Preprocessing Techniques

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Automated Reasoning and Satisfiability

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Motivation

Subsumption

Variable Elimination

Bounded Variable Addition

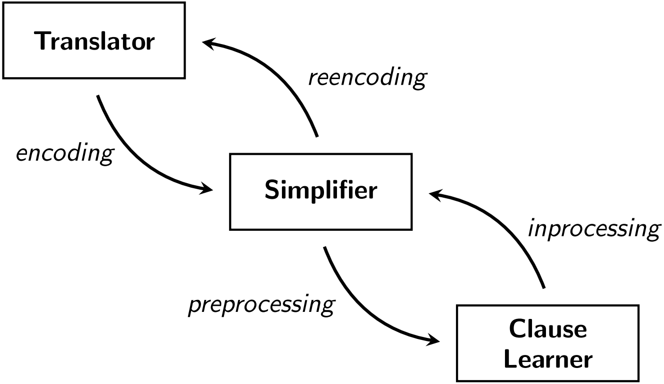
Blocked Clause Elimination

Hyper Binary Resolution

Unhiding Redundancy

Concluding Remarks

Interaction between different solving approaches



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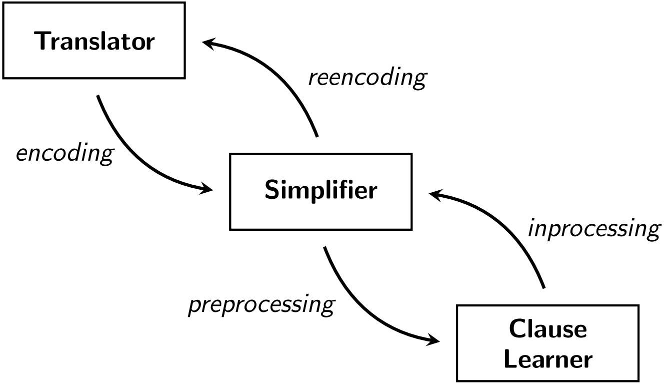

It all comes down to adding and removing redundant clauses

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A clause is redundant with respect to a formula if adding it to the formula preserves satisfiability.

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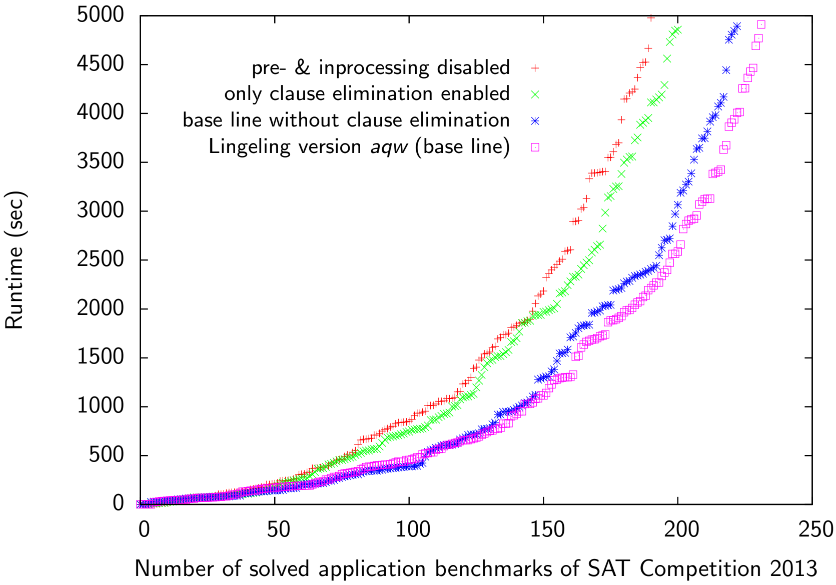
Challenge regarding redundant clauses:

- How to check redundancy in polynomial time?
- Ideally find redundant clauses in linear time

Preprocessing and Inprocessing in Practice



Tautologies and Subsumption

Definition (Tautology)

A clause C is a tautology if its contains two complementary literals x and $\overline{x}$.

Example

The clause $(a \vee b \vee \overline{b})$ *is a tautology.*

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ *subsumes clause* $(a \vee b \vee \overline{e})$*.*

Self-Subsuming Resolution

Self-Subsuming Resolution

$$\frac{C \lor x \qquad D \lor \overline{x}}{D}\; C \subseteq D \qquad \frac{(a \lor b \lor x) \qquad (a \lor b \lor e \lor \overline{x})}{(a \lor b \lor e)}$$

resolvent D subsumes second antecedent $D \lor \overline{x}$

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Example

Assume a CNF contains both antecedents

$\ldots (a \vee b \vee x)(a \vee b \vee e \vee \overline{x}) \ldots$

If D *is added, then* $D \vee \overline{x}$ *can be removed*

which in essence removes $\overline{x}$ *from* $D \vee \overline{x}$

$\ldots (a \vee b \vee x)(a \vee b \vee e \quad) \ldots$

Initially in the SATeLite preprocessor, [EenBiere'07]
now common in most solvers (i.e., as pre- and inprocessing)

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resolvent D subsumes second antecedent $D \vee \overline{x}$

Example: Remove literals using self-subsumption

$$\begin{array}{lll}
(a \vee b \vee e) & \wedge & (\overline{a} \vee b \vee e) \wedge \\
(\overline{a} \vee b \vee \overline{e}) & \wedge & (a \vee \overline{b} \vee e) \wedge \\
(\overline{a} \vee \overline{b} \vee f) & \wedge & (\overline{a} \vee \overline{b} \vee \overline{f}) \wedge \\
(a \vee \overline{e} \vee f) & \wedge & (a \vee \overline{e} \vee \overline{f})
\end{array}$$

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\end{array}$$

Self-Subsuming Example

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$$\frac{C \lor x \qquad D \lor \bar{x}}{D}\; C \subseteq D \qquad \frac{(a \lor b \lor x) \qquad (a \lor b \lor e \lor \bar{x})}{(a \lor b \lor e)}$$

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Implementing Subsumption

Definition (Subsumption)

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Example

The clause $(a \lor b)$ *subsumes clause* $(a \lor b \lor \overline{e})$.

