Maximum Satisfiability

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Automated Reasoning and Satisfiability
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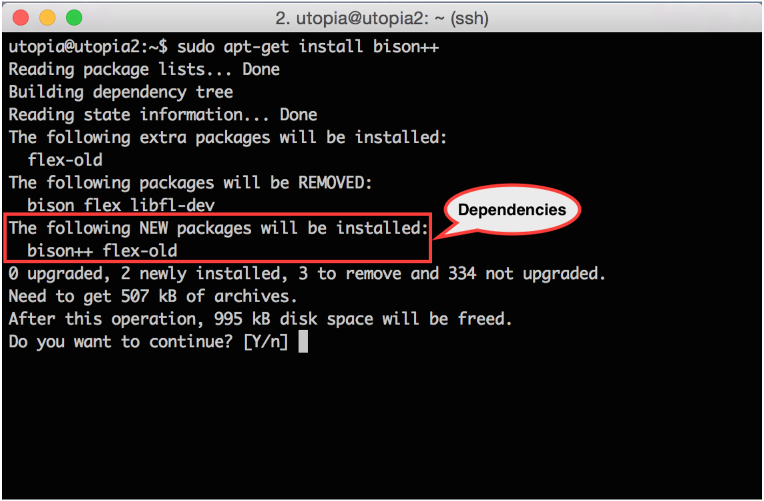
What is Boolean Satisfiability?

- **Fundamental problem** in Computer Science
  - The first problem to be proven NP-Complete
  - Has a wide range of applications
- Formula:
  - $\varphi = (\bar{x}_2 \vee \bar{x}_1) \wedge (x_2 \vee \bar{x}_3) \wedge (x_1) \wedge (x_3)$
- Boolean Satisfiability (SAT):
  - Is there an assignment of true or false values to variables such that φ evaluates to true?

Software Package Upgradeability Problem

```
2. utopia@utopia2: ~ (ssh)
utopia@utopia2:~$ sudo apt-get install bison++
Reading package lists... Done
Building dependency tree
Reading state information... Done
The following extra packages will be installed:
  flex-old
The following packages will be REMOVED:
  bison flex libfl-dev
The following NEW packages will be installed:
  bison++ flex-old
0 upgraded, 2 newly installed, 3 to remove and 334 not upgraded.
Need to get 507 kB of archives.
After this operation, 995 kB disk space will be freed.
Do you want to continue? [Y/n]
```

Software Package Upgradeability Problem



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Conflicts

Software Package Upgradeability Problem

Package	Dependencies	Conflicts
p_1	$\{p_2 \vee p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
p_4	$\{p_2 \wedge p_3\}$	$\{\}$

- Set of packages we want to install: $\{p_1, p_2, p_3, p_4\}$
- Each package p_i has a set of **dependencies**:
  - Packages that must be installed for p_i to be installed
- Each package p_i has a set of **conflicts**:
  - Packages that cannot be installed for p_i to be installed

NP Completeness

"I can't find an efficient algorithm, but neither can all these famous people."

NP Completeness

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- Giving up?
  - The problem is NP-hard, so let's develop heuristics or approximation algorithms.

NP Completeness

"I can't find an efficient algorithm, but neither can all these famous people."

- Giving up?
  - The problem is NP-hard, so let's develop heuristics or approximation algorithms.
- No! Current tools can find solutions for **very large** problems!

Software Package Upgradeability Problem as SAT

Package	Dependencies	Conflicts
p_1	$\{p_2 \vee p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
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How can we encode this problem to Boolean Satisfiability?

Software Package Upgradeability Problem as SAT

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How can we encode this problem to Boolean Satisfiability?

(Hint) Encode dependencies, conflicts, and installing all packages

Software Package Upgradeability Problem as SAT

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p_1	$\{p_2 \vee p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
p_4	$\{p_2 \wedge p_3\}$	$\{\}$

How can we encode this problem to Boolean Satisfiability?

- Encoding dependencies:
  - $p_1 \Rightarrow (p_2 \vee p_3) \equiv (\bar{p}_1 \vee p_2 \vee p_3)$
  - $p_2 \Rightarrow p_3 \equiv (\bar{p}_2 \vee p_3)$
  - $p_3 \Rightarrow p_2 \equiv (\bar{p}_3 \vee p_2)$
  - $p_4 \Rightarrow (p_2 \wedge p_3) \equiv (\bar{p}_4 \vee p_2) \wedge (\bar{p}_4 \vee p_3)$

Software Package Upgradeability Problem as SAT

Package	Dependencies	Conflicts
p_1	$\{p_2 \vee p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
p_4	$\{p_2 \wedge p_3\}$	$\{\}$

How can we encode this problem to Boolean Satisfiability?

- Encoding conflicts:
  - $p_1 \Rightarrow \bar{p}_4 \equiv (\bar{p}_1 \vee \bar{p}_4)$
  - $p_3 \Rightarrow \bar{p}_4 \equiv (\bar{p}_3 \vee \bar{p}_4)$

Software Package Upgradeability Problem as SAT

Package	Dependencies	Conflicts
p_1	$\{p_2 \vee p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
p_4	$\{p_2 \wedge p_3\}$	$\{\}$

How can we encode this problem to Boolean Satisfiability?

- Encoding installing all packages:
  - $(p_1) \wedge (p_2) \wedge (p_3) \wedge (p_4)$

Software Package Upgradeability Problem as SAT

Formula φ:

Dependencies $\quad \bar{p}_1 \vee p_2 \vee p_3 \quad \bar{p}_2 \vee p_3 \quad \bar{p}_3 \vee p_2$

Software Package Upgradeability Problem as SAT

Formula φ:

Dependencies	$\bar{p}_1 \vee p_2 \vee p_3$	$\bar{p}_2 \vee p_3$	$\bar{p}_3 \vee p_2$	
Conflicts	$\bar{p}_4 \vee p_2$	$\bar{p}_4 \vee p_3$	$\bar{p}_1 \vee \bar{p}_4$	$\bar{p}_3 \vee \bar{p}_4$

Software Package Upgradeability Problem as SAT

Formula φ:

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Packages	p_1	p_2	p_3	p_4

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Software Package Upgradeability Problem as SAT

Formula φ:

Dependencies	$\bar{p}_1 \vee p_2 \vee p_3$	$\bar{p}_2 \vee p_3$	$\bar{p}_3 \vee p_2$	
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Packages	p_1	p_2	p_3	p_4

- $\varphi = (\bar{p}_1 \vee p_2 \vee p_3) \wedge (\bar{p}_2 \vee p_3) \wedge (\bar{p}_3 \vee p_2) \wedge (\bar{p}_4 \vee p_2) \wedge (\bar{p}_4 \vee p_3) \wedge (\bar{p}_1 \vee \bar{p}_4) \wedge (\bar{p}_3 \vee \bar{p}_4) \wedge (p_1) \wedge (p_2) \wedge (p_3) \wedge (p_4)$

Software Package Upgradeability Problem as SAT

Formula φ:

Dependencies	$\bar{p}_1 \vee p_2 \vee p_3$	$\bar{p}_2 \vee p_3$	$\bar{p}_3 \vee p_2$	
Conflicts	$\bar{p}_4 \vee p_2$	$\bar{p}_4 \vee p_3$	$\bar{p}_1 \vee \bar{p}_4$	$\bar{p}_3 \vee \bar{p}_4$
Packages	p_1	p_2	p_3	p_4

- Formula is unsatisfiable 
- Can you find an unsatisfiable subformula?

(Hint) There are several with 3 clauses!

Software Package Upgradeability Problem as SAT

Formula φ:

Dependencies	$\bar{p}_1 \vee p_2 \vee p_3$	$\bar{p}_2 \vee p_3$	$\bar{p}_3 \vee p_2$	
Conflicts	$\bar{p}_4 \vee p_2$	$\bar{p}_4 \vee p_3$	$\bar{p}_1 \vee \bar{p}_4$	$\bar{p}_3 \vee \bar{p}_4$
Packages	p_1	p_2	p_3	p_4

- Formula is unsatisfiable
- We cannot install all packages
- How many packages can we install?

What is Maximum Satisfiability?

- Maximum Satisfiability (MaxSAT):
  - Clauses in the formula are either **soft** or **hard**
  - Hard clauses: **must** be satisfied
    (e.g. conflicts, dependencies)
  - Soft clauses: **desirable** to be satisfied
    (e.g. package installation)
- **Goal**: Maximize number of satisfied soft clauses

How to encode Software Package Upgradeability?

Software Package Upgradeability problem as MaxSAT:

- What are the hard constraints?
  - **(Hint)** Dependencies, conflicts or installation packages?
- What are the soft constraints?
  - **(Hint)** Dependencies, conflicts or installation packages?

How to encode Software Package Upgradeability?

Software Package Upgradeability problem as MaxSAT:

- What are the hard constraints?
  - Dependencies and conflicts
- What are the soft constraints?
  - Installation of packages

Software Package Upgradeability Problem as MaxSAT

MaxSAT Formula:

φ_h (Hard):	$\bar{p}_1 \vee p_2 \vee p_3$	$\bar{p}_2 \vee p_3$	$\bar{p}_3 \vee p_2$	
	$\bar{p}_4 \vee p_2$	$\bar{p}_4 \vee p_3$	$\bar{p}_1 \vee \bar{p}_4$	$\bar{p}_3 \vee \bar{p}_4$
φ_s (Soft):	p_1	p_2	p_3	p_4

- Dependencies and conflicts are encoded as hard clauses
- Installation of packages are encoded as soft clauses
- **Goal:** maximize the number of installed packages

Software Package Upgradeability Problem as MaxSAT

MaxSAT Formula:

φ_h (Hard):	$\bar{p}_1 \vee p_2 \vee p_3$	$\bar{p}_2 \vee p_3$	$\bar{p}_3 \vee p_2$	
	$\bar{p}_4 \vee p_2$	$\bar{p}_4 \vee p_3$	$\bar{p}_1 \vee \bar{p}_4$	$\bar{p}_3 \vee \bar{p}_4$
φ_s (Soft):	p_1	p_2	p_3	p_4

- Dependencies and conflicts are encoded as hard clauses
- Installation of packages are encoded as soft clauses
- **Optimal solution** (3 out 4 packages are installed)

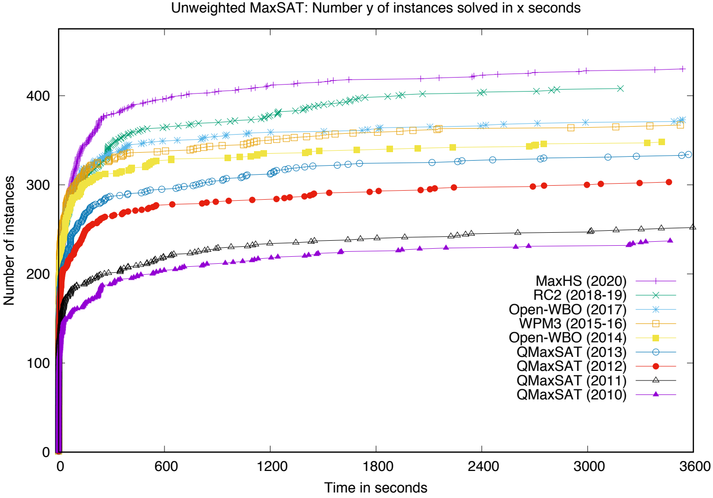
Why is MaxSAT Important?

