x,y)=(-0.0000, 0.0000)
i=17 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=18 x=0.7682 y=0.6948 f(x,y)=(-0.0000, 0.0000)
i=19 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
i=20 x=0.7682 y=0.6948 f(x,y)=(0.0000, 0.0000)
```

We can implement our example in a few lines of code

```
from math import *

def f(x, y):
    eq1 = cos(y) - x
    eq2 = sin(x) - y
    return (eq1, eq2)

def g(x, y):
    x = cos(y)
    y = sin(x)
    return (x, y)

def fpi(x0, y0, n):
    '''Solves the system of equations by fixed point iteration
    starting at x0 and stopping after n iterations. Also
    includes an auxiliary function f to test at each value.'''
    x = x0
    y = y0
    for i in range(n):
        ox, oy = f(x,y)
        print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
        x,y = g(x,y)
    i += 1
    print("i=%2d x=%.4f y=%.4f f(x,y)=(%.4f, %.4f)" % (i, x, y, ox, oy))
    return x,y

if __name__ == "__main__":
    x,y = fpi(-1, -1, 20)
```

# VALUE ITERATION

# RL Terminology

**Question:** Match each term (on the left) to the corresponding statement or definition (on the right)

## Terms:

- A. a reward function
- B. a transition probability
- C. a policy
- D. state/action/reward triples
- E. a value function
- F. transition function
- G. an optimal policy
- H. Matt's favorite statement

## Statements:

- 1. gives the expected future discounted reward of a state
- 2. maps from states to actions
- 3. quantifies immediate success of agent
- 4. is a deterministic map from state/action pairs to states
- 5. quantifies the likelihood of landing a new state, given a state/action pair
- 6. is the desired output of an RL algorithm
- 7. can be influenced by trading off between exploitation/exploration

# RL: Optimal Value Function & Policy

- Bellman Equations:
- Optimal policy:
  - Given  $V^*$ ,  $R(s, a)$ ,  $p(s' | s, a)$ ,  $\gamma$  we can compute this!
- Optimal value function:
  - System of  $|\mathcal{S}|$  equations and  $|\mathcal{S}|$  variables (each variable is some  $V^*(s)$  for some state  $s$ )
  - Can be written without  $\pi^*$



# RL: Optimal Value Function & Policy

- Bellman Equations:

$$V^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s' | s, \pi(s)) V^\pi(s')$$

- Optimal policy:

- Given  $V^*$ ,  $R(s, a)$ ,  $p(s' | s, a)$ ,  $\gamma$  we can compute this!

$$\pi^*(s) = \operatorname{argmax}_{a \in \mathcal{A}} \underbrace{R(s, a)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in \mathcal{S}} p(s' | s, a) V^*(s')}_{\text{(Discounted) Future reward}}$$

- Optimal value function:

$$V^*(s) = \max_{a \in \mathcal{A}} R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s' | s, a) V^*(s')$$

- System of  $|\mathcal{S}|$  equations and  $|\mathcal{S}|$  variables (each variable is some  $V^*(s)$  for some state  $s$ )
- Can be written without  $\pi^*$

# Example: Path Planning

# Value Iteration

**Algorithm:**

**Example:**

# Value Iteration

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**Algorithm 1** Value Iteration

---

1: **procedure** VALUEITERATION( $R(s, a)$  reward function,  $p(\cdot|s, a)$  transition probabilities)  
2:     Initialize value function  $V(s) = 0$  or randomly  
3:     **while** not converged **do**  
4:         **for**  $s \in \mathcal{S}$  **do**  
5:              $V(s) = \max_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')$   
6:         Let  $\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')$ ,  $\forall s$   
7:     **return**  $\pi$

---

Variant 1: without  $Q(s, a)$  table

# Value Iteration

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## Algorithm 1 Value Iteration

---