Forward Subsumption

for each clause C in formula Γ **do**
  if C is subsumed by a clause D in $\Gamma \setminus C$ **then**
    remove C from Γ

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for each clause C in formula Γ **do**
  remove all clauses D in Γ that are subsumed by C

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Backward Subsumption

for each clause C in formula Γ **do**
  pick a literal x in C
  remove all clauses D in Γ_x that are subsumed by C

Variable Elimination [DavisPutnam'60]

Definition (Resolution)

Given two clauses $C = (x \vee a_1 \vee \cdots \vee a_i)$ and $D = (\overline{x} \vee b_1 \vee \cdots \vee b_j)$, the *resolvent* of C and D on variable x (denoted by $C \bowtie_x D$) is $(a_1 \vee \cdots \vee a_i \vee b_1 \vee \cdots \vee b_j)$

Resolution on sets of clauses Γ_x and $\Gamma_{\overline{x}}$ (denoted by $\Gamma_x \bowtie_x \Gamma_{\overline{x}}$) generates all non-tautological resolvents of $C \in \Gamma_x$ and $D \in \Gamma_{\overline{x}}$.

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Proof procedure [DavisPutnam60]

VE is a complete proof procedure. Applying VE until fixpoint results in either the empty formula (satisfiable) or empty clause (unsatisfiable)

Example VE by clause distribution [DavisPutnam'60]

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Example of clause distribution

	Γ_x: $(x \vee c)$	Γ_x: $(x \vee \overline{d})$	Γ_x: $(x \vee \overline{a} \vee \overline{b})$
$\Gamma_{\overline{x}}$: $(\overline{x} \vee a)$	$(a \vee c)$	$(a \vee \overline{d})$	$(a \vee \overline{a} \vee \overline{b})$
$\Gamma_{\overline{x}}$: $(\overline{x} \vee b)$	$(b \vee c)$	$(b \vee \overline{d})$	$(b \vee \overline{a} \vee \overline{b})$
$\Gamma_{\overline{x}}$: $(\overline{x} \vee \overline{e} \vee f)$	$(c \vee \overline{e} \vee f)$	$(\overline{d} \vee \overline{e} \vee f)$	$(\overline{a} \vee \overline{b} \vee \overline{e} \vee f)$

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	$(\overline{x} \vee \overline{e} \vee f)$	$(c \vee \overline{e} \vee f)$	$(\overline{d} \vee \overline{e} \vee f)$	$(\overline{a} \vee \overline{b} \vee \overline{e} \vee f)$

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In the example: $|\Gamma_x \bowtie \Gamma_{\overline{x}}| > |\Gamma_x| + |\Gamma_{\overline{x}}|$

Exponential growth of clauses in general

VE by substitution [EenBiere07]

General idea

Detect gates (or definitions) $x = \mathrm{GATE}(a_1, \ldots, a_n)$ in the formula and use them to reduce the number of added clauses

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Possible gates

gate	G_x	$G_{\overline{x}}$
$\mathrm{AND}(a_1, \ldots, a_n)$	$(x \vee \overline{a}_1 \vee \cdots \vee \overline{a}_n)$	$(\overline{x} \vee a_1), \ldots, (\overline{x} \vee a_n)$
$\mathrm{OR}(a_1, \ldots, a_n)$	$(x \vee \overline{a}_1), \ldots, (x \vee \overline{a}_n)$	$(\overline{x} \vee a_1 \vee \cdots \vee a_n)$
$\mathrm{ITE}(c, t, f)$	$(x \vee \overline{c} \vee \overline{t}), (x \vee c \vee \overline{f})$	$(\overline{x} \vee \overline{c} \vee t), (\overline{x} \vee c \vee f)$

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Variable elimination by substitution [EenBiere07]

Let $R_x = \Gamma_x \setminus G_x$; $R_{\overline{x}} = \Gamma_{\overline{x}} \setminus G_{\overline{x}}$.

Replace $\Gamma_x \wedge \Gamma_{\overline{x}}$ by $G_x \bowtie_x R_{\overline{x}} \wedge G_{\overline{x}} \bowtie_x R_x$.
Always less than $\Gamma_x \bowtie_x \Gamma_{\overline{x}}$!

VE by substitution [EenBiere'07]

Example of gate extraction: $x = \text{AND}(a, b)$

$$\Gamma_x = (x \lor c) \land (x \lor \overline{d}) \land (x \lor \overline{a} \lor \overline{b})$$
$$\Gamma_{\overline{x}} = (\overline{x} \lor a) \land (\overline{x} \lor b) \land (\overline{x} \lor \overline{e} \lor f)$$

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Example of substitution

		R_x		G_x
		$(x \vee c)$	$(x \vee \overline{d})$	$(x \vee \overline{a} \vee \overline{b})$
$G_{\overline{x}}$	$(\overline{x} \vee a)$	$(a \vee c)$	$(a \vee \overline{d})$	
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$R_{\overline{x}}$	$(\overline{x} \vee \overline{e} \vee f)$			$(\overline{a} \vee \overline{b} \vee \overline{e} \vee f)$

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$G_{\overline{x}}$	$(\overline{x} \vee b)$	$(b \vee c)$	$(b \vee \overline{d})$	
$R_{\overline{x}}$	$(\overline{x} \vee \overline{e} \vee f)$			$(\overline{a} \vee \overline{b} \vee \overline{e} \vee f)$

using substitution: $|\Gamma_x \bowtie \Gamma_{\overline{x}}| < |\Gamma_x| + |\Gamma_{\overline{x}}|$

Bounded Variable Addition

Main Idea

Given a CNF formula Γ, can we construct a (semi)logically equivalent Γ' by introducing a new variable $x \notin VAR(\Gamma)$ such that $|\Gamma'| < |\Gamma|$?

