Structure and Interpretation of Music Concpets

Class10 – Tonal Arithmetics.

The Tonal Music Pitch and Interval Domain: Terminology, Arithmetics, Physical mappings.

The domain of notes and intervals in tonal music is created by a rich set of pitch-classes (interval-classes), that is much larger than the set of physical pitch-classes to which they are mapped. The set of *Tonal-Pitch-Classes* (*TPC*) (*Tonal-Interval-Classes* (*TIC*)) is generated by extending the western twelve tone selection with an additional seven tone selection – called the *diatonic points.* The overall TPC (TIC) domain is a set of pairs, that correspond to the diatonic points, and amount of alteration. Since the diatonic points map in a 1 : 1 manner to specific 12-tone pitch-classes, these pairwise values map also to the 12-tone pitch-classes. However, the mapping is no longer 1 : 1.

These class notes include the definition of the TPC and TIC concepts, the *Tonal-Pitch* and *Tonal-Interval* concepts, and their arithmetics. The definition of the concepts is done by providing a single abstraction for both concepts: *Tonal-Class* and *Tonal-Element.* The arithmetics is obtained from a mapping of the Tonal-Class values to 12-tone pitch-classes, and from two enumerations of the Tonal-Element values:

1. *diatonic_value*, that enumerates only the diatonic points in each octave – and hence, every octave includes only 7 elements.

2. *chromatic_value*, that enumerates all 12 pitch-classes in each octave – and hence, every octave includes 12 elements.

This way we obtain a uniform, single arithmetics for all Tonal-Pitch and Tonal-Interval computations, therby avoiding the need to provide separate arithmetics for the different types. The arithmetics characterizes the set of Tonal-Elements as a commutative group, and supports further operations and characterizations.

1 Terminology – Types

1.1 Tonal-Pitch-Class, Tonal-Interval-Class, Tonal-Class

Tonal-Pitch-Class

1. $Diatonic\_Pitch = \{C,\ D,\ E,\ F,\ G,\ A,\ B\}$

2. $Pitch\_Alteration = \{\flat\flat,\ \flat,\ \natural,\ \sharp,\ \sharp\sharp\}$

3. $Tonal\_Pitch\_Class\ (TPC) = Diatonic\_Pitch \times Pitch\_Alteration$

For example: $(C, \sharp)$ for the C sharp pitch class.

Tonal-Interval-Class

1. $Diatonic\_Interval = \{Prime,\ Second,\ Third,\ Fourth,\ Fifth,\ Sixth,\ Seventh\}$
2. $Interval\_Alteration = \{Diminished,\ Minor,\ Major,\ Perfect,\ Augmented\}$
3. $Tonal\_Interval\_Class\ (TIC) = Diatonic\_Interval \times Interval\_Alteration$

For example: $(Fifth, Perfect)$ for the perfect fifth interval class.

Tonal-Class

1. $Diatonic\_Point = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\}$
2. $Alteration = \{-2,\ -1,\ 0,\ 1,\ 2\}$
3. $Tonal\_Class\ (TC) = Diatonic\_Point \times Alteration$

For example: $(0, -1)$ for the $(C, \sharp)$ pitch-class, or for the $(Prime, Diminished)$ interval-class.

Selectors for the Tonal Class type:
$diatonic : Tonal\_Class \rightarrow Diatonic\_Point.$
$alteration : Tonal\_Class \rightarrow Alteration.$

1.2 Tonal-pitch, Tonal-Interval, Tonal-Element

1. $Octave = I$, the set of integers.
   $Positive\_Octave = N$, (including 0).
2. $Tonal\_pitch = Octave \times TPC.$
   The intention is that octave number 0 marks the middle octave (midi-pitch $60 - 72$).
3. $Tonal\_Interval = \{up, down\} \times Octave \times TIC.$
   For example: (down, 1, (Fifth, Perfect)) marks an interval of a whole octave + a perfect fifth in the downword direction.
4. $Tonal\_Element = Octave \times TC.$
   For example, $(-3, (0, -1))$ is a tonal element that denotes the tonal pitch Cb, 3 octaves below middle C – $(3, (C, \flat))$, or the donword tonal interval of 3 octaves and augmented prime – $(downword,\ 3, (Prime,\ Augmented))$.

2 Mapping the Tonal Level to a Diatonic and Twelve Tone Enumeration

Diatonic Points

$value : Diatonic\_Point \rightarrow Music\_Pitch\_Class$,
defined as follows:

dp	0	1	2	3	4	5	6
value(dp)	0	2	4	5	7	9	11

Tonal Class Mapping

The $value$ mapping is extended to values of the $Tonal\_Class$ set:
$value : Tonal\_Class \rightarrow Music\_Pitch\_Class \cup \{-2,\ -1,\ 12,\ 13\}$,
defined by: $value(d, a) = value(d) + a$.