- Many real-world applications can be encoded to MaxSAT:
  - Software package upgradeability
  - Error localization in C code

  - Wedding planning!

  - ...
- MaxSAT algorithms are **very effective** for solving real-word problems

The MaxSAT (r)evolution



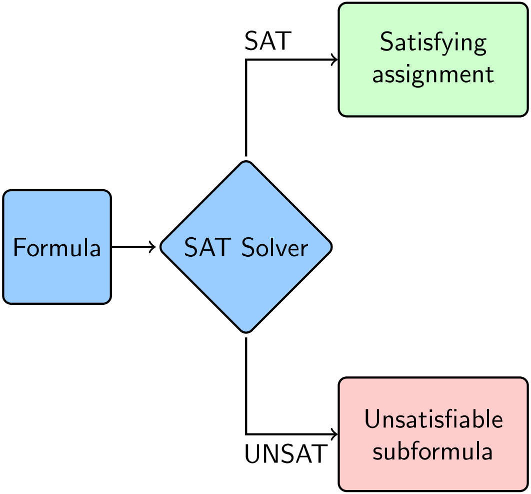
Comparing some of the best solvers from 2010-2020:

- In 2020: 81% more instances solved than in 2010!
- On same computer, same set of benchmarks

Outline

- MaxSAT Algorithms:
  - **Upper bound search** on the number of unsatisfied soft clauses
  - **Lower bound search** on the number of unsatisfied soft clauses
- **Partitioning** in MaxSAT:
  - Use the structure of the problem to guide the search
- Using MaxSAT solvers

SAT Solvers


SAT
Satisfying assignment
Formula
SAT Solver
UNSAT
Unsatisfiable subformula

Satisfying assignment

Formula:

$$x_1 \qquad x_2 \vee \bar{x}_1 \qquad \bar{x}_3 \vee x_1 \qquad \bar{x}_3 \vee \bar{x}_1 \qquad x_2 \vee \bar{x}_3$$

- Satisfying assignment:
  - Assignment to the variables that evaluates the formula to true

Satisfying assignment

Formula:

$$x_1 \quad x_2 \vee \bar{x}_1 \quad \bar{x}_3 \vee x_1 \quad \bar{x}_3 \vee \bar{x}_1 \quad x_2 \vee \bar{x}_3$$

- Satisfying assignment:
  - Assignment to the variables that evaluates the formula to true
  - $\mu = \{x_1 = 1, x_2 = 1, x_3 = 0\}$

Unsatisfiable subformula

Formula:

$$x_1 \qquad x_3 \qquad x_2 \lor \bar{x}_1 \qquad \bar{x}_3 \lor x_1 \qquad \bar{x}_2 \lor \bar{x}_1 \qquad x_2 \lor \bar{x}_3$$

- Formula is unsatisfiable

Unsatisfiable subformula

Formula:

$$x_1 \quad x_3 \quad x_2 \vee \bar{x}_1 \quad \bar{x}_3 \vee x_1 \quad \bar{x}_2 \vee \bar{x}_1 \quad x_2 \vee \bar{x}_3$$

- Formula is unsatisfiable
- Unsatisfiable subformula (core):
  - $\varphi' \subseteq \varphi$, such that φ' is unsatisfiable

MaxSAT Algorithms

- MaxSAT algorithms build on SAT solver technology
- MaxSAT algorithms use constraints not defined in causal form:
  - AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$
  - General cardinality constraints, $\sum_{j=1}^{n} x_j \leq k$
  - Pseudo-Boolean constraints, $\sum_{j=1}^{n} a_j x_j \leq k$
- Efficient encodings to CNF
  - Sinz, Totalizer, ...

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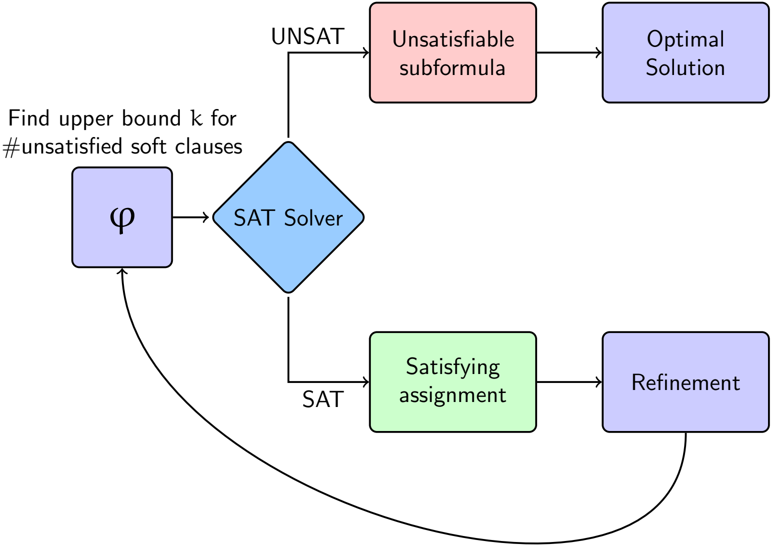
Upper Bound Search for MaxSAT



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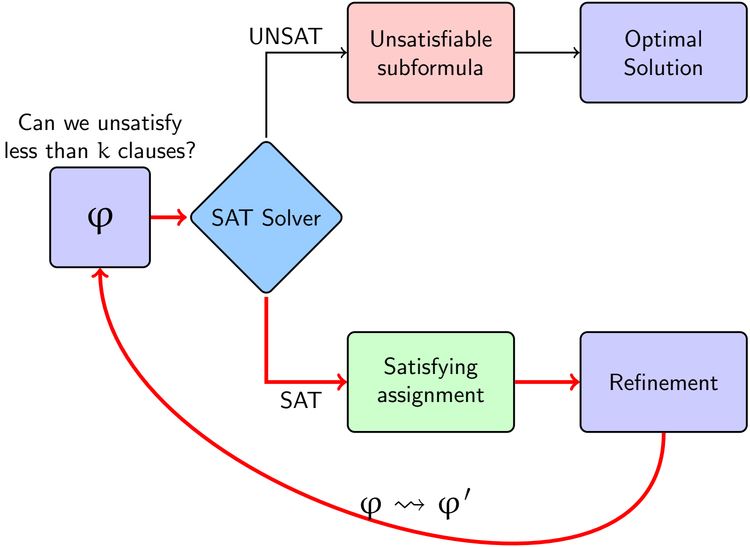

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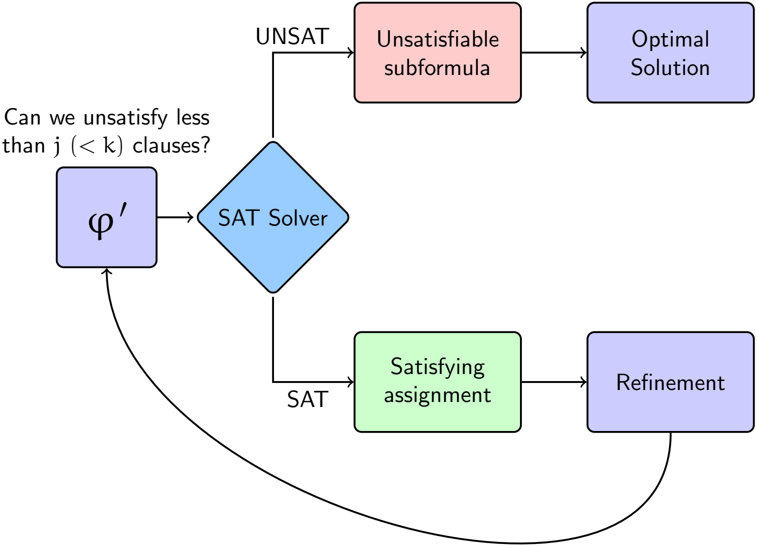

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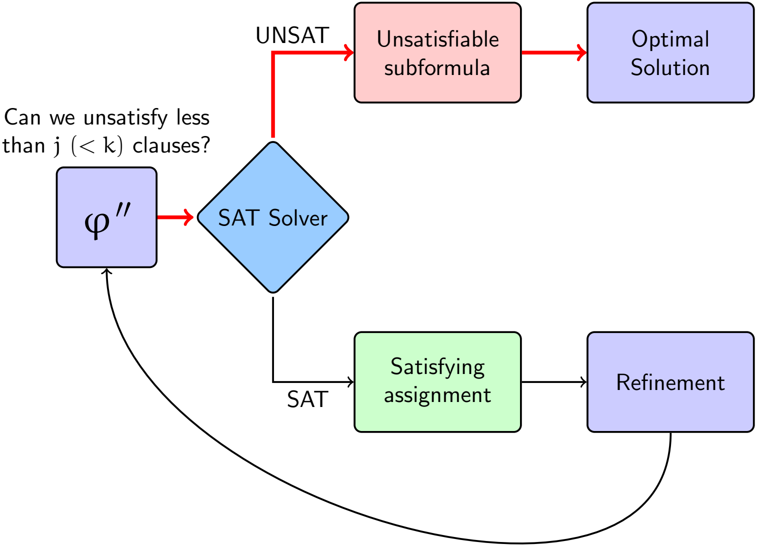

Linear Search Algorithms SAT-UNSAT

Partial MaxSAT Formula:

φ_h (Hard):		$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	
φ_s (Soft):	x_1	x_3	$x_2 \vee \bar{x}_1$	$\bar{x}_3 \vee x_1$

Linear Search Algorithms SAT-UNSAT

Partial MaxSAT Formula:

φ_h : $\bar{x}_2 \vee \bar{x}_1$ $x_2 \vee \bar{x}_3$

φ_s : $x_1 \vee r_1$ $x_3 \vee r_2$ $x_2 \vee \bar{x}_1 \vee r_3$ $\bar{x}_3 \vee x_1 \vee r_4$

- Relax all soft clauses
- Relaxation variables:
  - $V_R = \{r_1, r_2, r_3, r_4\}$
  - If a soft clause ω_i is **unsatisfied**, then $r_i = 1$
  - If a soft clause ω_i is **satisfied**, then $r_i = 0$

Linear Search Algorithms SAT-UNSAT

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$V_R = \{r_1, r_2, r_3, r_4\}$

- Formula is satisfiable
  - $\nu = \{x_1 = 1, x_2 = 0, x_3 = 0, r_1 = 0, r_2 = 1, r_3 = 1, r_4 = 0\}$
- **Goal:** Minimize number of relaxation variables assigned to 1

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Can we unsatisfy less than 2 soft clauses?

