Statistics for IT Managers
95-796, Fall 2012

Course Overview

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(see Blackboard for contact information!)

Statistics: why bother?

We have some problem we want to solve:
"Are book prices lower on the Internet?"
"What industry sectors are most profitable?"
"Should we invest in a new technology?"

Option 1: Rely on intuition ("Because users can more easily compare prices on the Internet, this will lead to more price competition and thus lower prices.")

Option 2: Collect and analyze real-world data to test whether your intuitions are correct.

Mass of data
Huge, unstructured, hard to interpret or use for decisions.

Statistics

Information
Brief, structured, interpretable, actionable.

1. "Gone with the Wind" $12 at Barnes and Noble (online)

2. "Statistics for Business and Economics" $100 at Amazon.com.

3. "Statistics for Business and Economics" $104 at B.Dalton.

(200,000 more records…)

Which methods to use?

How to apply them?
(by hand, by computer)

How to interpret results?

Descriptive statistics:
"For our data, prices are an average of $0.20 lower on the Internet."

Statistical inference:
"There (is / is not) a significant difference between textbook prices from online and physical retailers."

Goals of the course

- To provide individuals who aspire to IT management positions with the basic statistical tools for analyzing and interpreting data.
- By the end of this course, you should be able to correctly choose and apply the appropriate statistical methods for real-world problems related to IT management.
- Because most real-world datasets are too large to analyze by hand, you will be expected to learn and use the statistical software package Minitab.

Structure of the course

- 13 lectures divided into three modules:
  - Descriptive statistics and probability (4 lectures)
  - Hypothesis testing and inference (5 lectures)
  - Simple and multiple regression (4 lectures)
- Grades will be based on:
  - Three homeworks 30% (10% each)
  - Two mini-projects 30% (15% each)
  - Final exam 40%
- See syllabus on Blackboard for detailed schedule, and for course policies (cheating, late work, re-grades, e-mail questions).

Course textbook and slides

- Statistics for Business and Economics (11th ed.) by McClave, Benson, and Sincich.
  - Module 1 (Descriptive statistics and probability) covers Chapters 1-4.
  - Module 2 (Statistical inference) covers Chapters 5-7.
  - Module 3 (Regression) covers Chapters 10-11.
- Not all sections of these chapters will be covered. See syllabus for readings corresponding to each lecture.
- Slides for each module are available on Blackboard.

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Module 1: Descriptive Statistics and Probability (4 lectures)

Reading: Statistics for Business and Economics, Ch. 1-4

Basic definitions

- **Statistics** is the science of analyzing and interpreting data, i.e. transforming raw data into information.
- **Descriptive statistics** are used to organize and summarize data, and to present this information in a convenient and usable form.
  - Graphical displays (e.g. histograms, box plots)
  - Numerical summaries (e.g. mean, median, mode, variance)
- **Inferential statistics** use sample data to make estimates, decisions, predictions, or other generalizations about a larger set of data.
  - **Population**: data measuring some characteristic of all members of a group (“all teenage males who watch television”)
  - **Sample**: data on a representative subset of the population (“100 randomly sampled teenage males who watch television”)