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Reverse of Variable Elimination

For example, replace the clauses

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\begin{array}{ccc}
(a \lor c) & (a \lor \overline{d}) & \\
(b \lor c) & (b \lor \overline{d}) & \\
(c \lor \overline{e} \lor f) & (\overline{d} \lor \overline{e} \lor f) & (\overline{a} \lor \overline{b} \lor \overline{e} \lor f)
\end{array}
$$

by

$$
\begin{array}{ccc}
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\end{array}
$$

Challenge: how to find suitable patterns for replacement?

Factoring Out Subclauses

Example

Replace

$$(a \lor b \lor c \lor d) \quad (a \lor b \lor c \lor e) \quad (a \lor b \lor c \lor f)$$

by

$$(x \lor d) \quad (x \lor e) \quad (x \lor f) \quad (\overline{x} \lor a \lor b \lor c)$$

adds 1 variable and 1 clause *reduces* number of literals by 2

Not compatible with VE, which would eliminate x immediately!

... *so this does not work* ...

Bounded Variable Addition

Example

Smallest pattern that is compatible: Replace

$$(a \lor d) \quad (a \lor e)$$
$$(b \lor d) \quad (b \lor e)$$
$$(c \lor d) \quad (c \lor e)$$

by

$$(\bar{x} \lor a) \quad (\bar{x} \lor b) \quad (\bar{x} \lor c)$$
$$(x \lor d) \quad (x \lor e)$$

adds 1 variable *removes* 1 clause

Bounded Variable Addition

Possible Patterns

$$\begin{matrix} (X_1 \lor L_1) & \dots & (X_1 \lor L_k) \\ \vdots & & \vdots \\ (X_n \lor L_1) & \dots & (X_n \lor L_k) \end{matrix} \quad \equiv \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (X_i \lor L_j)$$

replaced by $\bigwedge_{i=1}^{n} (y \lor X_i) \land \bigwedge_{j=1}^{k} (\overline{y} \lor L_j)$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j

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$$\begin{matrix} (X_1 \vee L_1) & \dots & (X_1 \vee L_k) \\ \vdots & & \vdots \\ (X_n \vee L_1) & \dots & (X_n \vee L_k) \end{matrix} \quad \equiv \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k} (X_i \vee L_j)$$

replaced by $\bigwedge_{i=1}^{n} (y \vee X_i) \;\wedge\; \bigwedge_{j=1}^{k} (\overline{y} \vee L_j)$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j
- Reduction: $nk - n - k$ clauses are removed by replacement

Bounded Variable Addition on AtMostOne (1)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge \\
(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee \overline{x}_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_5) \wedge (\overline{x}_2 \vee \overline{x}_5) \wedge (\overline{x}_3 \vee \overline{x}_5) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee \overline{x}_6) \wedge (\overline{x}_2 \vee \overline{x}_6) \wedge (\overline{x}_3 \vee \overline{x}_6) \wedge (\overline{x}_4 \vee \overline{x}_6) \wedge (\overline{x}_5 \vee \overline{x}_6) \wedge \\
(\overline{x}_1 \vee \overline{x}_7) \wedge (\overline{x}_2 \vee \overline{x}_7) \wedge (\overline{x}_3 \vee \overline{x}_7) \wedge (\overline{x}_4 \vee \overline{x}_7) \wedge (\overline{x}_5 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee \overline{x}_8) \wedge (\overline{x}_2 \vee \overline{x}_8) \wedge (\overline{x}_3 \vee \overline{x}_8) \wedge (\overline{x}_4 \vee \overline{x}_8) \wedge (\overline{x}_5 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_9) \wedge (\overline{x}_2 \vee \overline{x}_9) \wedge (\overline{x}_3 \vee \overline{x}_9) \wedge (\overline{x}_4 \vee \overline{x}_9) \wedge (\overline{x}_5 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_{10}) \wedge (\overline{x}_2 \vee \overline{x}_{10}) \wedge (\overline{x}_3 \vee \overline{x}_{10}) \wedge (\overline{x}_4 \vee \overline{x}_{10}) \wedge (\overline{x}_5 \vee \overline{x}_{10})
\end{array}
$$

Bounded Variable Addition on AtMostOne (1)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge \\
(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee \overline{x}_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_5) \wedge (\overline{x}_2 \vee \overline{x}_5) \wedge (\overline{x}_3 \vee \overline{x}_5) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge \\
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(\overline{x}_1 \vee \overline{x}_7) \wedge (\overline{x}_2 \vee \overline{x}_7) \wedge (\overline{x}_3 \vee \overline{x}_7) \wedge (\overline{x}_4 \vee \overline{x}_7) \wedge (\overline{x}_5 \vee \overline{x}_7) \wedge \\
(\overline{x}_1 \vee \overline{x}_8) \wedge (\overline{x}_2 \vee \overline{x}_8) \wedge (\overline{x}_3 \vee \overline{x}_8) \wedge (\overline{x}_4 \vee \overline{x}_8) \wedge (\overline{x}_5 \vee \overline{x}_8) \wedge \\
(\overline{x}_1 \vee \overline{x}_9) \wedge (\overline{x}_2 \vee \overline{x}_9) \wedge (\overline{x}_3 \vee \overline{x}_9) \wedge (\overline{x}_4 \vee \overline{x}_9) \wedge (\overline{x}_5 \vee \overline{x}_9) \wedge \\
(\overline{x}_1 \vee \overline{x}_{10}) \wedge (\overline{x}_2 \vee \overline{x}_{10}) \wedge (\overline{x}_3 \vee \overline{x}_{10}) \wedge (\overline{x}_4 \vee \overline{x}_{10}) \wedge (\overline{x}_5 \vee \overline{x}_{10})
\end{array}
$$

Replace $(\overline{x}_i \vee \overline{x}_j)$ with $i \in \{1..5\}, j \in \{6..10\}$ by $(\overline{x}_i \vee y)$, $(\overline{x}_j \vee \overline{y})$