```
1: procedure VALUEITERATION( $R(s, a)$  reward function,  $p(\cdot|s, a)$ 
   transition probabilities)
2:   Initialize value function  $V(s) = 0$  or randomly
3:   while not converged do
4:     for  $s \in \mathcal{S}$  do
5:       for  $a \in \mathcal{A}$  do
6:          $Q(s, a) = R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')$ 
7:          $V(s) = \max_a Q(s, a)$ 
8:   Let  $\pi(s) = \operatorname{argmax}_a Q(s, a)$ ,  $\forall s$ 
9:   return  $\pi$ 
```

---

Variant 2: with  $Q(s, a)$  table

# Synchronous vs. Asynchronous Value Iteration

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## Algorithm 1 Asynchronous Value Iteration

---

```
1: procedure ASYNCHRONOUSVALUEITERATION( $R(s, a), p(\cdot|s, a)$ )
2:   Initialize value function  $V(s) = 0$  or randomly
3:   while not converged do
4:     for  $s \in \mathcal{S}$  do
5:        $V(s) = \max_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')$ 
6:   Let  $\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s'), \forall s$ 
7:   return  $\pi$ 
```

---

**asynchronous updates:** compute and update  $V(s)$  for each state one at a time

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## Algorithm 1 Synchronous Value Iteration

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```
1: procedure SYNCHRONOUSVALUEITERATION( $R(s, a), p(\cdot|s, a)$ )
2:   Initialize value function  $V(s)^{(0)} = 0$  or randomly
3:    $t = 0$ 
4:   while not converged do
5:     for  $s \in \mathcal{S}$  do
6:        $V(s)^{(t+1)} = \max_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')^{(t)}$ 
7:      $t = t + 1$ 
8:   Let  $\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s'), \forall s$ 
9:   return  $\pi$ 
```

---

**synchronous updates:** compute all the fresh values of  $V(s)$  from all the stale values of  $V(s)$ , then update  $V(s)$  with fresh values

# Value Iteration Convergence

very abridged

**Theorem 1** (Bertsekas (1989))

*$V$  converges to  $V^*$ , if each state is visited infinitely often*

Holds for both asynchronous and synchronous updates

**Theorem 2** (Williams & Baird (1993))

*if  $\max_s |V^{t+1}(s) - V^t(s)| < \epsilon$*

*then  $\max_s |V^{t+1}(s) - V^*(s)| < \frac{2\epsilon\gamma}{1-\gamma}, \forall s$*

Provides reasonable stopping criterion for value iteration

**Theorem 3** (Bertsekas (1987))

*greedy policy will be optimal in a finite number of steps (even if not converged to optimal value function!)*

Often greedy policy converges well before the value function

# **POLICY ITERATION**



# Policy Iteration

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**Algorithm 1** Policy Iteration

---

- 1: **procedure** POLICYITERATION( $R(s, a)$  reward function,  $p(\cdot|s, a)$  transition probabilities)
- 2:     Initialize policy  $\pi$  randomly
- 3:     **while** not converged **do**
- 4:         Solve Bellman equations for fixed policy  $\pi$

$$V^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, \pi(s)) V^\pi(s'), \quad \forall s$$

- 5:         Improve policy  $\pi$  using new value function

$$\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) V^\pi(s')$$

- 6:     **return**  $\pi$
-

# Policy Iteration

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## Algorithm 1 Policy Iteration

---

1: **procedure** POLICYITERATION( $R(s, a)$ ,  $\gamma$ ,  $p(\cdot|s, a)$   
transition probabilities)

2: Initialize policy  $\pi$  randomly

3: **while** not converged **do**

4: Solve Bellman equations for fixed policy  $\pi$

$$V^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, \pi(s)) V^\pi(s'), \quad \forall s$$

5: Improve policy  $\pi$  using new value function

$$\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) V^\pi(s')$$

6: **return**  $\pi$

---

Compute value function for fixed policy is easy

System of  $|\mathcal{S}|$  equations and  $|\mathcal{S}|$  variables

Greedy policy w.r.t. current value function

Greedy policy might **remain the same** for a particular state if there is no better action

# Policy Iteration Convergence

## Question:

How many policies are there for a finite sized state and action space?