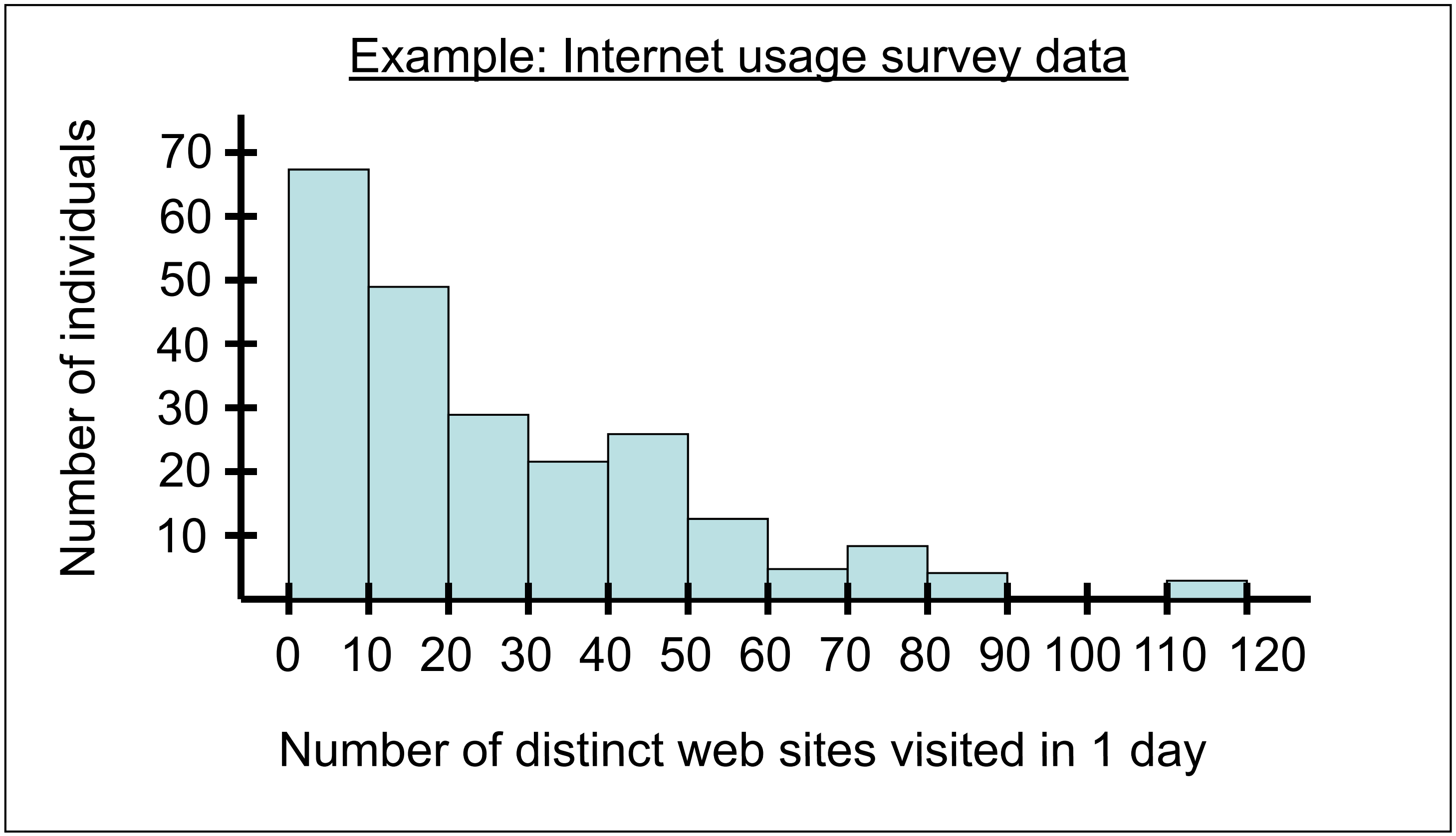
What can we conclude about the population, based on our sample?

Data types

- **Qualitative (or categorical) data:** each data point is classified into one of a given set of categories.
  - **Nominal data:** categories do not have a given order.
    - Animal type: {dog, cat, bird, fish}.
  - **Ordinal data:** categories have a given order.
    - Movie ranking: 1-5 stars.
- **Quantitative (or numerical) data:** each data point is measured on a naturally occurring numerical scale.
  - Height, weight, income, etc.

Histograms

- One of the many graphical methods for displaying numerical data.
- Shows counts or percentages of data in each interval.



Numerical descriptive statistics

- **Measures of the center of the data**
  - Mean, median, mode
- **Measures of variability**
  - Variance, standard deviation, range, interquartile range
- Some advantages of numerical statistics:
  - More succinct than graphical methods
  - Less subject to distortion
  - Form the basis for statistical inferences
- Any disadvantages?

Measures of the center

- **Mean:** the average of all values.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{(x_1 + x_2 + \ldots + x_n)}{n}$$

x_i = value of the i^{th} observation
n = total number of observations

- **Median:** the "middle" number when measurements are arranged in ascending (or descending) order.
- **Mode:** the most common value.