Bounded Variable Addition on AtMostOne (2)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$
\begin{array}{l}
(\overline{x}_1 \lor \overline{x}_2) \land (\overline{x}_9 \lor \overline{x}_{10}) \land (\overline{x}_8 \lor \overline{x}_{10}) \land (\overline{x}_7 \lor \overline{x}_{10}) \land (\overline{x}_6 \lor \overline{x}_{10}) \land \\
(\overline{x}_1 \lor \overline{x}_3) \land (\overline{x}_2 \lor \overline{x}_3) \land (\overline{x}_8 \lor \overline{x}_9) \land (\overline{x}_7 \lor \overline{x}_9) \land (\overline{x}_6 \lor \overline{x}_9) \land \\
(\overline{x}_1 \lor \overline{x}_4) \land (\overline{x}_2 \lor \overline{x}_4) \land (\overline{x}_3 \lor \overline{x}_4) \land (\overline{x}_7 \lor \overline{x}_8) \land (\overline{x}_6 \lor \overline{x}_8) \land \\
(\overline{x}_1 \lor \overline{x}_5) \land (\overline{x}_2 \lor \overline{x}_5) \land (\overline{x}_3 \lor \overline{x}_5) \land (\overline{x}_4 \lor \overline{x}_5) \land (\overline{x}_6 \lor \overline{x}_7) \land \\
(\overline{x}_1 \lor y) \land (\overline{x}_2 \lor y) \land (\overline{x}_3 \lor y) \land (\overline{x}_4 \lor y) \land (\overline{x}_5 \lor y) \land \\
(\overline{x}_6 \lor \overline{y}) \land (\overline{x}_7 \lor \overline{y}) \land (\overline{x}_8 \lor \overline{y}) \land (\overline{x}_9 \lor \overline{y}) \land (\overline{x}_{10} \lor \overline{y})
\end{array}
$$

Bounded Variable Addition on AtMostOne (2)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge$$
$$(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge$$
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$$(\overline{x}_1 \vee \overline{x}_5) \wedge (\overline{x}_2 \vee \overline{x}_5) \wedge (\overline{x}_3 \vee \overline{x}_5) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge$$
$$(\overline{x}_1 \vee y) \wedge (\overline{x}_2 \vee y) \wedge (\overline{x}_3 \vee y) \wedge (\overline{x}_4 \vee y) \wedge (\overline{x}_5 \vee y) \wedge$$
$$(\overline{x}_6 \vee \overline{y}) \wedge (\overline{x}_7 \vee \overline{y}) \wedge (\overline{x}_8 \vee \overline{y}) \wedge (\overline{x}_9 \vee \overline{y}) \wedge (\overline{x}_{10} \vee \overline{y})$$

Replace matched pattern

$$(\overline{x}_1 \vee z) \wedge (\overline{x}_2 \vee z) \wedge (\overline{x}_3 \vee z) \wedge$$
$$(\overline{x}_4 \vee \overline{z}) \wedge (\overline{x}_5 \vee \overline{z}) \wedge (y \vee \overline{z})$$

Bounded Variable Addition on AtMostOne (3)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$(\overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_9 \vee \overline{x}_{10}) \wedge (\overline{x}_8 \vee \overline{x}_{10}) \wedge (\overline{x}_7 \vee \overline{x}_{10}) \wedge (\overline{x}_6 \vee \overline{x}_{10}) \wedge$$
$$(\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (\overline{x}_8 \vee \overline{x}_9) \wedge (\overline{x}_7 \vee \overline{x}_9) \wedge (\overline{x}_6 \vee \overline{x}_9) \wedge$$
$$(\overline{x}_1 \vee z) \wedge (\overline{x}_2 \vee z) \wedge (\overline{x}_3 \vee z) \wedge (\overline{x}_7 \vee \overline{x}_8) \wedge (\overline{x}_6 \vee \overline{x}_8) \wedge$$
$$(\overline{x}_4 \vee \overline{z}) \wedge (\overline{x}_5 \vee \overline{z}) \wedge (y \vee \overline{z}) \wedge (\overline{x}_4 \vee \overline{x}_5) \wedge (\overline{x}_6 \vee \overline{x}_7) \wedge$$
$$(\overline{x}_4 \vee y) \wedge (\overline{x}_5 \vee y) \wedge (\overline{x}_6 \vee \overline{y}) \wedge (\overline{x}_7 \vee \overline{y}) \wedge (\overline{x}_8 \vee \overline{y})$$
$$(\overline{x}_9 \vee \overline{y}) \wedge (\overline{x}_{10} \vee \overline{y})$$

Bounded Variable Addition on AtMostOne (3)

Example encoding of AtMostOne $(x_1, x_2, \ldots, x_n)$

$$(\bar{x}_1 \vee \bar{x}_2) \wedge (\bar{x}_9 \vee \bar{x}_{10}) \wedge (\bar{x}_8 \vee \bar{x}_{10}) \wedge (\bar{x}_7 \vee \bar{x}_{10}) \wedge (\bar{x}_6 \vee \bar{x}_{10}) \wedge$$
$$(\bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee \bar{x}_9) \wedge (\bar{x}_6 \vee \bar{x}_9) \wedge$$
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$$(\bar{x}_4 \vee \bar{z}) \wedge (\bar{x}_5 \vee \bar{z}) \wedge (y \vee \bar{z}) \wedge (\bar{x}_4 \vee \bar{x}_5) \wedge (\bar{x}_6 \vee \bar{x}_7) \wedge$$
$$(\bar{x}_4 \vee y) \wedge (\bar{x}_5 \vee y) \wedge (\bar{x}_6 \vee \bar{y}) \wedge (\bar{x}_7 \vee \bar{y}) \wedge (\bar{x}_8 \vee \bar{y})$$
$$(\bar{x}_9 \vee \bar{y}) \wedge (\bar{x}_{10} \vee \bar{y})$$

Replace matched pattern

$$(\bar{x}_6 \vee w) \wedge (\bar{x}_7 \vee w) \wedge (\bar{x}_8 \vee w) \wedge$$
$$(\bar{x}_9 \vee \bar{w}) \wedge (\bar{x}_{10} \vee \bar{w}) \wedge (\bar{y} \vee \bar{w})$$

Blocked Clauses [Kullmann 1999]

Definition (Blocked Clause)

A clause $(C \vee x)$ is a blocked on x w.r.t. a CNF formula Γ if for every clause $(D \vee \overline{x}) \in \Gamma$, resolvent $C \vee D$ is a tautology.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{e}) \wedge (\overline{a} \vee e)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{e}$.
Third clause is blocked by e

Blocked Clauses [Kullmann 1999]

Definition (Blocked Clause)

A clause $(C \lor x)$ is a blocked on x w.r.t. a CNF formula Γ if for every clause $(D \lor \overline{x}) \in \Gamma$, resolvent $C \lor D$ is a tautology.