## Question:

Suppose policy iteration is shown to improve the policy at every iteration. Can you bound the number of iterations it will take to converge? If yes, what is the bound? If no, why not?

# Value Iteration vs. Policy Iteration

- Value iteration requires  $O(|A| |S|^2)$  computation per iteration
- Policy iteration requires  $O(|A| |S|^2 + |S|^3)$  computation per iteration
- In practice, policy iteration converges in fewer iterations

---

## Algorithm 1 Value Iteration

```
1: procedure VALUEITERATION( $R(s, a)$  reward function,  $p(\cdot|s, a)$ 
   transition probabilities)
2:   Initialize value function  $V(s) = 0$  or randomly
3:   while not converged do
4:     for  $s \in \mathcal{S}$  do
5:        $V(s) = \max_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')$ 
6:   Let  $\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V(s')$ ,  $\forall s$ 
7:   return  $\pi$ 
```

---

---

## Algorithm 1 Policy Iteration

```
1: procedure POLICYITERATION( $R(s, a)$  reward function,  $p(\cdot|s, a)$ 
   transition probabilities)
2:   Initialize policy  $\pi$  randomly
3:   while not converged do
4:     Solve Bellman equations for fixed policy  $\pi$ 
       
$$V^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, \pi(s))V^\pi(s'), \forall s$$

5:     Improve policy  $\pi$  using new value function
       
$$\pi(s) = \operatorname{argmax}_a R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a)V^\pi(s')$$

6:   return  $\pi$ 
```

---

# **STOCHASTIC REWARDS AND VALUE ITERATION**

# Q&A

**Q:** What if the rewards are also stochastic?

**A:** No problem. Everything we've been doing here still works just fine.

The Q-Learning algorithm doesn't need to change at all.

Let's consider how value iteration would look slightly different though...

# RL: Components

## From the Environment (i.e. the MDP)

- State space,  $\mathcal{S}$
- Action space,  $\mathcal{A}$
- Reward function,  $R(s, a, s')$ ,  $R : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow \mathbb{R}$
- Transition probabilities,  $p(s' | s, a)$ 
  - Deterministic transitions:

$$p(s' | s, a) = \begin{cases} 1 & \text{if } \delta(s, a) = s' \\ 0 & \text{otherwise} \end{cases}$$

where  $\delta(s, a)$  is a transition function

Markov Assumption

$$p(s_{t+1} | s_t, a_t, \dots, s_1, a_1) \\ = p(s_{t+1} | s_t, a_t)$$

## From the Model

- Policy,  $\pi : \mathcal{S} \rightarrow \mathcal{A}$
- Value function,  $V^\pi : \mathcal{S} \rightarrow \mathbb{R}$ 
  - Measures the expected total payoff of starting in some state  $s$  and executing policy  $\pi$

# Markov Decision Processes (MDP)

In RL, the source of our data is an MDP:

1. Start in some initial state  $s_0 \in \mathcal{S}$
2. For time step  $t$ :
  1. Agent observes state  $s_t \in \mathcal{S}$
  2. Agent takes action  $a_t \in \mathcal{A}$  where  $a_t = \pi(s_t)$
  3. Agent receives reward  $r_t \in \mathbb{R}$  where  $r_t = R(s_t, a_t, s_{t+1})$
  4. Agent transitions to state  $s_{t+1} \in \mathcal{S}$  where  $s_{t+1} \sim p(s' | s_t, a_t)$
3. Total reward is  $\sum_{t=0}^{\infty} \gamma^t r_t$ 
  - The value  $\gamma$  is the “discount factor”, a hyperparameter  $0 < \gamma < 1$

- Makes the same Markov assumption we used for HMMs! The next state only depends on the current state and action.
- *Def.:* we **execute** a policy  $\pi$  by taking action  $a = \pi(s)$  when in state  $s$