Example dataset: 1, 1, 2, 2, 2, 3, 4, 4, 5, 16

Mean = (1 + 1 + 2 + 2 + 2 + 3 + 4 + 4 + 5 + 16) / 10 = 4

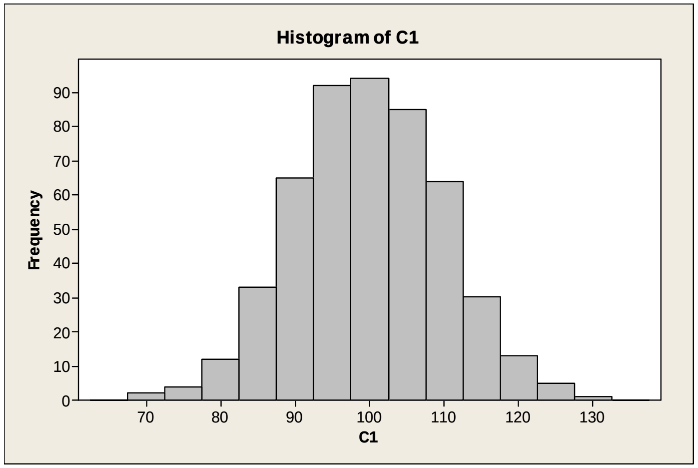
Median = (2 + 3) / 2 = 2.5

Mode = 2

Notice that the mean is more affected by outlier values than the median!

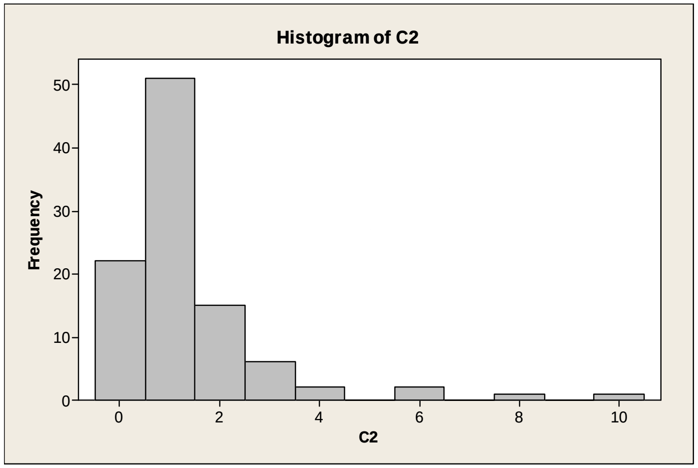
Skewed distributions

- A distribution is **symmetric** if mean = median.
- A distribution is **positively skewed** if mean > median.
- A distribution is **negatively skewed** if mean < median.



500 values generated from N(100,10)
Mean = 99.83, Median = 99.91

Approximately symmetric



100 values generated from F(3,5)
Mean = 1.37, Median = 0.88

Positively skewed

Measures of variability

- **Range:** the difference between the smallest and largest observations.
- **Interquartile range:** the difference between the 25th and 75th percentiles, where the kth percentile is a value such that k% of the observations are below that value and (100-k)% of the observations are above that value.