Partial MaxSAT Formula:

φ_h :		$\bar{x}_2 \lor \bar{x}_1$	$x_2 \lor \bar{x}_3$	
φ_s :	$x_1 \lor r_1$	$x_3 \lor r_2$	$x_2 \lor \bar{x}_1 \lor r_3$	$\bar{x}_3 \lor x_1 \lor r_4$

$\mu = 2 \qquad V_R = \{r_1, r_2, r_3, r_4\}$

- r_2 and r_3 were assigned truth value 1:
  - Current solution unsatisfies 2 soft clauses
- Can less than 2 soft clauses be unsatisfied?

Can we unsatisfy less than 2 soft clauses?

Partial MaxSAT Formula:

φ_h : $\quad \bar{x}_2 \vee \bar{x}_1 \quad x_2 \vee \bar{x}_3 \quad \text{CNF}(\sum_{r_i \in V_R} r_i \leq 1)$

φ_s : $\quad x_1 \vee r_1 \quad x_3 \vee r_2 \quad x_2 \vee \bar{x}_1 \vee r_3 \quad \bar{x}_3 \vee x_1 \vee r_4$

$\mu = 2 \qquad V_R = \{r_1, r_2, r_3, r_4\}$

- Add cardinality constraint that excludes solutions that unsatisfies 2 or more soft clauses:
  - $\text{CNF}(r_1 + r_2 + r_3 + r_4 \leq 1)$

Can we unsatisfy less than 2 soft clauses? No!

Partial MaxSAT Formula:

φ_h :	$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	$\text{CNF}(\sum_{r_i \in V_R} r_i \leq 1)$	
φ_s :	$x_1 \vee r_1$	$x_3 \vee r_2$	$x_2 \vee \bar{x}_1 \vee r_3$	$\bar{x}_3 \vee x_1 \vee r_4$

$\mu = 2 \qquad V_R = \{r_1, r_2, r_3, r_4\}$

- Formula is unsatisfiable:
  - There are no solutions that unsatisfy 1 or less soft clauses

Can we unsatisfy less than 2 soft clauses? No!

Partial MaxSAT Formula:

φ_h:		$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	
φ_s:	x_1	x_3	$x_2 \vee \bar{x}_1$	$\bar{x}_3 \vee x_1$

$\mu = 2 \qquad V_R = \{r_1, r_2, r_3, r_4\}$

- **Optimal solution**: given by the last model and corresponds to unsatisfying 2 soft clauses:
  - $\nu = \{x_1 = 1, x_2 = 0, x_3 = 0\}$

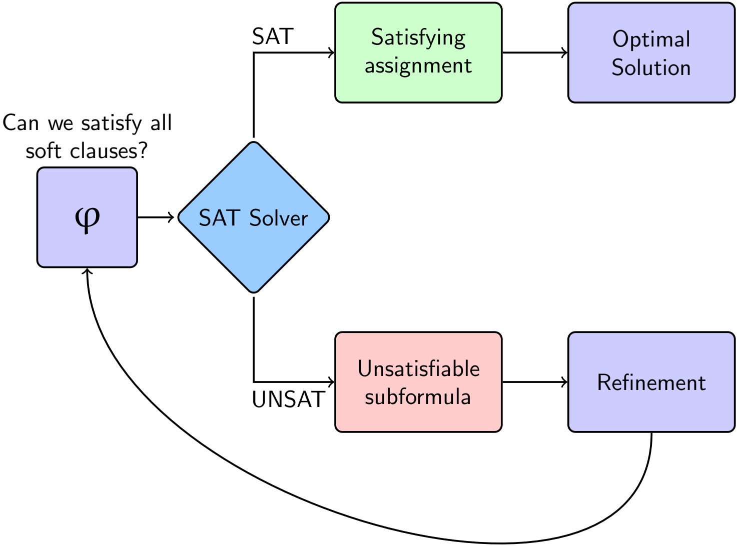
MaxSAT algorithms

- We have just seen a search on the **upper bound**
- What other kind of search can we do to find an optimal solution?

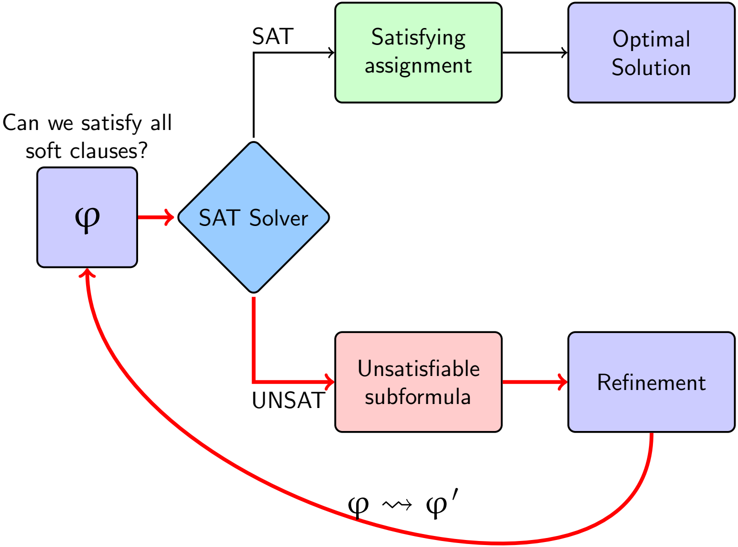
MaxSAT algorithms

- We have just seen a search on the **upper bound**
- What other kind of search can we do to find an optimal solution?
- What if we start searching from the **lower bound**?

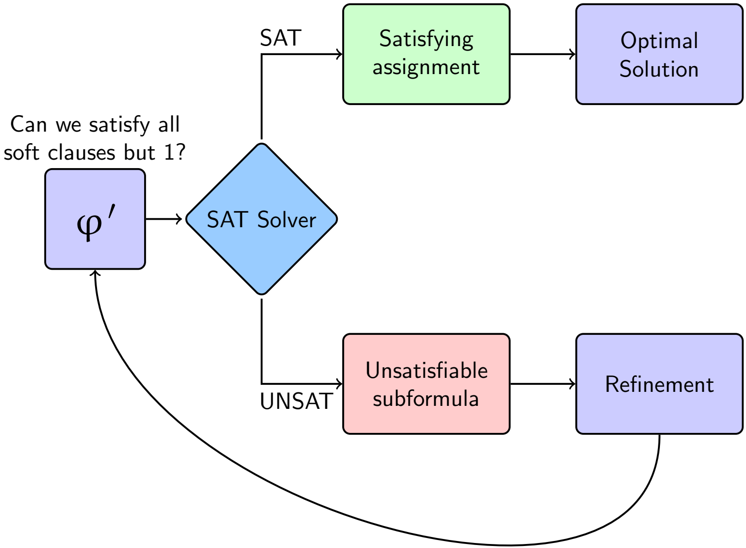
Lower Bound Search for MaxSAT



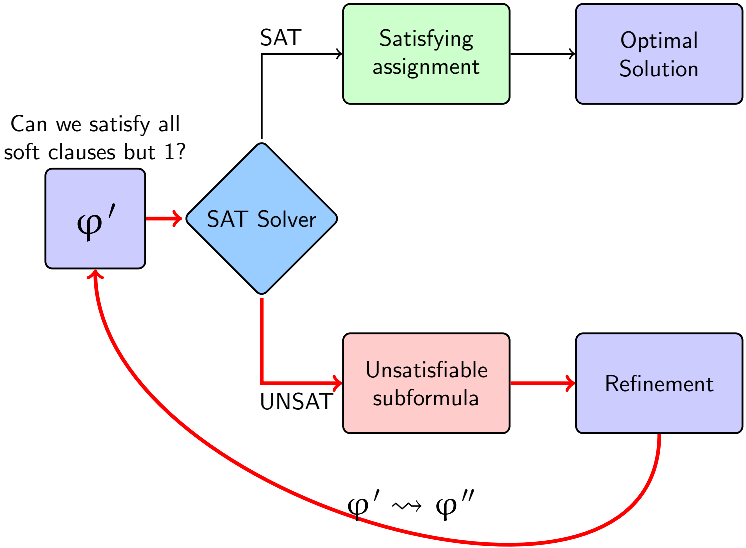
Lower Bound Search for MaxSAT



Lower Bound Search for MaxSAT



Lower Bound Search for MaxSAT



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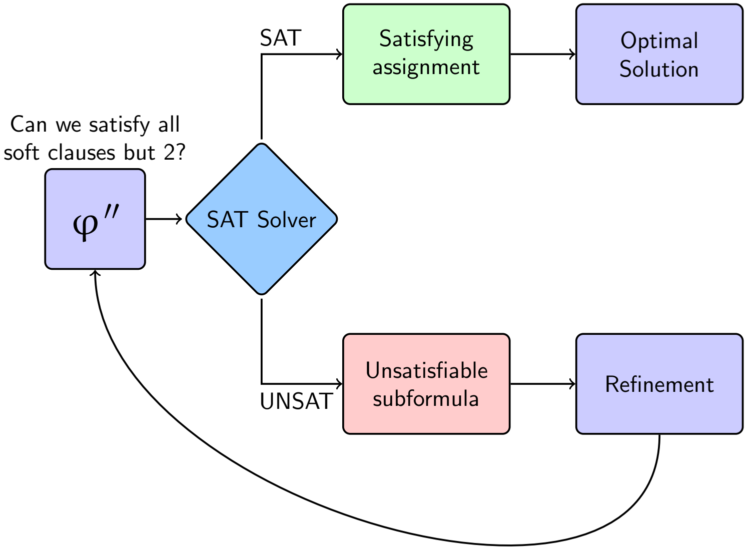
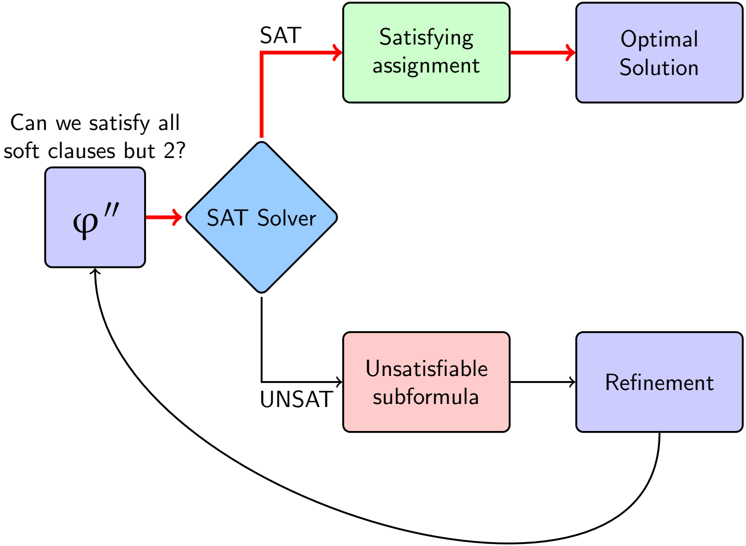