# Optimal Value Function

For the optimal policy function  $\pi^*$  we can compute its **value function** as:

$$\begin{aligned} V^{\pi^*}(s) &= V^*(s) \\ &= \mathbb{E}[R(s_0, \pi^*(s_0), s_1) + \gamma R(s_1, \pi^*(s_1), s_2) \\ &\quad + \gamma^2 R(s_2, \pi^*(s_2), s_3) \cdots \mid s_0 = s, \pi^*]. \end{aligned}$$

This **optimal value function** can be represented recursively as:

$$V^*(s) = \max_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} p(s'|s, a) (R(s, a, s') + \gamma V^*(s')).$$

If  $R(s, a, s') = R(s, a)$  (deterministic transition), then we have the form:

$$V^*(s) = \max_{a \in \mathcal{A}} \left\{ R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) V^*(s') \right\}.$$

# Value Iteration

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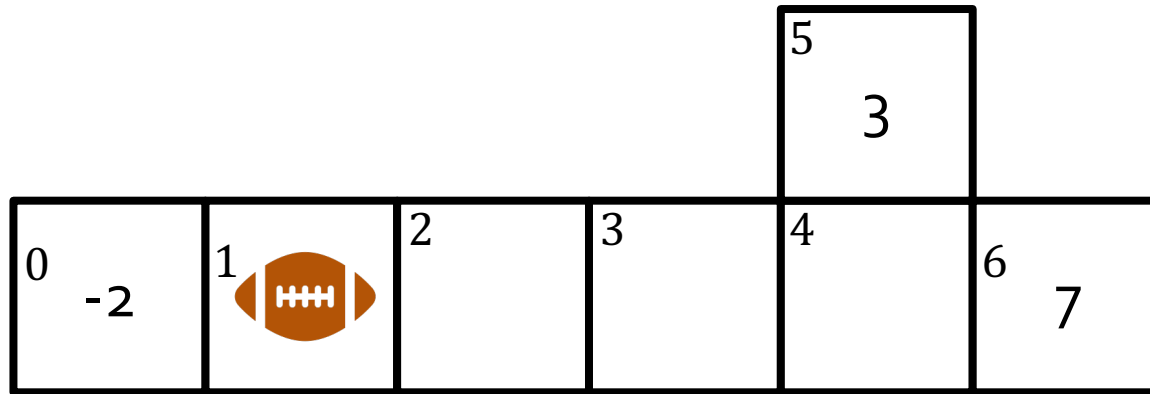
**Algorithm 1** Value Iteration with Stochastic Rewards

---

- 1: **procedure** VALUEITERATION( $R(s, a, s')$  reward function,  $p(\cdot|s, a)$  transition probabilities)
  - 2:     Initialize value function  $V(s) = 0$  or randomly
  - 3:     **while** not converged **do**
  - 4:         **for**  $s \in \mathcal{S}$  **do**
  - 5:              $V(s) = \max_a \sum_{s' \in \mathcal{S}} p(s'|s, a)(R(s, a, s') + \gamma V(s'))$
  - 6:     Let  $\pi(s) = \operatorname{argmax}_a \sum_{s' \in \mathcal{S}} p(s'|s, a)(R(s, a, s') + \gamma V(s'))$ ,  $\forall s$
  - 7:     **return**  $\pi$
- 

This is (more or less) fixed point iteration applied to the recursive definition of the optimal value function.