Example dataset: 1, 1, 2, 2, 2, 3, 4, 4, 5, 16

25th percentile = 2

75th percentile = 4

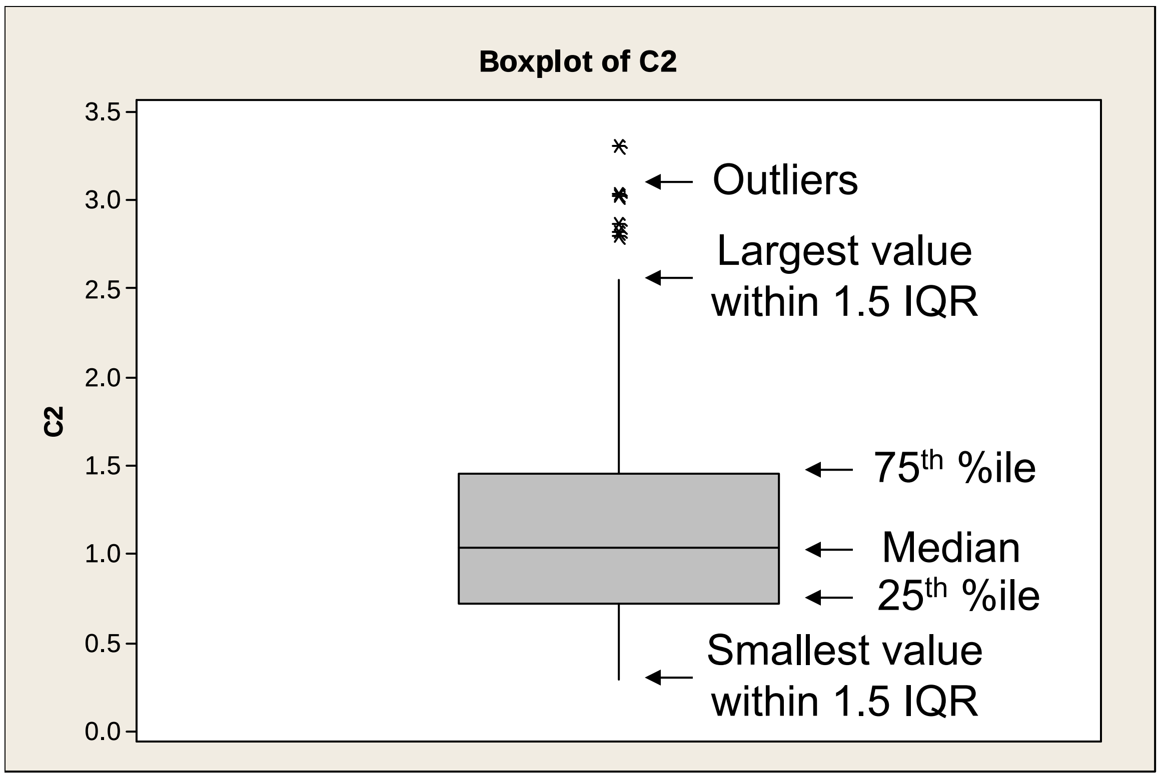
Range = 16 - 1 = 15.

Interquartile range = 4 - 2 = 2.

Like the median, the interquartile range is robust to outliers!

Box plots

- Make it easy to see the variability and skewness of a distribution, as well as any outliers (unexpected values).



Measures of variability

- **Variance:** the average squared deviation from the mean.
- **Standard deviation:** the square root of the variance.

Sample variance $s^2 = \dfrac{\sum (x_i - \bar{x})^2}{n - 1}$

Sample standard deviation $s = \sqrt{s^2}$

(n – 1) is used in the denominator instead of n.

This makes the sample variance s^2 an <u>unbiased estimator</u> of the population variance σ^2.

<u>Example dataset</u>: 1, 1, 2, 2, 2, 3, 4, 4, 5, 16

Mean = 4

Deviations: -3, -3, -2, -2, -2, -1, 0, 0, 1, 12

Squared deviations: 9, 9, 4, 4, 4, 1, 0, 0, 1, 144

Sample variance: $s^2 = (9 + 9 + 4 + 4 + 4 + 1 + 0 + 0 + 1 + 144) / (10 - 1) = \dfrac{176}{9}$

Sample standard deviation: $s = \sqrt{\dfrac{176}{9}} \approx 4.42$

Why measures of variability?

- Measures of the center tell us about our <u>expectation</u> (e.g. expected profit or loss).
- Measures of variability characterize our <u>risk</u> or <u>uncertainty</u> about this expectation.

<u>Scenario 1</u>: You are offered $5000.

Expected profit? Risk? Would you take this offer?

<u>Scenario 2</u>: You are offered a gamble on the flip of a fair coin.
If the coin comes up heads, you win $50K, otherwise you lose $40K.

Expected profit? Risk? Would you take this offer?