Lower Bound Search for MaxSAT



Linear Search Algorithms UNSAT-SAT

Partial MaxSAT Formula:

φ_h : $\bar{x}_2 \vee \bar{x}_1$ $\quad$ $x_2 \vee \bar{x}_3$

φ_s : $x_1 \vee r_1$ $\quad$ $x_3 \vee r_2$ $\quad$ $x_2 \vee \bar{x}_1 \vee r_3$ $\quad$ $\bar{x}_3 \vee x_1 \vee r_4$

- Relax all soft clauses
- Relaxation variables:
  - $V_R = \{r_1, r_2, r_3, r_4\}$
  - If a soft clause ω_i is **unsatisfied**, then $r_i = 1$
  - If a soft clause ω_i is **satisfied**, then $r_i = 0$

Can we satisfy all soft clauses?

Partial MaxSAT Formula:

φ_h : $\bar{x}_2 \vee \bar{x}_1$ $\quad x_2 \vee \bar{x}_3$ $\quad \text{CNF}(\sum_{r_i \in V_R} r_i \leq 0)$

φ_s : $x_1 \vee r_1$ $\quad x_3 \vee r_2$ $\quad x_2 \vee \bar{x}_1 \vee r_3$ $\quad \bar{x}_3 \vee x_1 \vee r_4$

$\mu = 2 \qquad V_R = \{r_1, r_2, r_3, r_4\}$

- Add cardinality constraint that excludes solutions that unsatisfies 1 or more soft clauses:
  - $\text{CNF}(r_1 + r_2 + r_3 + r_4 \leq 0)$

Can we satisfy all soft clauses but 1?

Partial MaxSAT Formula:

φ_h :	$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	$\text{CNF}(\sum_{r_i \in V_R} r_i \leq 0)$	
φ_s :	$x_1 \vee r_1$	$x_3 \vee r_2$	$x_2 \vee \bar{x}_1 \vee r_3$	$\bar{x}_3 \vee x_1 \vee r_4$

- Formula is unsatisfiable:
  - There are no solutions that unsatisfy 0 or less soft clauses
- Add cardinality constraint that excludes solutions that unsatisfies 2 or more soft clauses:
  - $\text{CNF}(r_1 + r_2 + r_3 + r_4 \leq 1)$

Can we satisfy all soft clauses but 2?

Partial MaxSAT Formula:

φ_h :	$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	$\text{CNF}(\sum_{r_i \in V_R} r_i \leq 1)$	
φ_s :	$x_1 \vee r_1$	$x_3 \vee r_2$	$x_2 \vee \bar{x}_1 \vee r_3$	$\bar{x}_3 \vee x_1 \vee r_4$

- Formula is unsatisfiable:
  - There are no solutions that unsatisfy 1 or less soft clauses
- Add cardinality constraint that excludes solutions that unsatisfies 3 or more soft clauses:
  - $\text{CNF}(r_1 + r_2 + r_3 + r_4 \leq 2)$

Can we satisfy all soft clauses but 2? Yes!

Partial MaxSAT Formula:

φ_h :	$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	CNF$(\sum_{r_i \in V_R} r_i \leq 2)$	
φ_s :	$x_1 \vee r_1$	$x_3 \vee r_2$	$x_2 \vee \bar{x}_1 \vee r_3$	$\bar{x}_3 \vee x_1 \vee r_4$

- Formula is satisfiable:
  - $\mu = \{x_1 = 1, x_2 = 0, x_3 = 0, r_1 = 0, r_2 = 1, r_3 = 1, r_4 = 0\}$
- Optimal solution unsatisfies 2 soft clauses

Unsatisfiability-based Algorithms

- What are the problems of this algorithm?
  (Hint) Number of relaxation variables? Size of the cardinality constraint? Other?

Unsatisfiability-based Algorithms

- What are the problems of this algorithm?
  (Hint) Number of relaxation variables? Size of the cardinality constraint? Other?
- We relax all soft clauses!
- The cardinality constraint contain as many literals as we have soft clauses!
- Can we do better?

Unsatisfiability-based Algorithms

Partial MaxSAT Formula:

φ_h (Hard):		$\bar{x}_2 \vee \bar{x}_1$	$x_2 \vee \bar{x}_3$	
φ_s (Soft):	x_1	x_3	$x_2 \vee \bar{x}_1$	$\bar{x}_3 \vee x_1$

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- Formula is unsatisfiable

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- Formula is unsatisfiable
- Identify an unsatisfiable core

Unsatisfiability-based Algorithms

Partial MaxSAT Formula:

$$\varphi_h: \quad \bar{x}_2 \vee \bar{x}_1 \quad x_2 \vee \bar{x}_3 \quad \text{CNF}(r_1 + r_2 \leq 1)$$

$$\varphi_s: \quad x_1 \vee r_1 \quad x_3 \vee r_2 \quad x_2 \vee \bar{x}_1 \quad \bar{x}_3 \vee x_1$$

- Relax non-relaxed soft clauses in unsatisfiable core:
  - Add cardinality constraint that excludes solutions that unsatisfies 2 or more soft clauses:
    - $\text{CNF}(r_1 + r_2 \leq 1)$
  - Relaxation on demand instead of relaxing all soft clauses eagerly

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Partial MaxSAT Formula:

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- Formula is satisfiable:
  - $\mu = \{x_1 = 1, x_2 = 0, x_3 = 0, r_1 = 0, r_2 = 1, r_3 = 1, r_4 = 0\}$
- Optimal solution unsatisfies 2 soft clauses

Unsatisfiability-based Algorithms

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Unsatisfiability-based Algorithms

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φ_s: $x_1 \vee r_1 \vee r_3$ $x_3 \vee r_2 \vee r_4$ $x_2 \vee \bar{x}_1 \vee r_5$ $\bar{x}_3 \vee x_1 \vee r_6$

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Challenges for Unsatisfiability-based MaxSAT Algorithms

- Unsatisfiable cores found by the SAT solver are **not minimal**

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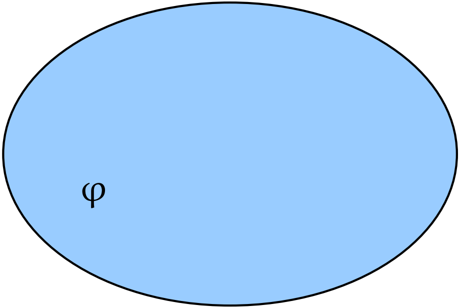

Formula φ

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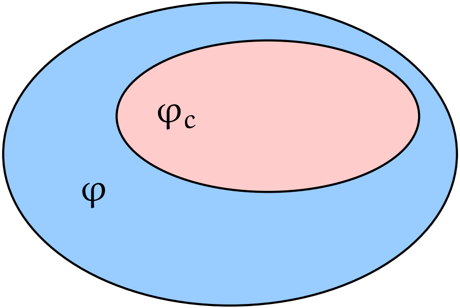

Formula φ

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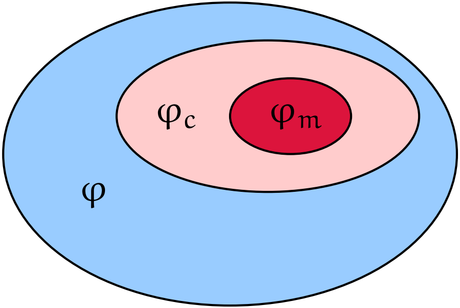

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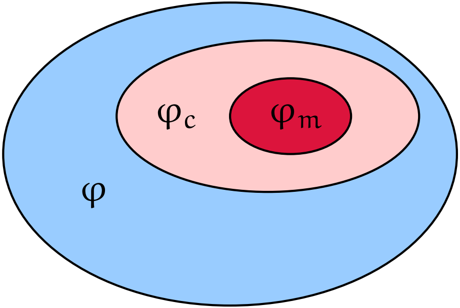


Formula φ

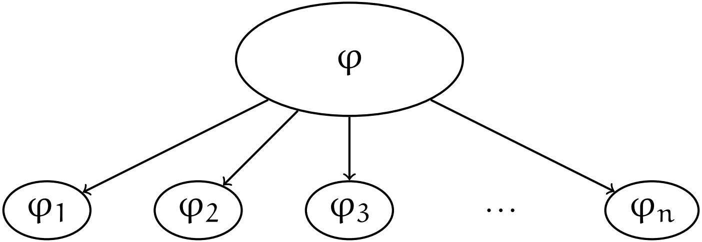
Unsatisfiable core φ_c

Minimal core φ_m

- **Minimizing** unsatisfiable cores is **computationally hard**

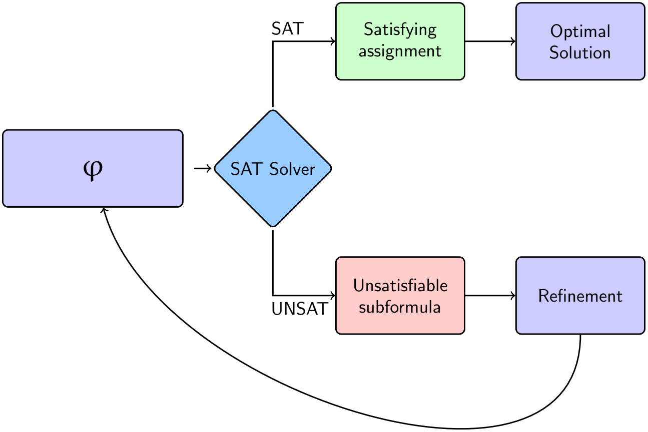
Partitioning in MaxSAT

- Partitioning in MaxSAT:
  - Partition the soft clauses into disjoint sets
  - Iteratively increase the size of the MaxSAT formula