Example

Consider the formula $(a \lor b) \land (a \lor \overline{b} \lor \overline{e}) \land (\overline{a} \lor e)$*.*
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{e}$*.*
Third clause is blocked by e

Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF Γ, remove C from Γ.

Example

Consider $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{e}) \wedge (\overline{a} \vee e)$*.*
After removing either $(a \vee \overline{b} \vee \overline{e})$ *or* $(\overline{a} \vee e)$*, the clause* $(a \vee b)$ *becomes blocked (no clause with either* $\overline{b}$ *or* $\overline{a}$*).*

An extreme case in which BCE *removes all clauses!*

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After removing either $(a \vee \overline{b} \vee \overline{e})$ *or* $(\overline{a} \vee e)$*, the clause* $(a \vee b)$ *becomes blocked (no clause with either* $\overline{b}$ *or* $\overline{a}$*).*

An extreme case in which BCE *removes all clauses!*

Proposition

BCE is confluent, i.e., has a unique fixpoint

- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

- BCE converts the Tseitin encoding to Plaisted Greenbaum
- BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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(y)

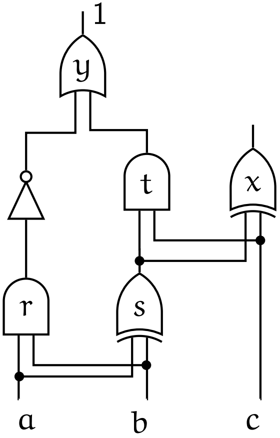
$(\bar{y} \vee t \vee \bar{r})$
$(y \vee \bar{t})$
$(y \vee r)$

$(\bar{x} \vee s \vee c)$
$(\bar{x} \vee \bar{s} \vee \bar{c})$
$(x \vee s \vee \bar{c})$
$(x \vee \bar{s} \vee c)$

$(t \vee \bar{s} \vee \bar{c})$
$(\bar{t} \vee s)$
$(\bar{t} \vee c)$

$(r \vee \bar{a} \vee \bar{b})$
$(\bar{r} \vee a)$
$(\bar{r} \vee b)$

$(\bar{s} \vee a \vee b)$
$(\bar{s} \vee \bar{a} \vee \bar{b})$
$(s \vee a \vee \bar{b})$
$(s \vee \bar{a} \vee b)$



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Example of circuit simplification by BCE on Tseitin encoding

(y)

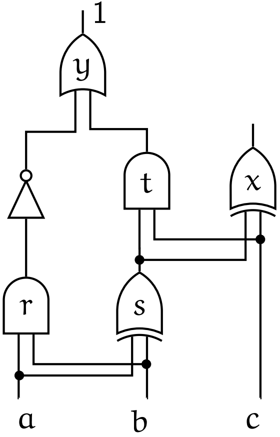
$(\overline{y} \lor t \lor \overline{r})$
~~$(y \lor \overline{t})$~~
~~$(y \lor r)$~~

$(\overline{x} \lor s \lor c)$
$(\overline{x} \lor \overline{s} \lor \overline{c})$
$(x \lor s \lor \overline{c})$
$(x \lor \overline{s} \lor c)$

$(t \lor \overline{s} \lor \overline{c})$
$(\overline{t} \lor s)$
$(\overline{t} \lor c)$

$(r \lor \overline{a} \lor \overline{b})$
$(\overline{r} \lor a)$
$(\overline{r} \lor b)$

$(\overline{s} \lor a \lor b)$
$(\overline{s} \lor \overline{a} \lor \overline{b})$
$(s \lor a \lor \overline{b})$
$(s \lor \overline{a} \lor b)$



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~~$(x \lor s \lor \bar{c})$~~
~~$(x \lor \bar{s} \lor c)$~~

$(t \lor \bar{s} \lor \bar{c})$
$(\bar{t} \lor s)$
$(\bar{t} \lor c)$

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$(\bar{r} \lor a)$
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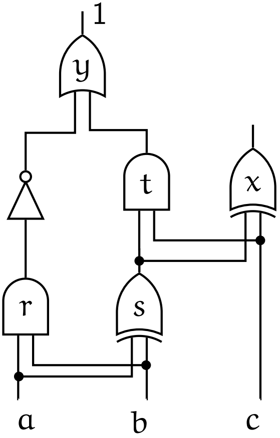

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~~$(x \lor s \lor \bar{c})$~~
~~$(x \lor \bar{s} \lor c)$~~

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$(\bar{t} \lor s)$
$(\bar{t} \lor c)$

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$(\bar{r} \lor a)$
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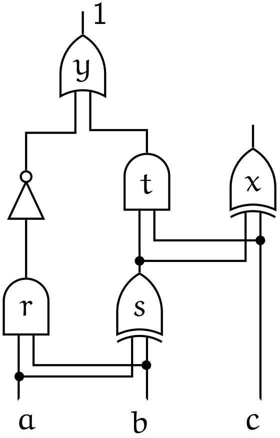

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~~$(x \lor s \lor \bar{c})$~~
~~$(x \lor \bar{s} \lor c)$~~

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$(\bar{t} \lor c)$

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~~$(\bar{r} \lor a)$~~
~~$(\bar{r} \lor b)$~~

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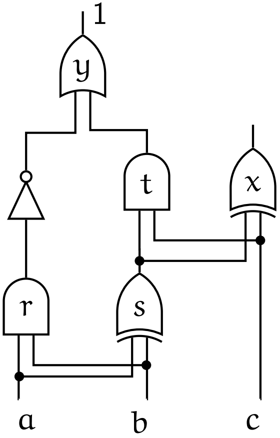