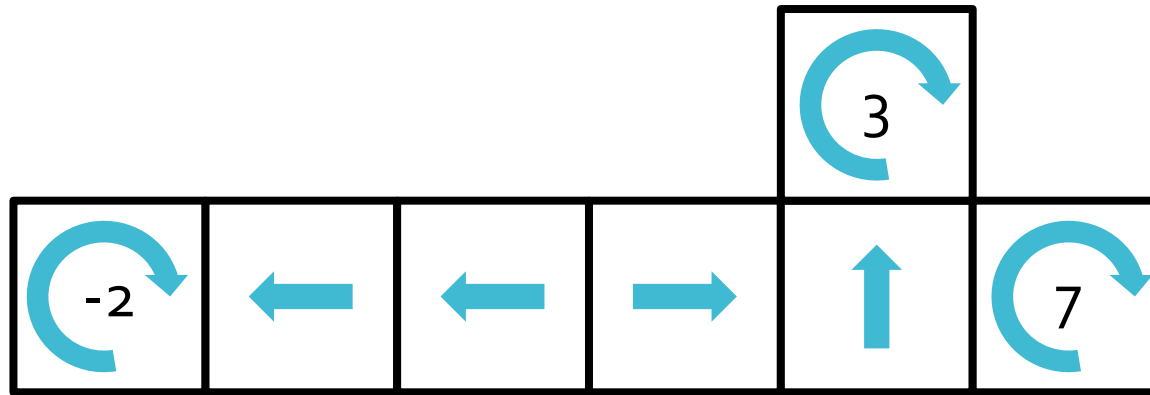
# RL: Value Function Example



$$R(s, a) = \begin{cases} -2 & \text{if entering state 0 (safety)} \\ 3 & \text{if entering state 5 (field goal)} \\ 7 & \text{if entering state 6 (touch down)} \\ 0 & \text{otherwise} \end{cases}$$