Tonal Element Mapping

Each tonal element value has two values associated with it: A $diatonic\_value$, that provides the diatonic position in a diatonic points axis, and $chromatic\_value$, that provides the position in a music pitch axis. The arithmetics of the tonal elements arises from coordination of these two values. The $diatonic\_value$ provides a count of the tonal element on an axis of diatonic points (base 7), and the $chromatic\_value$ provides the actual music pitch (a count on an axis of music pitch classes – base 12).

1. $diatonic\_value : Tonal\_Element \rightarrow I$:
   where,
   $diatonic\_value(o, tc) = 7 * o + diatonic(tc)$
   Example: $diatonic\_value(-1, (6, 2)) = 7 * (-1) + 6 = -1$

2. $chromatic\_value : TE \rightarrow I$:
   where,
   $chromatic\_value(o, tc) = 12 * o + value(tc)$
   Example: $chromatic\_value(-1, (6, 2)) = 12 * (-1) + 11 + 2 = 1$

3. $music\_pitch : TE \rightarrow Music\_Pitch$:
   where, $music\_pitch(o, tc) = (o + 5 + mod\_quotient(value(tc), 12),\ modulo(value(tc), 12))$
   where $mod\_quotient$ and $modulo$ are quotient and remainder functions that return always a positive "modulo" remainder, and a quotient that together account for the whole division. They are defined at the end of this document. Modulo is a primitive of Scheme.

3 Arithmetics

Proposition 1 *Given two integers dv and cv, they uniquely define a tonal element (o, tc) as follows :*

1. $o = \lfloor \frac{dv}{7} \rfloor$
2. $diatonic(tc) = modulo(dv, 7)$
3. $alteration = cv - chromatic\_value((o, (diatonic\_point(tc), 0)))$

Definition 1 *Addition of tonal elements:*
Let (o_1, tc_1), (o_2, tc_2) be tonal elements. Their addition, $(o, tc) = (o_1, tc_1) + (o_2, tc_2)$, is the tonal element that is uniquely defined (by the above Proposition) from:

- $dv = diatonic\_value(o_1, tc_1) + diatonic\_value(o_2, tc_2)$
- $cv = chromatic\_value(o_1, tc_1) + chromatic\_value(o_2, tc_2)$

Claim 1 *The set TE of tonal elements, together with the addition operation forms a commutative additive group.*

Proof:

1. **Commutativity**: By the commutativity of numeric addition, and the above proposition, $(o_1, tc_1) + (o_2, tc_2) = (o_2, tc_2) + (o_1, tc_1)$
2. **Associativity**: By the associativity of numeric addition, and the above proposition, $(o_1, tc_1) + ((o_2, tc_2) + (o_3, tc_3)) = ((o_1, tc_1) + (o_2, tc_2)) + (o_3, tc_3)$
3. **Unit element**: For every tonal element (o, tc): $(o, tc) + (0, (0, 0)) = (o, tc)$
4. **Inverse element**: The inverse element of a tonal element (o, tc) is defined by the above Proposition from:
   - $dv = -(diatonic\_value(o, tc))$
   - $cv = -(chromatic\_value(o, tc))$

   It is easy to verify that the inverse element, denoted $-(o, tc)$ indeed satisfies: $(o, tc) + (-(o, tc)) = (0, (0, 0))$

Definition 2 *Subtraction of tonal elements:*
Let (o_1, tc_1) and (o_2, tc_2) be tonal elements. Then: $(o_1, tc_1) - (o_2, tc_2) = (o_1, tc_1) + (-(o_2, tc_2))$

4 Translations from and to the Tonal Pitch and Tonal Interval to the Tonal Element Abstraction

1. Mapping Tonal-Pitch-Classes and Tonal-interval-Classes to Tonal-Classes, and back:

   (a) The translations $tpc \rightarrow tc$ and $tc \rightarrow tpc$ to and from Tonal-Pitch-Classes and Tonal-Classes are defined component-wise, in the following tables:

$diatonic\_tpc \leftrightarrow tc$							
$Diatonic\_Pitch$	C	D	E	F	G	A	B
$Diatonic\_Point$	0	1	2	3	4	5	6

$alteration\_tpc \leftrightarrow tc$					
$Pitch\_Alteration$	$\flat\flat$	$\flat$	$\natural$	$\sharp$	$\sharp\sharp$
$Alteration$	-2	-1	0	1	2

   For example : $tpc \rightarrow tc(G, \sharp\sharp) = (4, 2)$

   (b) The translations to and from Tonal-Interval-Classes and Tonal-Classes are defined component-wise, in the following tables:

$diatonic\_tic \leftrightarrow tc$							
$Diatonic\_Interval$	$Prime$	$Second$	$Third$	$Fourth$	$Fifth$	$Sixth$	$Seventh$
$Diatonic\_Point$	0	1	2	3	4	5	6

   For a $Diatonic\_Interval \in \{Prime,\ Fourth,\ Fifth\}$:

$alteration\_Perfect\_tic \leftrightarrow tc$			
$Interval\_Alteration$	$Diminished$	$Perfect$	$Augmented$
$Alteration$	-1	0	1