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$(\bar{t} \vee s)$
$(\bar{t} \vee c)$

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~~$(\bar{r} \vee a)$~~
~~$(\bar{r} \vee b)$~~

$(\bar{s} \vee a \vee b)$
$(\bar{s} \vee \bar{a} \vee \bar{b})$
~~$(s \vee a \vee \bar{b})$~~
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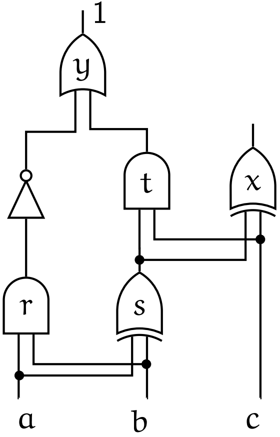

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~~$(x \lor s \lor \bar{c})$~~
~~$(x \lor \bar{s} \lor c)$~~

~~$(t \lor \bar{s} \lor \bar{c})$~~
$(\bar{t} \lor s)$
$(\bar{t} \lor c)$

~~$(r \lor \bar{a} \lor \bar{b})$~~
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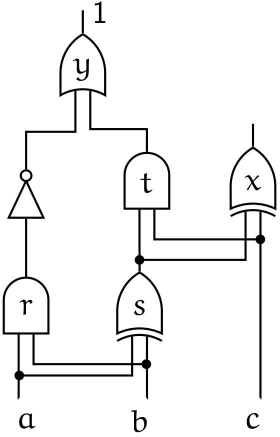

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BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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~~$(\bar{y} \vee t \vee \bar{r})$~~
~~$(y \vee \bar{t})$~~
~~$(y \vee r)$~~

~~$(\bar{x} \vee s \vee c)$~~
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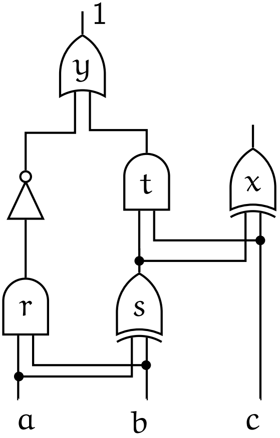

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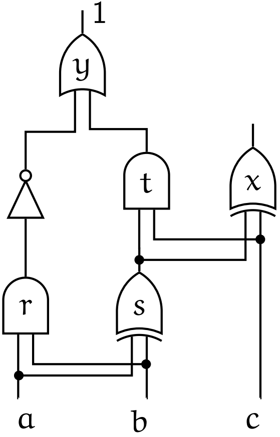

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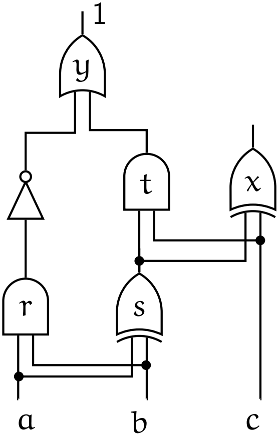

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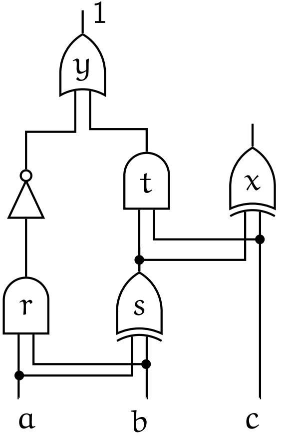


BCE: Solution Reconstruction

Input:

- stack S of eliminated blocked clauses
- formula Γ (without the blocked clauses)
- assignment α that satisfies Γ

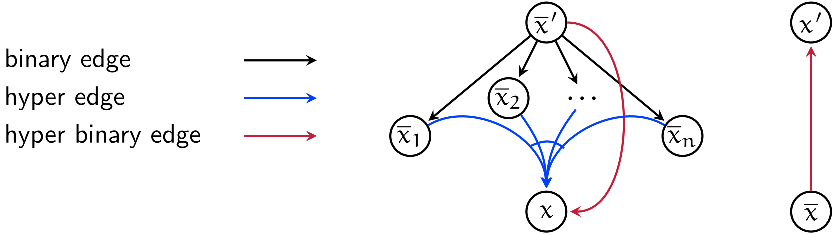
Output: an assignment that satisfies $\Gamma \wedge S$

```
while S.size () do
    ⟨C, l⟩ := S.pop ()
    if α falsifies C then α := α_l
end while
return α
```

Hyper Binary Resolution [Bacchus-AAAI02]

Definition (Hyper Binary Resolution Rule)

$$\frac{(x \vee x_1 \vee x_2 \vee \cdots \vee x_n) \quad (\bar{x}_1 \vee x') \quad (\bar{x}_2 \vee x') \quad \ldots \quad (\bar{x}_n \vee x')}{(x \vee x')}$$



Hyper Binary Resolution Rule:

- combines multiple resolution steps into one
- uses one n-ary clauses and multiple binary clauses
- special case *hyper unary resolution* where $x = x'$

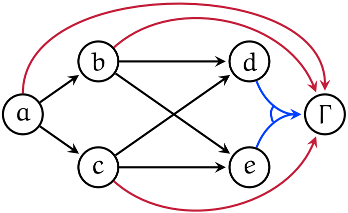
Hyper Binary Resolution (HBR)

Definition (Hyper Binary Resolution)

Apply the hyper binary resolution rule until fixpoint

Example

Consider
$(\overline{a} \vee b) \wedge (\overline{a} \vee c) \wedge (\overline{b} \vee d) \wedge (\overline{b} \vee e) \wedge (\overline{c} \vee d) \wedge (\overline{c} \vee e) \wedge (\overline{d} \vee \overline{e} \vee f)$.



hyper binary resolvents:
$(\overline{a} \vee f), (\overline{b} \vee f), (\overline{c} \vee f)$