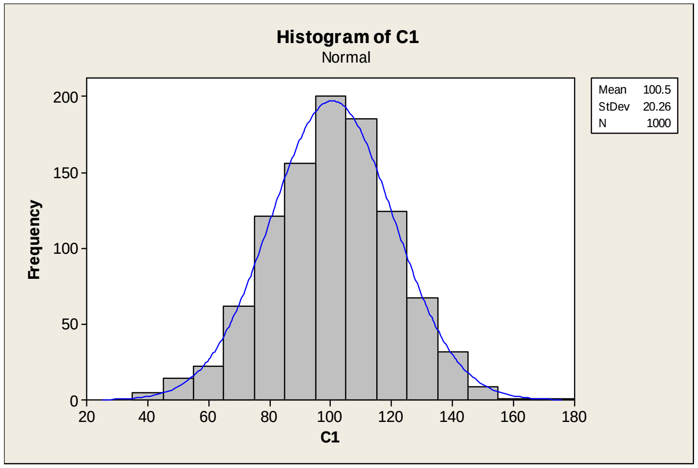
The empirical rule

- For symmetric, unimodal (“mound-shaped”) distributions:
  - Approximately 68% of the measurements will fall within 1 standard deviation of the mean.
  - Approximately 95% of the measurements will fall within 2 standard deviations of the mean.
  - Approximately 99.7% of the measurements will fall within 3 standard deviations of the mean.
- This rule is useful for:
  - Identifying outliers (erroneous data, unusual events)
  - Calibrating the likelihood of success.
  - “Guesstimating” the standard deviation.

<u>Example</u>: mean height of trees = 30 feet, standard deviation = 10 feet

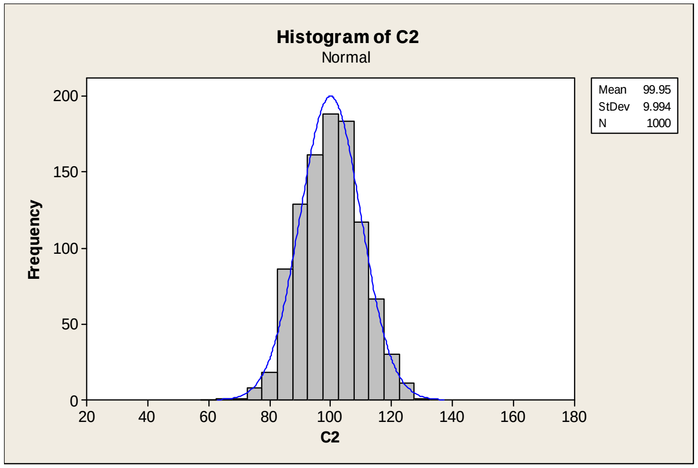
How likely are we to see a tree taller than 40 feet?
How likely are we to see a tree taller than 60 feet?

Examples of the empirical rule



1000 data points generated from N(100,20)

68% of the data should be between 80 and 120
95% of the data should be between 60 and 140
Almost all of the data should be between 40 and 160



1000 data points generated from N(100,10)

68% of the data should be between 90 and 110
95% of the data should be between 80 and 120
Almost all of the data should be between 70 and 130

Using Minitab

- Creating and listing data (p. 27-28)
- Graphing data (p. 110)
- Computing numerical descriptive statistics (p. 110-111)
- Generating a random sample (p. 170-171)

Why study probability?

- Basis for statistical inference:
  - Margin of error on opinion poll is +/- 4%.
  - Difference between test scores is significant at 5% level.
- Key element of business:
  - Expected profit, risk, uncertainty, etc.
- Key element of operations management :
  - Setting inventory level, delivery cycle, response time.
- Our intuitions about probabilities are terrible!

“98% of individuals who do not make a return visit to a web site are first-time visitors.”

“98% of first-time visitors will not make a return visit to a web site.”

Basic definitions

- **Probability of A**: a number P(A) between zero and one, indicating the likelihood of event A.
  - P(coin flip lands on heads) = ½
  - P(it will rain tomorrow) = 0.8
- Interpreting probability as **relative frequency**:

$$P(A) = \lim_{n \to \infty} \frac{\text{\# of times event A occurs in n trials}}{n}$$

- Probabilities can be **objective** or **subjective**.
- **Complement** of event A: the event that A does not occur, usually denoted by ~A, A^C, A', or $\overline{A}$.
  - Important rule: P(~A) = 1 – P(A).