- Advantages:
  - **Easier formulas** for the SAT solver
  - **Smaller unsatisfiable cores** at each iteration

Framework for Partitioning-based MaxSAT Algorithms



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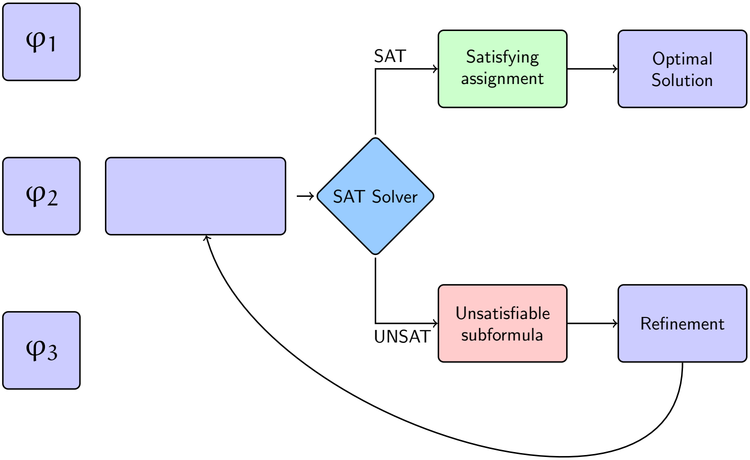

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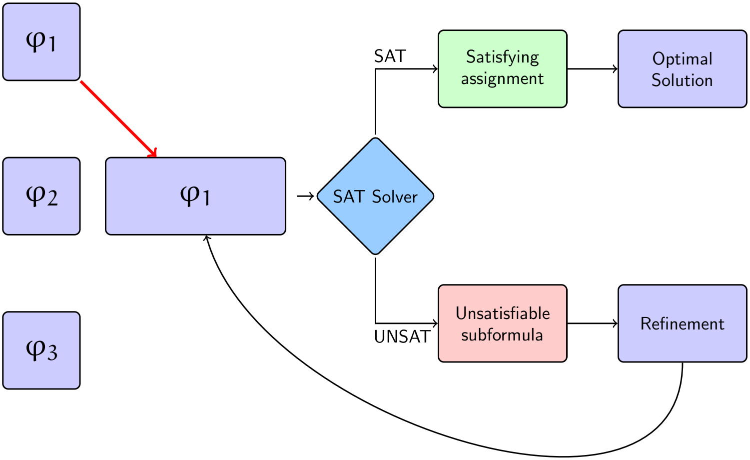

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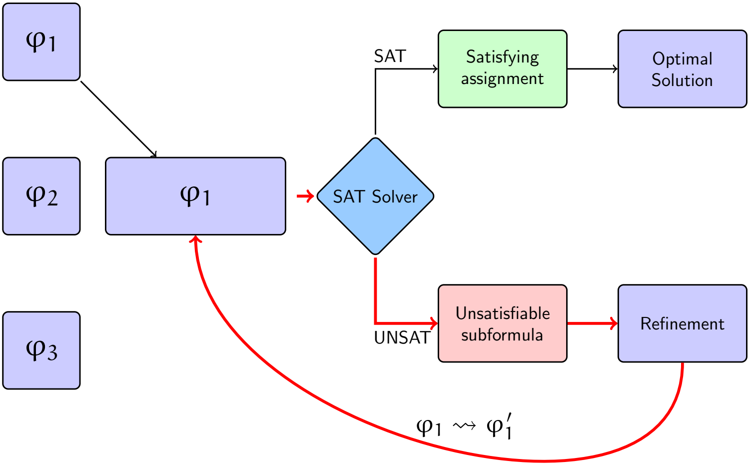

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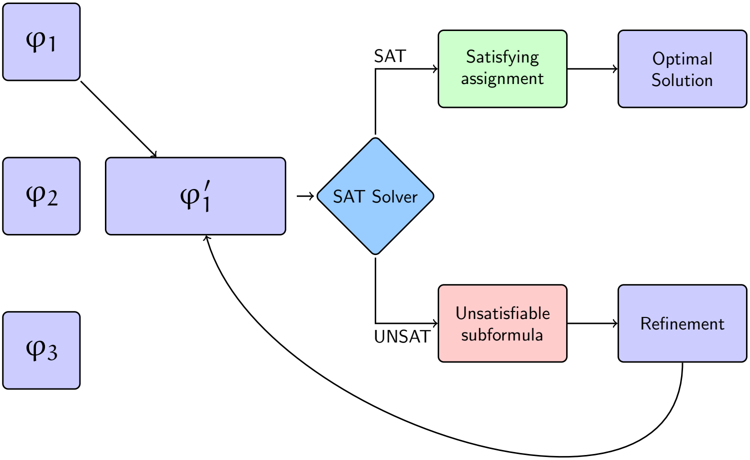

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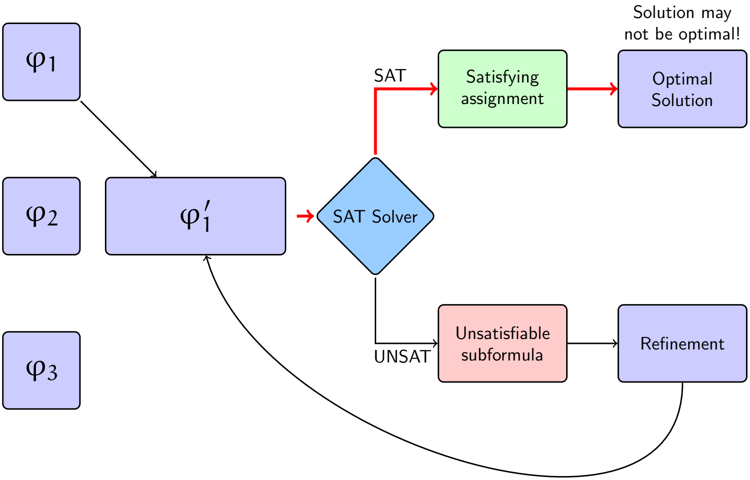

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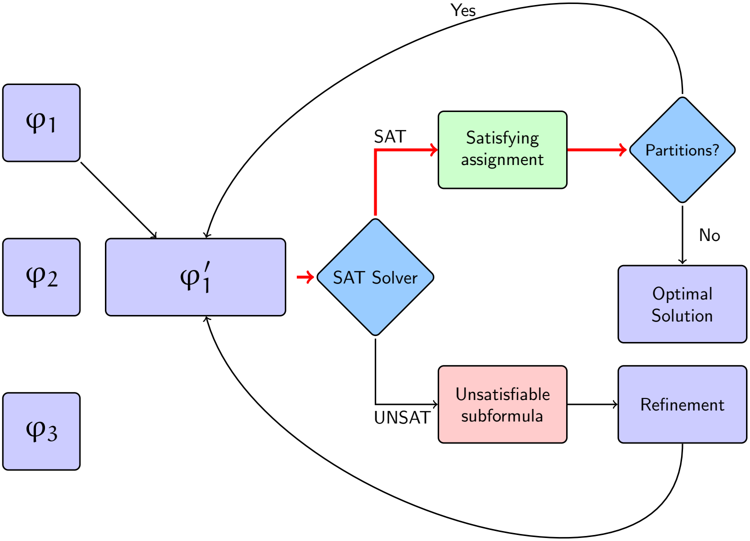

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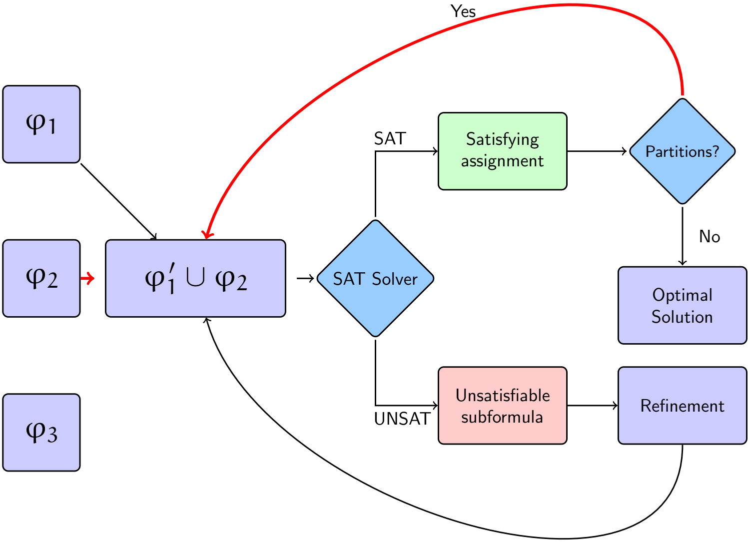

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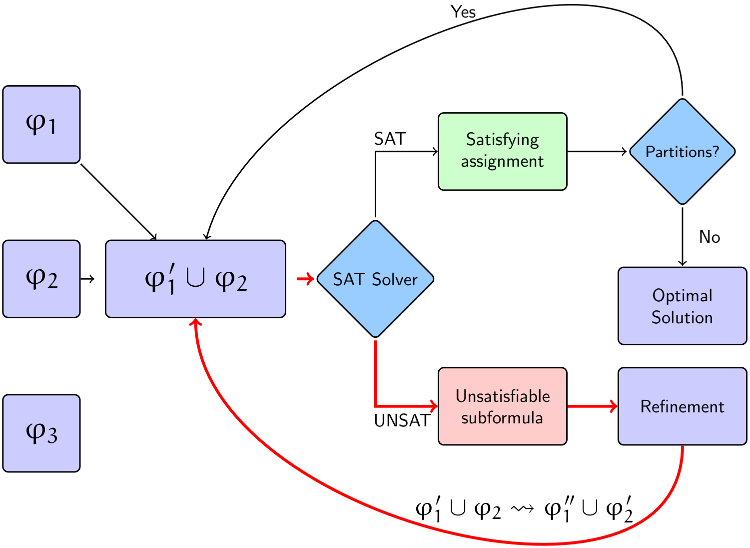