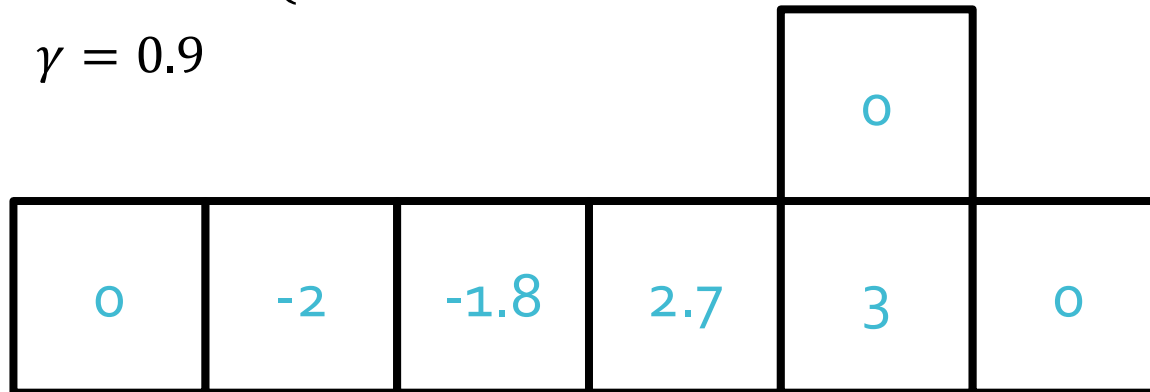
$$\gamma = 0.9$$

# RL: Value Function Example

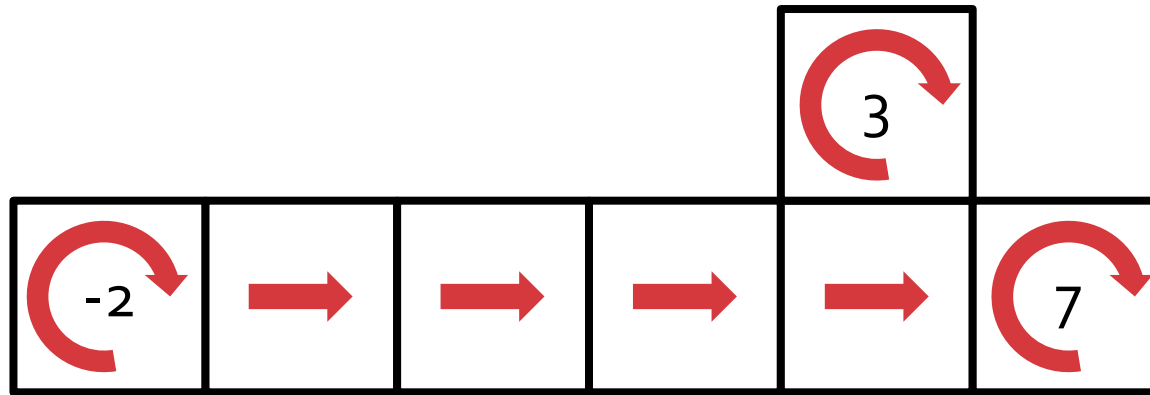


$$R(s, a) = \begin{cases} -2 & \text{if entering state 0 (safety)} \\ 3 & \text{if entering state 5 (field goal)} \\ 7 & \text{if entering state 6 (touch down)} \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma = 0.9$$

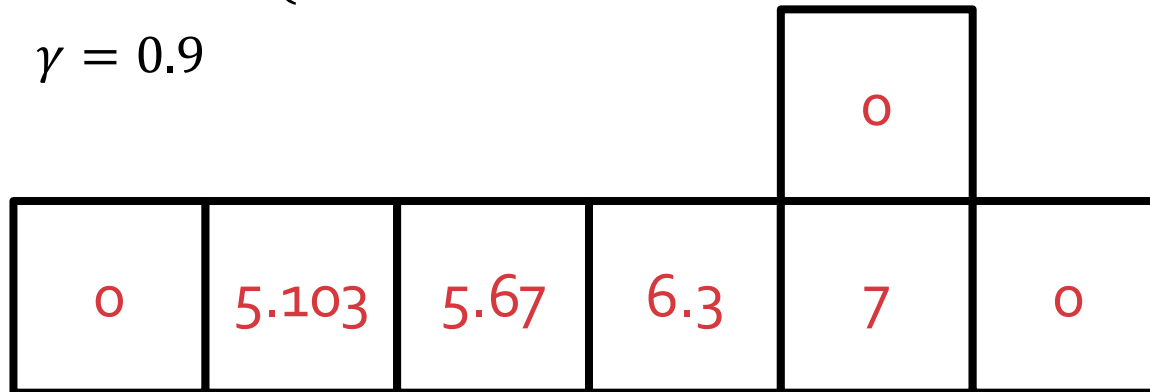


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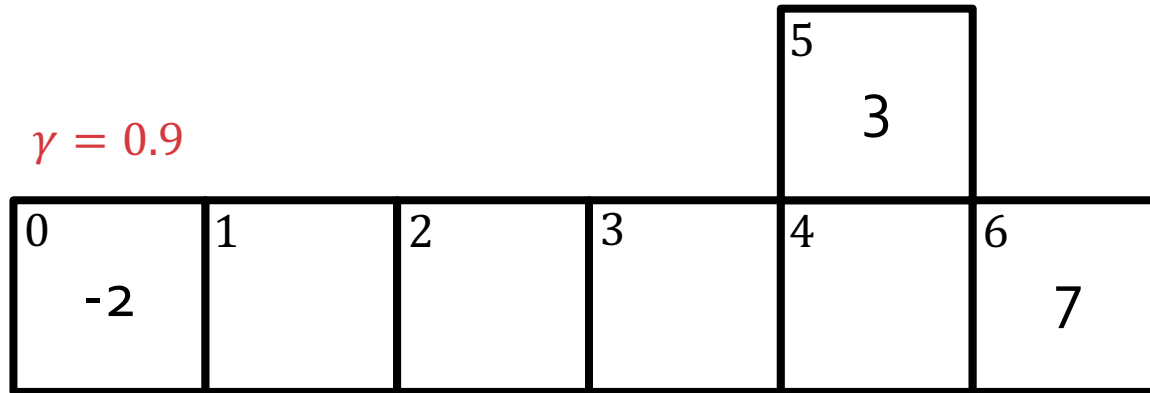