Combining probabilities

- Given two events A and B, the probability of both events occurring simultaneously is denoted by $P(A \cap B)$, i.e. the "probability of A and B."
- The probability of at least one of the two events occurring is denoted by $P(A \cup B)$, i.e. the "probability of A or B."
- Important rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Example: x = roll of a six-sided die. $P(\{x \text{ is even}\} \cup \{x \geq 3\})$
- Mutually exclusive events: $P(A \cap B) = 0$.
  - For mutually exclusive events, $P(A \cup B) = P(A) + P(B)$.
  - Example: x = roll of a six-sided die. $A = \{x \text{ is even}\}$, $B = \{x = 1\}$.
  - Example: A and ~A are mutually exclusive and exhaustive.

$P(A \cap \sim A) = 0$ $\qquad$ $P(A \cup \sim A) = 1$

Conditional probabilities

- Given that an event B has occurred, the probability that event A has also occurred is denoted by $P(A \mid B)$, i.e. the "probability of A given B."
  - Example: x = roll of a six-sided die. $P(\{x \text{ is even}\} \mid \{x \leq 5\})$
- Important rule: $P(A \mid B) = P(A \cap B) / P(B)$.
  - Note that $P(A \mid B) \neq P(B \mid A)$
  - Example: x = roll of a six-sided die. $P(\{x \leq 5\} \mid \{x \text{ is even}\})$
- Another way to express this rule:

  $$P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$$
- Given mutually exclusive and exhaustive events $B_1..B_n$:

  $$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) \\ &= P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + \ldots + P(A \mid B_n)\, P(B_n). \end{aligned}$$

Example: There are three coins in a box: one fair coin, one two-headed coin, and one biased coin with P(heads) = 2/3. If you draw one coin at random and flip it, what is the probability that it lands on heads?

Independent events

- Two events A and B are said to be <u>independent</u> if:
  $P(A \mid B) = P(A \mid \sim B) = P(A)$, and $P(B \mid A) = P(B \mid \sim A) = P(B)$.
- In other words, two events are independent if the occurrence (or non-occurrence) of one event does not change the probability that the other will occur.
- Independent or dependent?
  - <u>Example 1</u>: A = heads on first toss of a fair coin, B = tails on second toss of that coin.
  - <u>Example 2</u>: A = individual knows Java programming, B = that individual is an engineer.
  - <u>Example 3</u>: A = heads on first toss of a fair coin, B = tails on first toss of that coin.
- If A and B are independent:
  $P(A \cap B) = P(A \mid B)\,P(B) = P(A)\,P(B)$.
- More generally, for independent events $A_1..A_n$:
  $P(A_1 \cap \ldots \cap A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$.

Bayes' Theorem

- A way of figuring out a conditional probability P(A | B) if we have the opposite conditional probability, P(B | A).
- In fact, we have to know the probabilities P(B | A) and P(B | ~A), as well as the "prior probability" P(A).

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A \cap B) + P(\sim A \cap B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \sim A)P(\sim A)}$$

- More generally, given mutually exclusive and exhaustive events $A_1..A_n$:

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + \ldots + P(B \mid A_n)P(A_n)}$$

<u>Example</u>: There are three coins in a box: one fair coin, one two-headed coin, and one biased coin with P(heads) = 2/3. You draw one coin at random and flip it: it lands on heads. What is the probability that it is the fair coin?

Random variables

- **Sample space**: the set of all possible outcomes of a statistical experiment.
  - Flipping three coins: HHH, HHT, …, TTT
- **Random variable**: a variable that assigns a numerical value to each possible outcome.
  - Number of heads flipped: 3 if HHH, 2 if HHT, etc.
- Random variables can be **discrete** or **continuous**:
  - **Discrete variable** can take a countable number of values (e.g. number of heads flipped = 0, 1, 2, or 3).
  - **Continuous variable** can take an uncountable number of values (e.g. height, weight, response time).

Discrete random variables

- **Probability mass function p(x)** specifies the probability associated with each possible value of the discrete random variable x.
  - Example: x = number of heads in three coin flips.

$p(0) = 1/8$	{TTT}
$p(1) = 3/8$	{TTH, THT, HTT}
$p(2) = 3/8$	{THH, HTH, HHT}
$p(3) = 1/8$	{HHH}

- We must have $p(x) \geq 0$ for all x, and $\sum p(x) = 1$.
- **Mean** (or **expected value**): $\mu = \sum x\, p(x)$.
- **Variance**: $\sigma^2 = \sum (x - \mu)^2\, p(x)$.
- **Standard deviation**: $\sigma = \sqrt{\sigma^2}$

What are the mean and standard deviation of x for the coin flip example?