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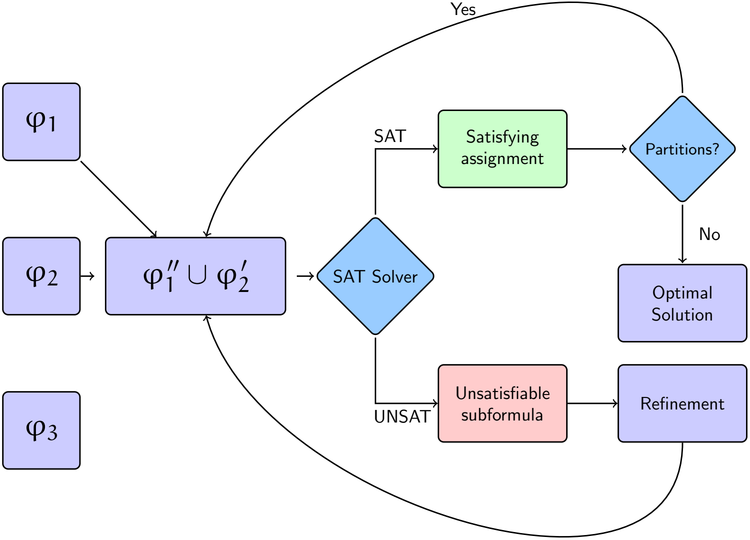

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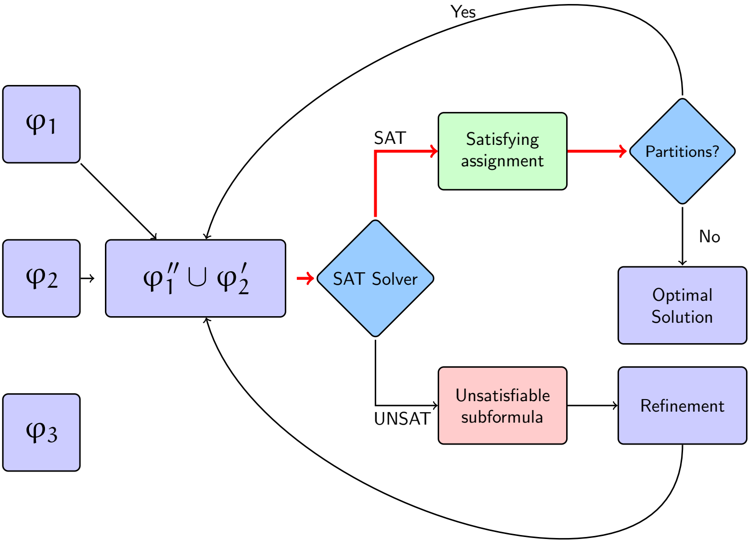

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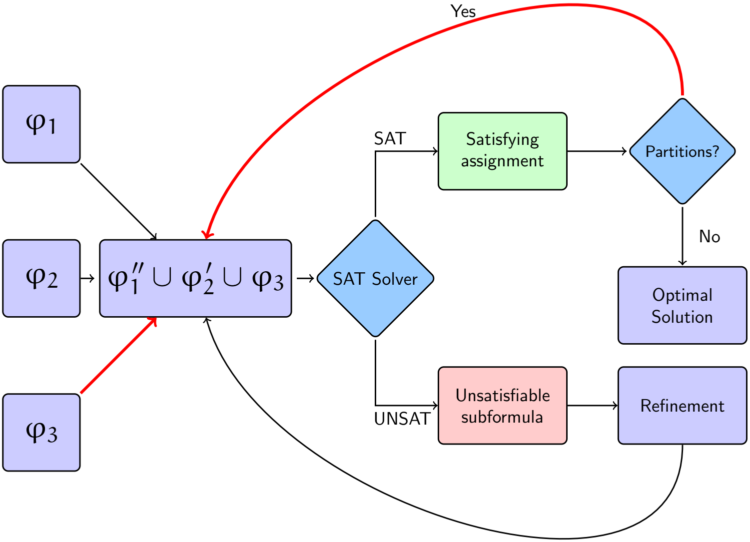

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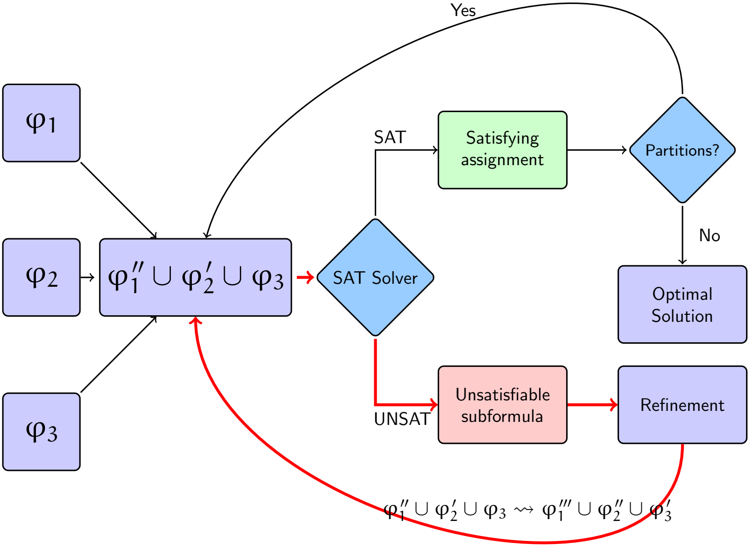

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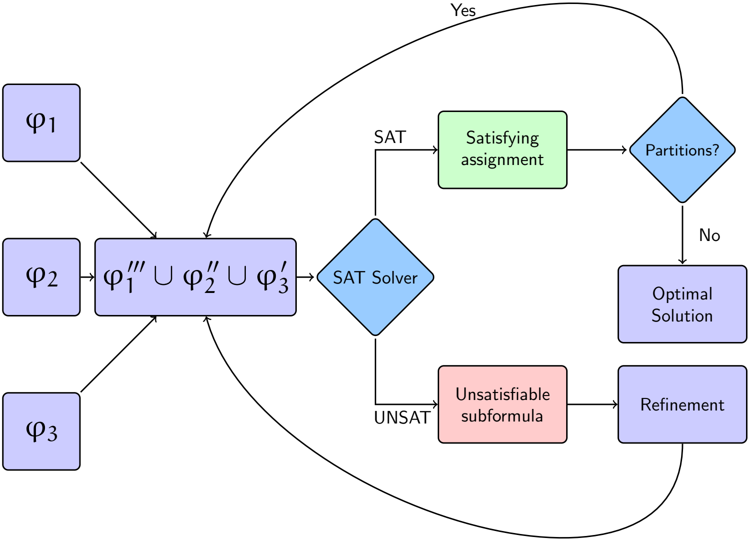

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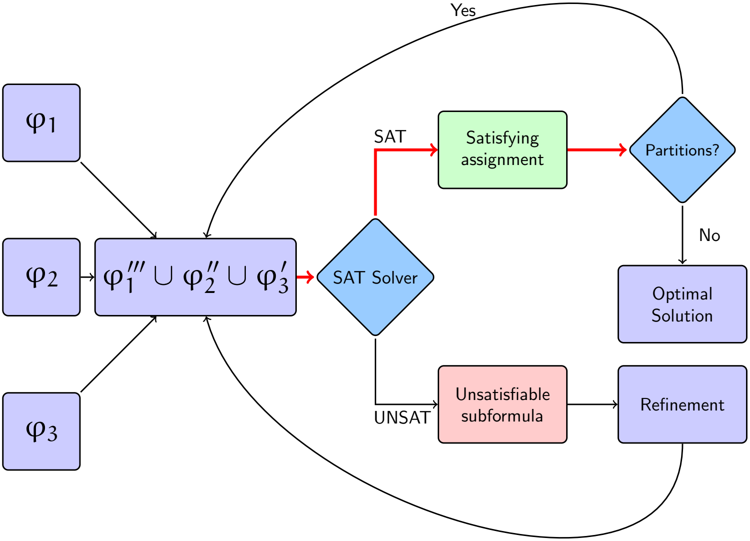

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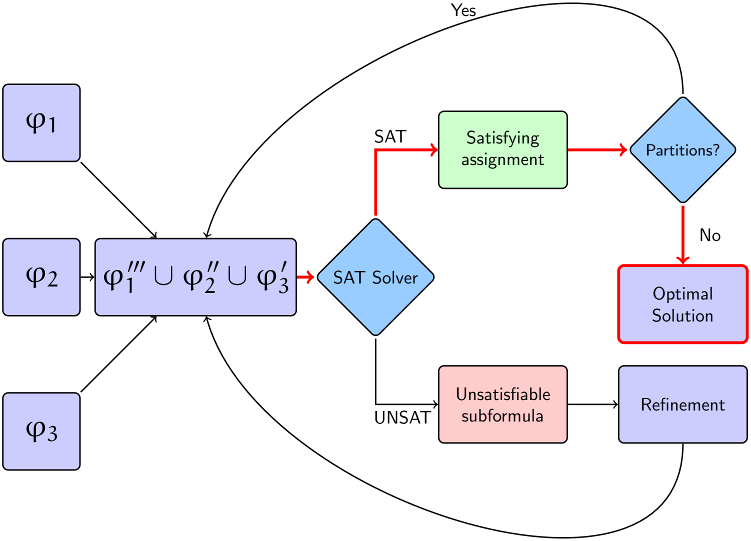
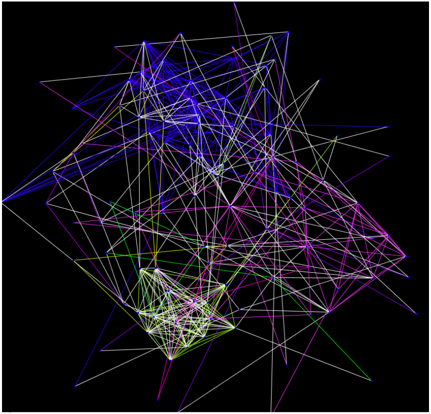


How to Partition Soft Clauses?