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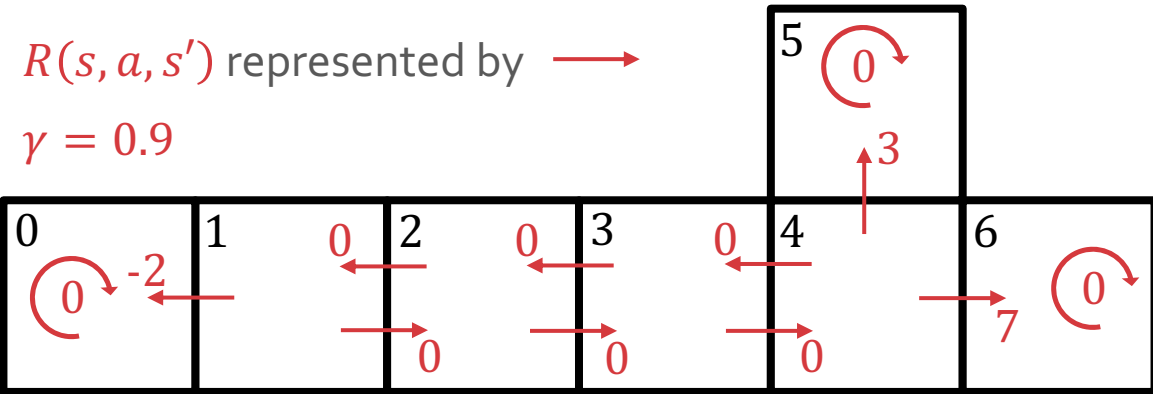


# Example: Stochastic Transitions and Rewards



$$R(s, a, s') = \begin{cases} -2 & \text{if entering state 0 (safety)} \\ 3 & \text{if entering state 5 (field goal)} \\ 7 & \text{if entering state 6 (touch down)} \\ 0 & \text{otherwise} \end{cases}$$