Sampling of random variables

- Let us assume that we perform the “three coin flip” experiment 80 times, and count the number of heads x for each experiment:
  - <u>We expect</u>: 10 {x=0}, 30 {x=1}, 30 {x=2}, 10 {x=3}. (Mean = 1.5, Variance = 0.75)
  - <u>First trial</u>: 12 {x=0}, 22 {x=1}, 31 {x=2}, 15 {x=3}. (Mean = 1.61, Variance = 0.92)
  - <u>Second trial</u>: 12 {x=0}, 27 {x=1}, 32 {x=2}, 9 {x=3} (Mean = 1.47, Variance = 0.78)
- Notice that the sample proportions are close, but not equal, to the expected proportions p(x).
- As the number of trials increases, the sample proportions will converge to their expectations, as will the sample mean and sample variance.

“Law of Large Numbers”

A practice problem

- An insurance company sells hurricane damage insurance to a Florida homeowner for \$1,000/year. In a given year, there is a 95% chance of no damage, 4% chance of minor (\$20,000) damage, and a 1% chance of major (\$80,000) damage.
  - Let x = the insurance company's profit. What is p(x)?

    $p(1{,}000) = 0.95$, $p(-19{,}000) = 0.04$, $p(-79{,}000) = 0.01$.
  - What is the probability that the insurance company will make a profit in a given year?

    $P(x > 0) = 95\%$.
  - What is the company's expected yearly profit? Is this a profitable policy for the insurance company?

    $0.95(\$1{,}000) + 0.04(-\$19{,}000) + 0.01(-\$79{,}000) = -\$600$.

    Not profitable!

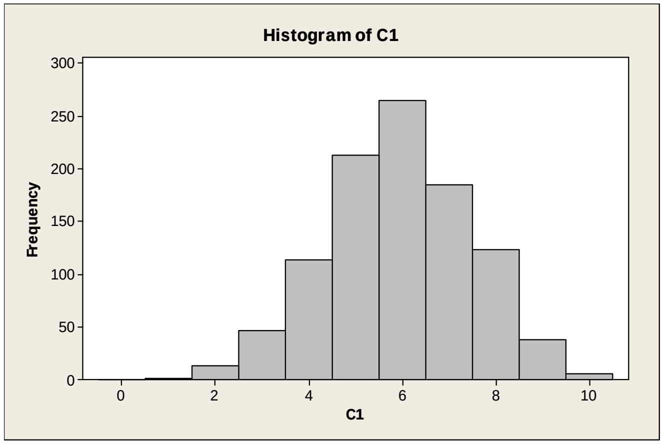
The binomial distribution

- Given an experiment with probability p of success. Let random variable x denote the number of successes in n independent trials.
- Then x follows a **binomial distribution**, x ~ Bin(n,p).

$$p(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \text{ for } 0 \le x \le n$$

- For example, we have a weighted coin with P(heads) = 0.6.

  Let x = the number of heads in 10 trials.



x ~ Bin(10,0.6)

<u>For x ~ Bin(n,p)</u>

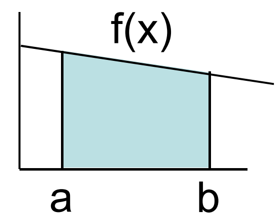
Mean of x: $\mu = np$.

Variance of x: $\sigma^2 = np(1-p)$

Continuous random variables

- **Probability density function f(x)** specifies the probability associated with each range of the continuous random variable x:

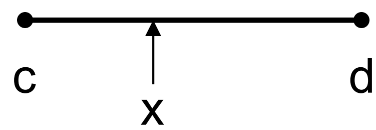
$$P(a \le x \le b) = \int_a^b f(x)dx$$ ← Area under the curve f(x), from a to b



- We must have f(x) ≥ 0 for all x, and $\int f(x)dx = 1$.
- **Mean** (or **expected value**): $\mu = \int x\, f(x)dx$
- **Variance**: $\sigma^2 = \int (x-\mu)^2 f(x)dx$
- **Standard deviation**: $\sigma = \sqrt{\sigma^2}$