- **Graph representation** of the MaxSAT formula:
  - Vertices: Variables
  - Edges: Between variables that appear in the same clause



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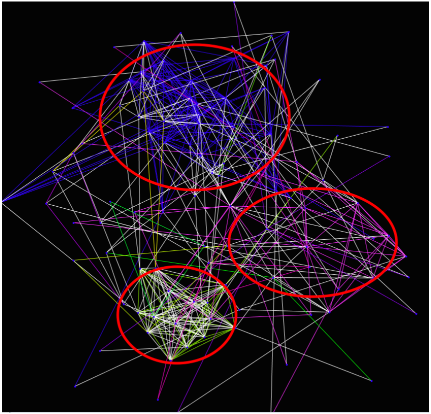

Graph representations for MaxSAT

- There are many ways to represent MaxSAT as a graph:
  - Clause-Variable Incidence Graph (CVIG)
  - Variable Incidence Graph (VIG)
  - Hypergraph
  - Resolution Graph
  - ...

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MaxSAT Formulas as Resolution-based Graphs

- MaxSAT solvers rely on the identification of **unsatisfiable cores**
- How can we capture sets of clauses that are closely related and are likely to result in unsatisfiable cores?
  - Represent MaxSAT formulas as **resolution graphs**!
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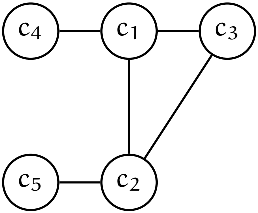

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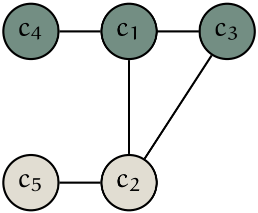
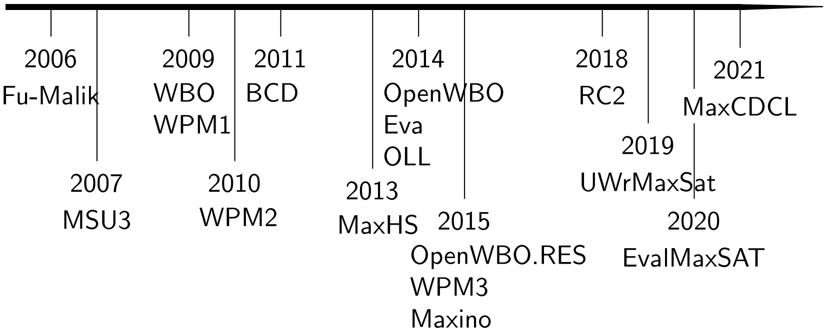


Unsatisfiability-based Algorithms

Timeline

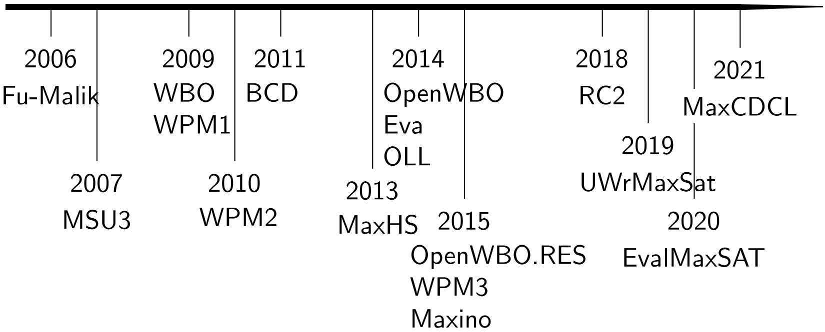


Fu-Malik

- First core-guided algorithm for MaxSAT
- Uses multiple relaxation variables per soft clause
- Only requires AtMost1 constraints

Unsatisfiability-based Algorithms

Timeline

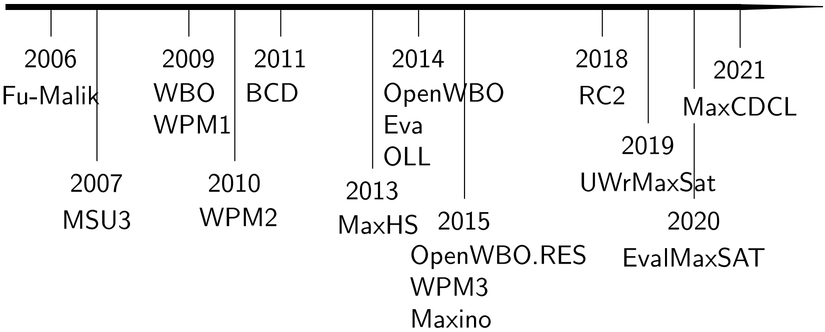


MSU3

- Uses one relaxation variable per soft clause
- Requires cardinality / pseudo-Boolean constraints

Unsatisfiability-based Algorithms

Timeline

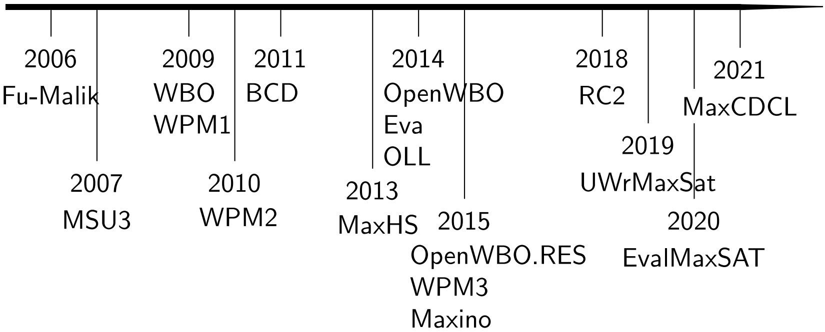


WBO
WPM1

- Generalizes Fu-Malik algorithm to weighted problems
- Efficient implementation of the Fu-Malik algorithm

Unsatisfiability-based Algorithms

Timeline

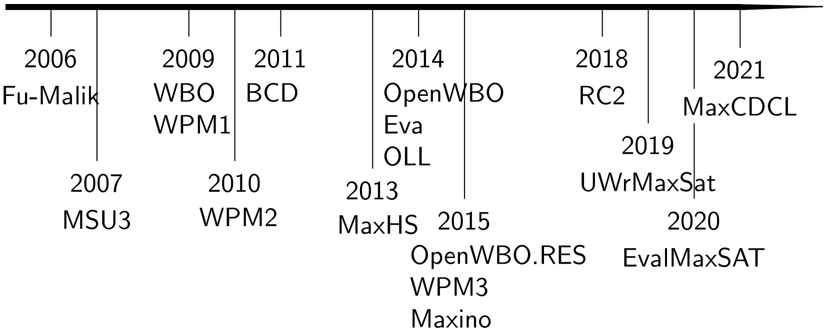


WPM2

- Only one relaxation per soft clause
- Group intersecting cores into disjoint covers
- Uses a cardinality constraint per cover

Unsatisfiability-based Algorithms

Timeline

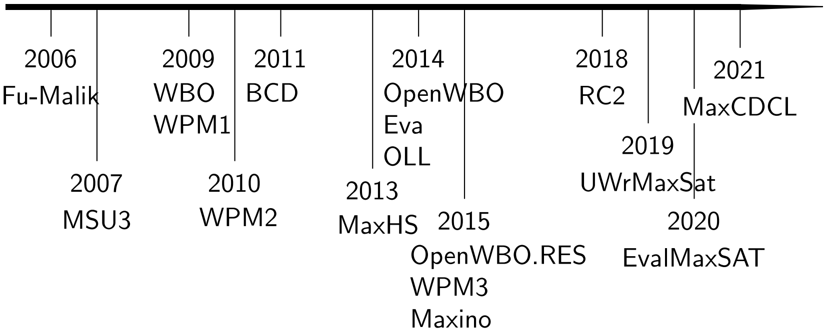


BCD

- Uses binary search in core-guided algorithms

Unsatisfiability-based Algorithms

Timeline

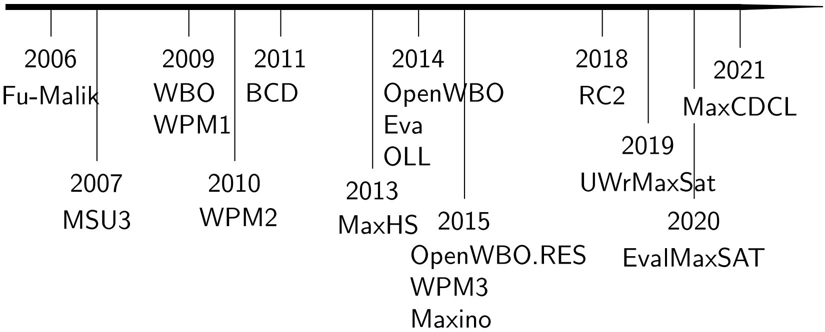


MaxHS

- Based on Hitting Sets
- Combines SAT and MIP solvers

Unsatisfiability-based Algorithms

Timeline

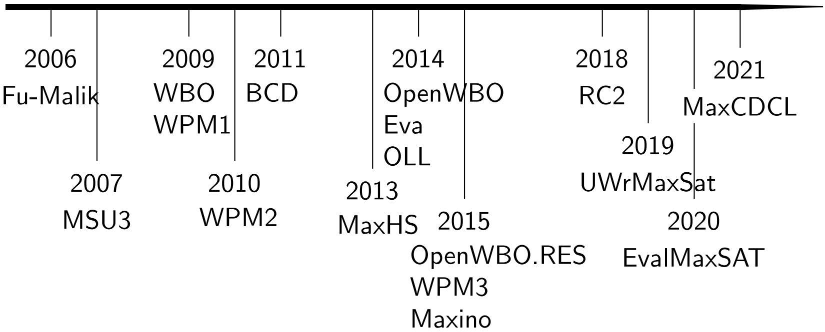


OpenWBO

- Improves the MSU3 algorithm with incremental construction of cardinality constraints
- Efficient implementation of the MSU3 algorithm

Unsatisfiability-based Algorithms

Timeline

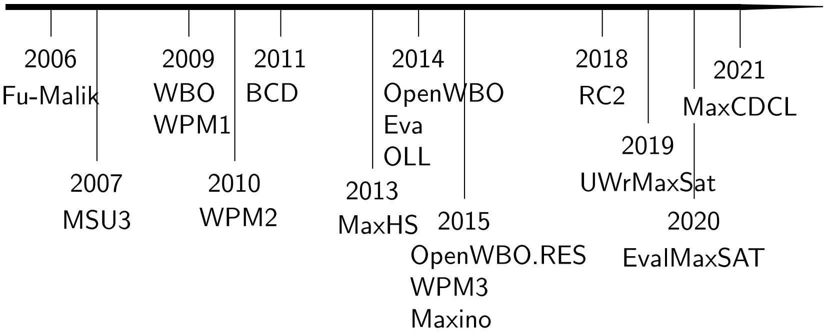


Eva

- Uses MaxSAT resolution to refine the formula instead of using AtMost1 constraints