   For a $Diatonic\_Interval \in \{Second,\ Third,\ Sixth,\ Seventh\}$:

$alteration\_MajMin\_tic \leftrightarrow tc$				
$Interval\_Alteration$	$Diminished$	$Minor$	$Major$	$Augmented$
$Alteration$	-2	-1	0	1

   For example : $tic \rightarrow tc(Fourth, Augmented) = (3, 1)$

2. Translations from and to Tonal Pitches and Tonal Elements:

   - $tp \rightarrow te(o, tpc) = (o, tpc \rightarrow tc(tpc))$

- $te \rightarrow tp(o, tc) = (o, tc \rightarrow tpc(tc))$

3. Translations from and to Tonal Intervals and Tonal Elements:

   - $ti \rightarrow te(dir, o, tic) =$
     if $dir = up$: $(o, tic \rightarrow tc(tic))$
     if $dir = down$: $-(ti \rightarrow te(up, o, tic))$
     where $-$ is the inverse operation of $+$ on TE.

   - $te \rightarrow ti(o, tc) =$
     if $o \geq 0$: $(up, o, tc \rightarrow tic(tc))$
     if $o < 0$: $(te \rightarrow ti(-(o, tc)))_{/dir=down}$
     where $-$ is the inverse operation of $+$ on TE, and
     for a $Tonal\_Interval$ ti, $ti_{/dir=val}$ denotes a $Tonal\_Interval$ that is equal to ti except for $dir = val$.

Example 1 *Translations: Tonal Elements, Tonal Pitches, Tonal Intervals:*

- $ti \rightarrow te((up, 1, (Fifth, Diminished))) = (1, (4, -1))$
- $ti \rightarrow te((down, 1, (Fifth, Diminished))) = -(1, (4, -1)) = (-2, (3, 1))$
- $te \rightarrow tp((0, (4, 1))) = (0, (G, \natural))$
- $te \rightarrow ti((0, (6, 0))) = (up, 0, (Seventh, Major))$

Example 2 *Tonal Arithmetics:*

1. $Tonal\_Pitch + Tonal\_Interval = Tonal\_Pitch$:
   $(0, (G, \natural)) + (up, 0, (Fourth, Perfect)) =$
   $te \rightarrow tp[tp \rightarrow te((0, (G, \natural))) + ti \rightarrow te((UP, 0, (Fourth, Perfect)))] =$
   $te \rightarrow tp[(0, (4, 0)) + (0, (3, 0))] = te \rightarrow tp((1, (0, 0))) = (1, (C, \natural))$

2. $Tonal\_Interval + Tonal\_Interval = Tonal\_Interval$:
   $(up, 0, (Third, Major)) + (up, 0, (Third, Minor)) =$
   $te \rightarrow ti[ti \rightarrow te((up, 0, (Third, Major))) + ti \rightarrow te((up, 0, (Third, Minor)))] =$
   $te \rightarrow ti[(0, (2, 0)) + (0, (2, -1))] = te \rightarrow ti((0, (4, 0))) = (up, 0, (Fifth, Perfect))$

3. $Tonal\_Interval - Tonal\_Interval = Tonal\_Interval$:
   $(up, 0, (Seventh, Minor)) - (up, 0, (Third, Minor)) =$
   $te \rightarrow ti[ti \rightarrow te((up, 0, (Seventh, Minor))) + ti \rightarrow te((up, 0, (Third, Minor)))] =$
   $te \rightarrow ti[(0, (6, -1)) - (0, (2, -1))] = te \rightarrow ti[(0, (6, -1)) + (-(0, (2, -1)))] =$
   $te \rightarrow ti[(0, (6, -1)) + (-1, (5, 0))] = te \rightarrow ti((0, (4, 0))) = (up, 0, (Fifth, Perfect))$

4. Tonal_Pitch − Tonal_Pitch = Tonal_Interval:

$(0,(G,\natural)) - (1,(C,\natural)) =$

$te \to ti[tp \to te((0,(G,\natural))) - tp \to te((1,(C,\natural)))] =$

$te \to ti[(0,(4,0)) - (1,(0,0))] = te \to ti[(0,(4,0)) + (-(1,(0,0)))] =$

$te \to ti[(0,(4,0)) + (-1,(0,0))] = te \to ti((0,(3,0))) = (up, 0, (Fourth, Perfect))$

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Scheme procedures for mod-quotient and modulo.
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(define (mod-quotient n1 n2)
   (numerator (/ (- n1 (modulo n1 n2))
                 n2)))

(define (true-mod-quotient-modulo n1 n2)
        (= n1 (+ (* (mod-quotient n1 n2) n2)
                 (modulo n1 n2))))

> (mod-quotient 50 12)
4
> (modulo 50 12)
2
> (modulo -50 12)
10
> (mod-quotient -50 12)
-5
> (true-mod-quotient-modulo 50 12)
#t
> (true-mod-quotient-modulo -50 12)
#t
> (true-mod-quotient-modulo 0 12)
#t
```