HBR is confluent, i.e., has a unique fixpoint

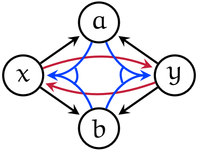
Structural Hashing of AND-gates via HBR

gate g	$g \Rightarrow f(g_1, \ldots g_n)$ "positive"	$g \Leftarrow f(g_1, \ldots g_n)$ "negative"
$g := \text{OR}(g_1, \ldots, g_n)$	$(\bar{g} \vee g_1 \vee \cdots \vee g_n)$	$(g \vee \overline{g_1}),\ldots,(g \vee \overline{g_n})$
$g := \text{AND}(g_1, \ldots, g_n)$	$(\bar{g} \vee g_1),\ldots,(\bar{g} \vee g_n)$	$(g \vee \overline{g_1} \vee \cdots \vee \overline{g_n})$
$g := \text{XOR}(g_1, g_2)$	$(\bar{g} \vee \overline{g_1} \vee \overline{g_2}),\ (\bar{g} \vee g_1 \vee g_2)$	$(g \vee \overline{g_1} \vee g_2),\ (g \vee g_1 \vee \overline{g_2})$
$g := \text{ITE}(g_1, g_2, g_3)$	$(\bar{g} \vee \overline{g_1} \vee g_2),\ (\bar{g} \vee g_1 \vee g_3)$	$(g \vee \overline{g_1} \vee \overline{g_2}),\ (g \vee g_1 \vee \overline{g_3})$

Definition (Structural Hashing of AND-gates)

Given a Boolean circuit with two equivalent gates, merge the gates.

Example



$x = \text{AND(a,b)} : \ (\bar{x} \vee a) \wedge (\bar{x} \vee b) \wedge (x \vee \bar{a} \vee \bar{b})$

$y = \text{AND(a,b)} : \ (\bar{y} \vee a) \wedge (\bar{y} \vee b) \wedge (y \vee \bar{a} \vee \bar{b})$

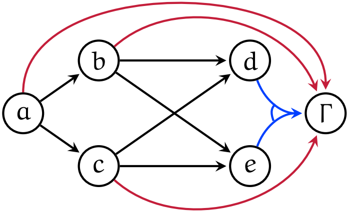
the two HBRs $(\bar{x} \vee y)$ and $(x \vee \bar{y})$ express that $x = y$

Non-transitive Hyper Binary Resolution (NHBR)

A problem with classic HBR is that it adds many transitive binary clauses

Example

Consider
$(\overline{a} \lor b) \land (\overline{a} \lor c) \land (\overline{b} \lor d) \land (\overline{b} \lor e) \land (\overline{c} \lor d) \land (\overline{c} \lor e) \land (\overline{d} \lor \overline{e} \lor f)$.



adding $(\overline{b} \lor f)$ or $(\overline{c} \lor f)$
makes $(\overline{a} \lor f)$ transitive

Solution [HeuleJärvisaloBiere 2013]

Add only non-transitive hyper binary resolvents
Can be implemented using an alternative unit propagation style

Space Complexity of NHBR: Quadratic

Question regarding complexity [Biere 2009]

- Are there formulas where the transitively reduced hyper binary resolution closure is quadratic in size w.r.t. to the size of the original?
- where size = #clauses or size = #literals or size = #variables

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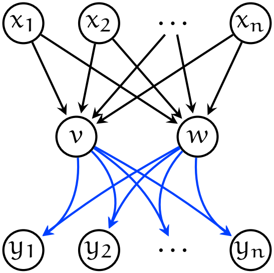
Yes!

Consider the formula $\Gamma_n = \bigwedge_{1 \leq i \leq n} \left((\overline{x}_i \vee v) \wedge (\overline{x}_i \vee w) \wedge (\overline{v} \vee \overline{w} \vee y_i) \right)$



#variables: $2n + 2$

#clauses: $3n$

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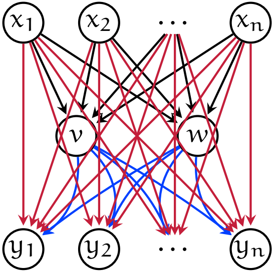

#variables: $2n + 2$

#clauses: $3n$

#literals: $7n$

n^2 hyper binary resolvents: $(\overline{x}_i \vee y_j)$ for $1 \leq i, j \leq n$

Redundancy

Redundant clauses:

- Removal of $C \in \Gamma$ preserves unsatisfiability of Γ
- Assign all $x \in C$ to false and check for a conflict in $\Gamma \setminus \{C\}$

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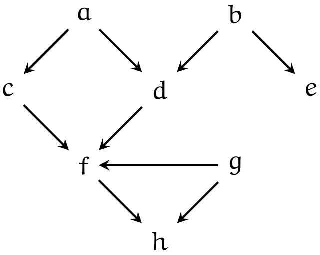
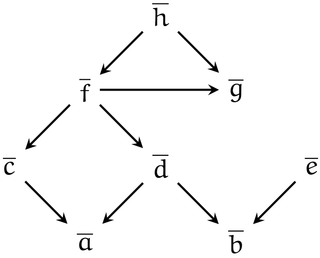
Redundancy elimination during pre- and in-processing

- Distillation [JinSomenzi2005]
- ReVivAl [PietteHamadiSaïs2008]
- Unhiding [HeuleJärvisaloBiere2011]

Unhide: Binary implication graph (BIG)

unhide: use the binary clauses to detect redundant clauses and literals

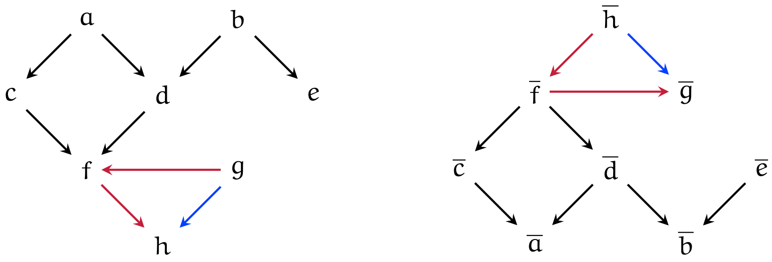




$$(\overline{a} \lor c) \land (\overline{a} \lor d) \land (\overline{b} \lor d) \land (\overline{b} \lor e) \land$$
$$(\overline{c} \lor f) \land (\overline{d} \lor f) \land (\overline{g} \lor f) \land (\overline{f} \lor h) \land$$
$$(\overline{g} \lor h) \land \underbrace{(\overline{a} \lor \overline{e} \lor h) \land (\overline{b} \lor \overline{c} \lor h) \land (a \lor b \lor c \lor d \lor e \lor f \lor g \lor h)}_{\text{non binary clauses}}$$