Unsatisfiability-based Algorithms

Timeline

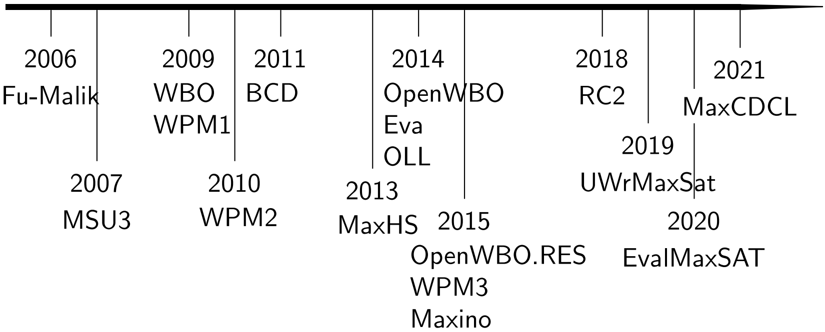


OLL
WPM3

- Introduce new variables to represent cardinality constraints
- $d = r_1 + r_2 + r_3 \leq 1$
- Soft clause (d, 1) is introduced

Unsatisfiability-based Algorithms

Timeline



OpenWBO.RES

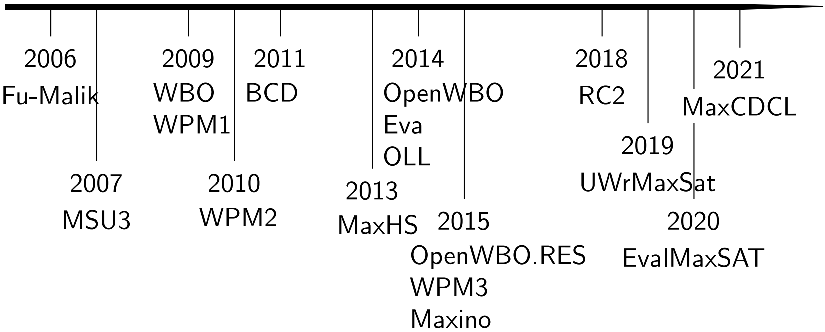
- Uses resolution-based graphs to partition soft clauses

OpenWBO.RES
Maxino

- Construction of the cardinality constraint uses core structure

Unsatisfiability-based Algorithms

Timeline

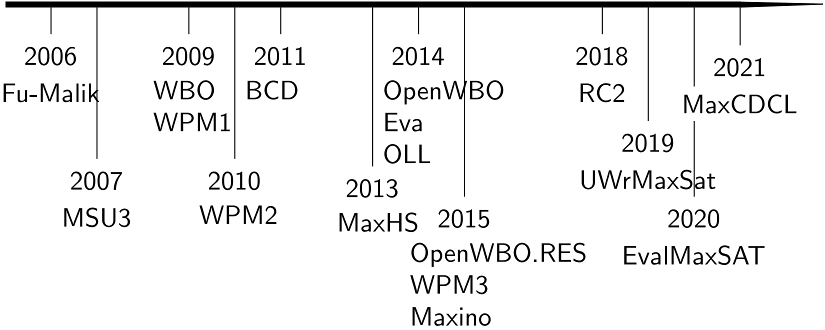


RC2
UWrMaxSat
EvalMaxSAT

- Efficient implementations of the OLL algorithm
- OLL algorithm is currently the most used one

Unsatisfiability-based Algorithms

Timeline



MaxCDCL

- Combines CDCL with Branch-and-Bound
- Lookahead estimate LB on the number of falsified soft clauses

Want to try MaxSAT solving?

- Java:
  - **SAT4J**
  - http://www.sat4j.org/
- Python:
  - **RC2**
  - Best solver in 2018 and 2019!
  - SAT solvers written in C++
  - https://pysathq.github.io
- http://maxsat-evaluations.github.io
  - Modify a solver today and enter this year competition!

Standard Solver Input Format: DIMACS WCNF

- Variables indexed from 1 to n
- Negation: -
  - -3 stands for $\bar{x}_3$
- 0: special end-of-line character
- One special header "p"-line:
  `p wcnf #vars #clauses top`
  - #vars: number of variables
  - #clauses: number of clauses
  - top: "weight" of hard clauses
- Clauses represented as lists of integers
  - Weight is the first number
  - $(\bar{x}_3 \vee x_1 \vee \bar{x}_{45})$, weight 2:
    `2 -3 1 -45 0`
- Clause is hard if weight is equal to top

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  - $(\bar{x}_3 \vee x_1 \vee \bar{x}_{45})$, weight 2:
    `2 -3 1 -45 0`
- Clause is hard if weight is equal to top
- New format removes header
  - Special symbol for hard clauses ('h')

Standard Solver Input Format: DIMACS WCNF

Example: pointer analysis domain (pa-2.wcnf):

```
p wcnf 17997976 23364255 9223372036854775807
142 -11393180 12091478 0
200 -12496389 -1068725 13170751 0
209 -8854604 -8854942 -8854943 -8253894 9864153 0
174 -9406753 -8105076 11844088 0
200 -10403325 -8104972 12524177 0
142 -11987544 12096893 0
37 -10981341 -10980973 10838652 0
209 -9578314 -9579250 -9579251 -8254733 9578317 0
209 -8868994 -8870298 -8870299 -8254157 8868997 0
209 -9387012 -9387508 -9387509 -8253943 9387015 0
174 -9834074 -8106628 12074710 0
200 -10726788 -8105074 12909526 0
...
9223372036854775807 -13181184 0
9223372036854775807 -13181215 0
...  truncated 763 MB
```

Standard Solver Input Format: DIMACS WCNF

Example: pointer analysis domain (pa-2.wcnf):

```
142 -11393180 12091478 0
200 -12496389 -1068725 13170751 0
209 -8854604 -8854942 -8854943 -8253894 9864153 0
174 -9406753 -8105076 11844088 0
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142 -11987544 12096893 0
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209 -9578314 -9579250 -9579251 -8254733 9578317 0
209 -8868994 -8870298 -8870299 -8254157 8868997 0
209 -9387012 -9387508 -9387509 -8253943 9387015 0
174 -9834074 -8106628 12074710 0
200 -10726788 -8105074 12909526 0
...
h -13181184 0
h -13181215 0
...
```

New simplified format!

Push-Button Solver Technology

Example: $ open-wbo pa-2.wcnf

Push-Button Solver Technology

Example: $ open-wbo pa-2.wcnf

```
c Open-WBO: a Modular MaxSAT Solver
c Version:  MaxSAT Evaluation 2016
c Authors:  Ruben Martins, Vasco Manquinho, Ines Lynce
c Contributors:  Miguel Neves, Saurabh Joshi, Mikolas Janota
...
c |Problem Type:  Weighted
c |Number of variables:  17,997,976
c |Number of hard clauses:  8,237,870
c |Number of soft clauses:  15,126,385
c |Parse time:  5.60 s
...
o 4699
o 4609
o 143
s OPTIMUM FOUND
c Total time:  361.26 s v 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15...
...17997976
```

Want to know more about MaxSAT?

Email me! Always happy to talk about MaxSAT!

- rubenm@cs.cmu.edu

Check the tutorial slides and webpage presented at ECAI'20:

- Advances in Maximum Satisfiability: https://ecai20-maxsat-tutorial.github.io/

Chapter in the 2nd edition of the Handbook of Satisfiability:

- Maximum Satisfiability: https://www.cs.cmu.edu/~rubenm/papers/p02c24-mxm.pdf

References

MaxSAT solvers:

[Fu-Malik] Z. Fu, S. Malik. On Solving the Partial MAX-SAT Problem. SAT 2006: 252-265.

[MSU3] J. Marques-Silva, J. Planes. On using unsatisfiability for solving Maximum Satisfiability. Technical report 2007

[WBO] V. Manquinho, J. Marques-Silva, J. Planes. Algorithms for Weighted Boolean Optimization. SAT 2009: 495-508

[WPM1] Carlos Ansótegui, Maria Luisa Bonet, and Jordi Levy. Solving (weighted) partial MaxSAT through satisfiability testing. SAT 2009: 427–440

[WPM2] Carlos Ansótegui, Maria Luisa Bonet, and Jordi Levy. A new algorithm for weighted partial MaxSAT. AAAI 2010

[BC2] Federico Heras, António Morgado, and Joao Marques-Silva. Core-guided binary search algorithms for maximum satisfiability. AAAI 2011

[OpenWBO] R. Martins, S. Joshi, V. Manquinho, I. Lynce. Incremental Cardinality Constraints for MaxSAT. CP 2014: 531-548

[OLL] António Morgado, Carmine Dodaro, and Joao Marques-Silva. Core-guided MaxSAT with soft cardinality constraints. CP 2014: 564–573

[OpenWBO.RES] R. Martins, V. Manquinho, I. Lynce. Exploiting Resolution-Based Representations for MaxSAT Solving. SAT 2015: 272-286

[MaxHS] Jessica Davies, Fahiem Bacchus: Postponing Optimization to Speed Up MAXSAT Solving. CP 2013: 247-262

[RC2] Alexey Ignatiev, António Morgado, Joao Marques-Silva: PySAT: A Python Toolkit for Prototyping with SAT Oracles. SAT 2018: 428-437

References

Cardinality and Pseudo-Boolean Encodings:

C. Sinz. Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. CP 2005: 827-831

N. Manthey, T. Philipp, P. Steinke. A More Compact Translation of Pseudo-Boolean Constraints into CNF Such That Generalized Arc Consistency Is Maintained. KI 2014: 123-134

T. Philipp, P. Steinke. PBLib - A Library for Encoding Pseudo-Boolean Constraints into CNF. SAT 2015: 9-16 `http://tools.computational-logic.org/content/pblib.php`

Community Structure:

C. Ansótegui, J. Giráldez-Cru, Jordi Levy. The Community Structure of SAT Formulas. SAT 2012: 410-423

Web pages of interest:

MaxSAT Evaluation: `http://www.maxsat.udl.cat/`
Open-WBO: `http://sat.inesc-id.pt/open-wbo/`
SAT4J: `http://www.sat4j.org/`
RC2: `https://pysathq.github.io`
MaxHS: `http://www.maxhs.org/`
SATGraf: `https://bitbucket.org/znewsham/satgraf`