# Example: Stochastic Transitions and Rewards

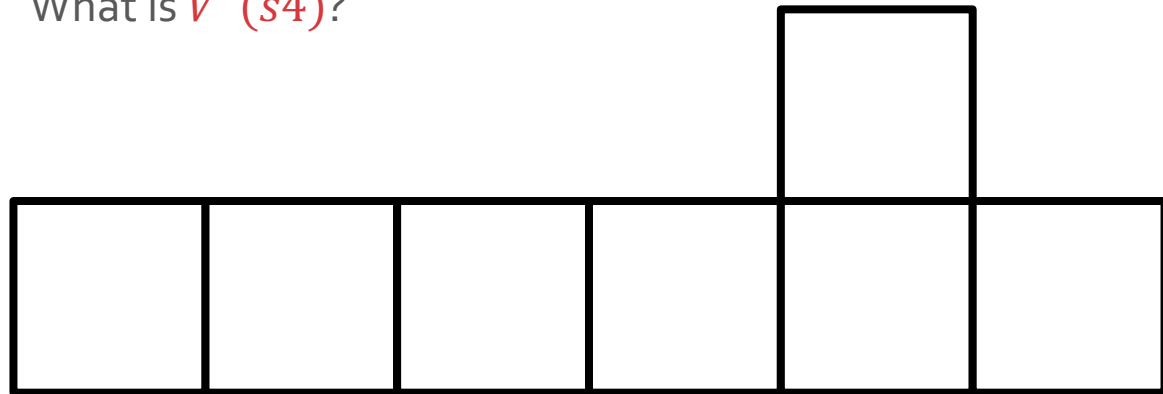


Suppose

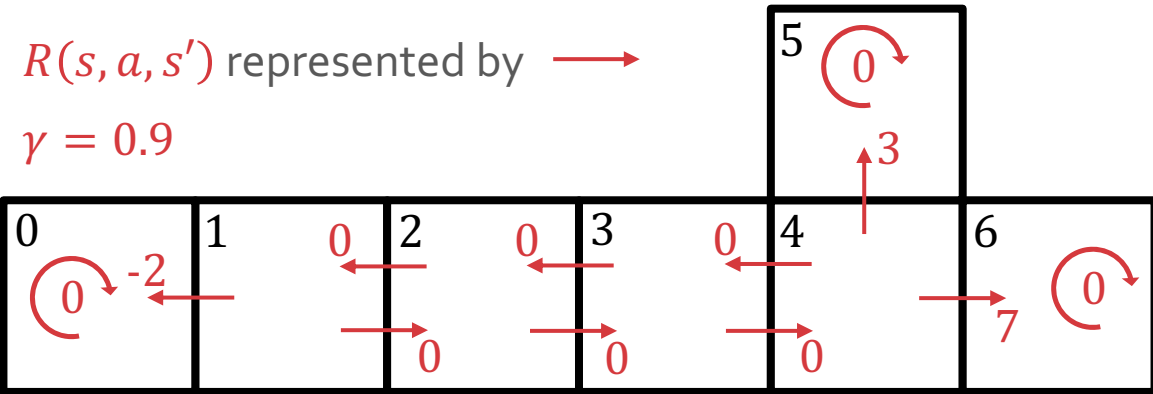
- $p(s_6 | s_4, a) = 0.5$
- $p(s_5 | s_4, a) = 0.5$

$\gamma = 0.9$

What is  $V^*(s_4)$ ?



# Example: Stochastic Transitions and Rewards

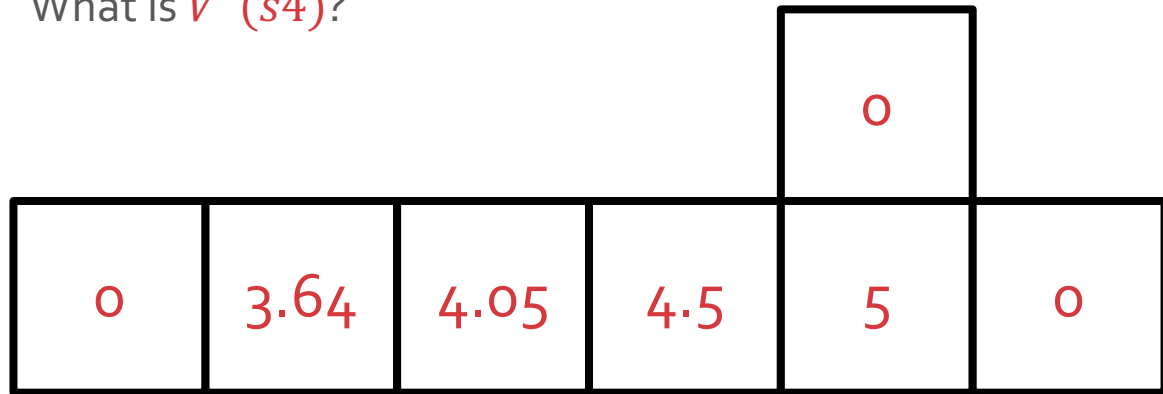


Suppose

- $p(s_6 | s_4, a) = 0.5$
- $p(s_5 | s_4, a) = 0.5$

$\gamma = 0.9$

What is  $V^*(s_4)$ ?





# Learning Objectives

## Reinforcement Learning: Value and Policy Iteration

*You should be able to...*

1. Compare the reinforcement learning paradigm to other learning paradigms
2. Cast a real-world problem as a Markov Decision Process
3. Depict the exploration vs. exploitation tradeoff via MDP examples
4. Explain how to solve a system of equations using fixed point iteration
5. Define the Bellman Equations
6. Show how to compute the optimal policy in terms of the optimal value function
7. Explain the relationship between a value function mapping states to expected rewards and a value function mapping state-action pairs to expected rewards
8. Implement value iteration
9. Implement policy iteration
10. Contrast the computational complexity and empirical convergence of value iteration vs. policy iteration
11. Identify the conditions under which the value iteration algorithm will converge to the true value function
12. Describe properties of the policy iteration algorithm