Unhide: Transitive reduction (TRD)

transitive reduction: remove shortcuts in the binary implication graph



$(\overline{a} \vee c) \wedge (\overline{a} \vee d) \wedge (\overline{b} \vee d) \wedge (\overline{b} \vee e) \wedge$

$(\overline{c} \vee f) \wedge (\overline{d} \vee f) \wedge (\overline{g} \vee f) \wedge (\overline{f} \vee h) \wedge$

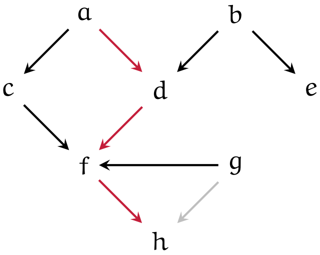
~~$(\overline{g} \vee h)$~~ $\wedge (\overline{a} \vee \overline{e} \vee h) \wedge (\overline{b} \vee \overline{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$

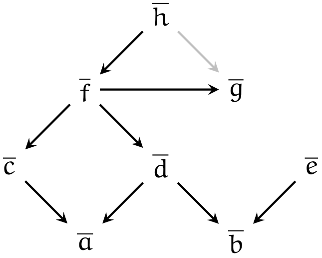
TRD

$g \rightarrow f \rightarrow h$

Unhide: Hidden tautology elimination (HTE) (1)

HTE removes clauses that are subsumed by an implication in BIG





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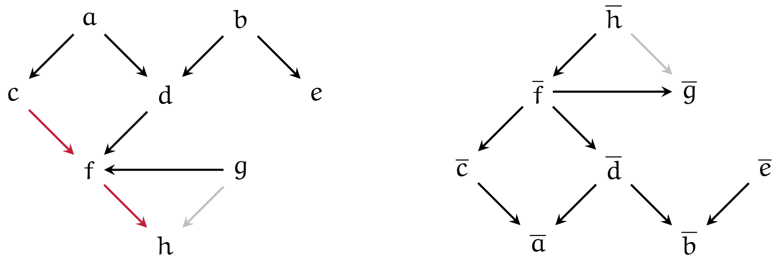
$\cancel{(\bar{a} \vee \bar{e} \vee h)} \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$

HTE

$a \rightarrow d \rightarrow f \rightarrow h$

Unhide: Hidden tautology elimination (HTE) (2)

HTE removes clauses that are subsumed by an implication in BIG



$(\bar{a} \lor c) \land (\bar{a} \lor d) \land (\bar{b} \lor d) \land (\bar{b} \lor e) \land$

$(\bar{c} \lor f) \land (\bar{d} \lor f) \land (\bar{g} \lor f) \land (\bar{f} \lor h) \land$

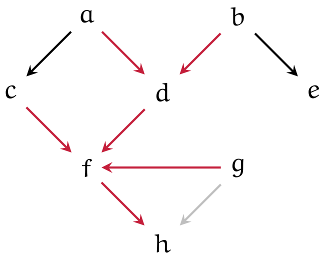
~~$(\bar{b} \lor \bar{c} \lor h)$~~ $\land (a \lor b \lor c \lor d \lor e \lor f \lor g \lor h)$

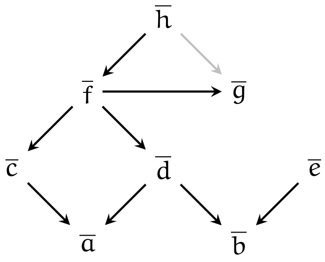
HTE

$c \rightarrow f \rightarrow h$

Unhide: Hidden literal elimination (HLE)

HLE removes literal using the implication in BIG





$(\bar{a} \lor c) \land (\bar{a} \lor d) \land (\bar{b} \lor d) \land (\bar{b} \lor e) \land$

$(\bar{c} \lor f) \land (\bar{d} \lor f) \land (\bar{g} \lor f) \land (\bar{f} \lor h) \land$

(~~$a \lor b \lor c \lor d \lor$~~ $e \lor$ ~~$f \lor g \lor$~~ h)

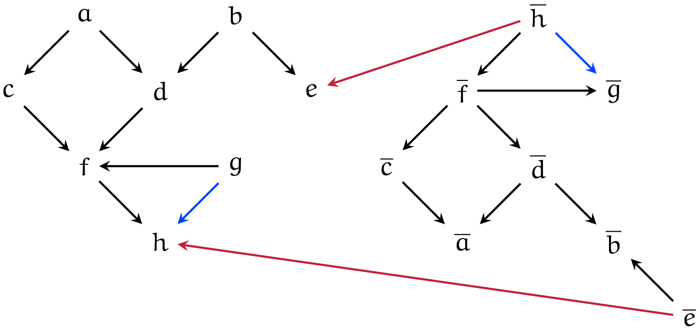
HLE

all but e imply h

also b implies e

Unhide: TRD + HTE + HLE

unhide: redundancy elimination removes and adds arcs from BIG(F)



$$(\overline{a} \vee c) \wedge (\overline{a} \vee d) \wedge (\overline{b} \vee d) \wedge (\overline{b} \vee e) \wedge$$
$$(\overline{c} \vee f) \wedge (\overline{d} \vee f) \wedge (\overline{g} \vee f) \wedge (\overline{f} \vee h) \wedge (e \vee h)$$

Many Techniques

Many pre- or in-processing techniques in SAT solvers:

- (Self-)Subsumption
- Variable Elimination
- Blocked Clause Elimination
- Hyper Binary Resolution
- Bounded Variable Addition
- Equivalent Literal Substitution
- Failed Literal Elimination
- Autarky Reasoning
- ...

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... and the list is growing:

- Propagation Redundant Clauses [CADE'17]