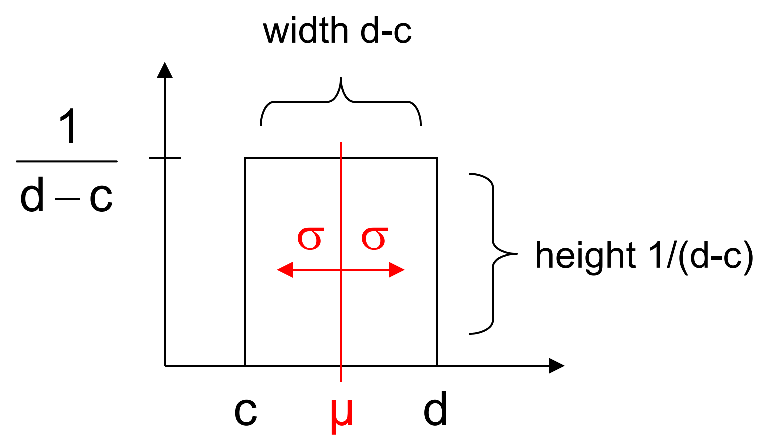
The uniform distribution

- Choose a point on the interval [c,d], where each point on the interval is equally likely.



$x \sim \text{Uniform}(c,d)$

$$f(x) = \begin{cases} \dfrac{1}{d-c} & \text{if } c \le x \le d \\ 0 & \text{otherwise} \end{cases}$$



Mean: $\mu = (c + d) / 2$

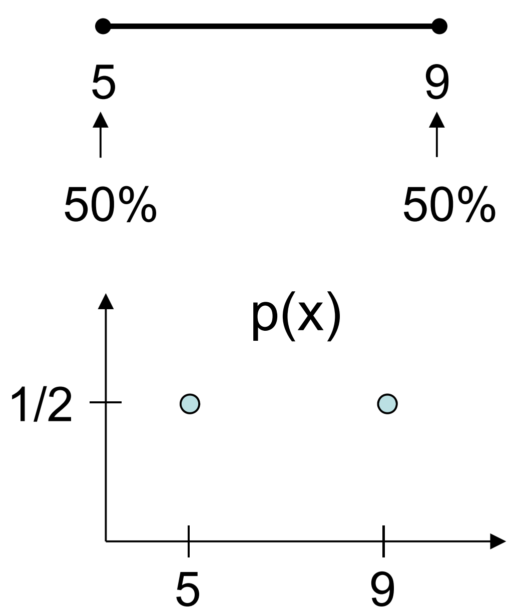
Variance: $\sigma^2 = (d - c)^2 / 12$

Std. dev.: $\sigma = (d - c) / \sqrt{12}$

<u>Example</u>: if product weights are uniformly distributed on [1,1.5], what is the probability that a product will have weight > 1.2?

Comparison of discrete and continuous random variables

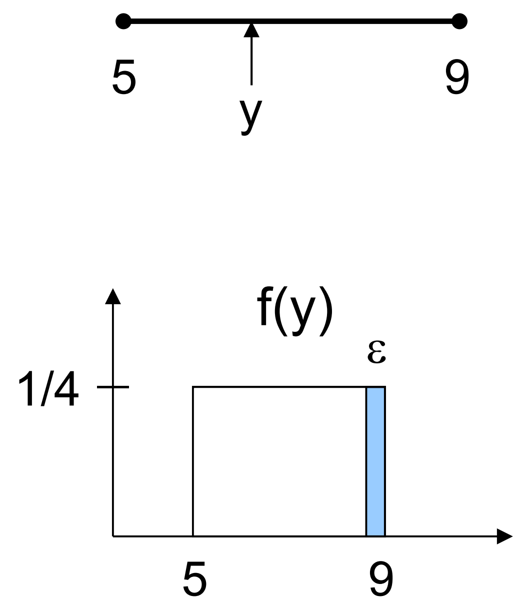
x ~ Endpoints(5,9)



Probability mass function

Sum of values = 1

y ~ Uniform(5,9)



Probability density function

Area under curve = 1

Pr(x = 5) = Pr(x = 9) = ½.

Pr(9 – ε ≤ x ≤ 9) = ε / 4.

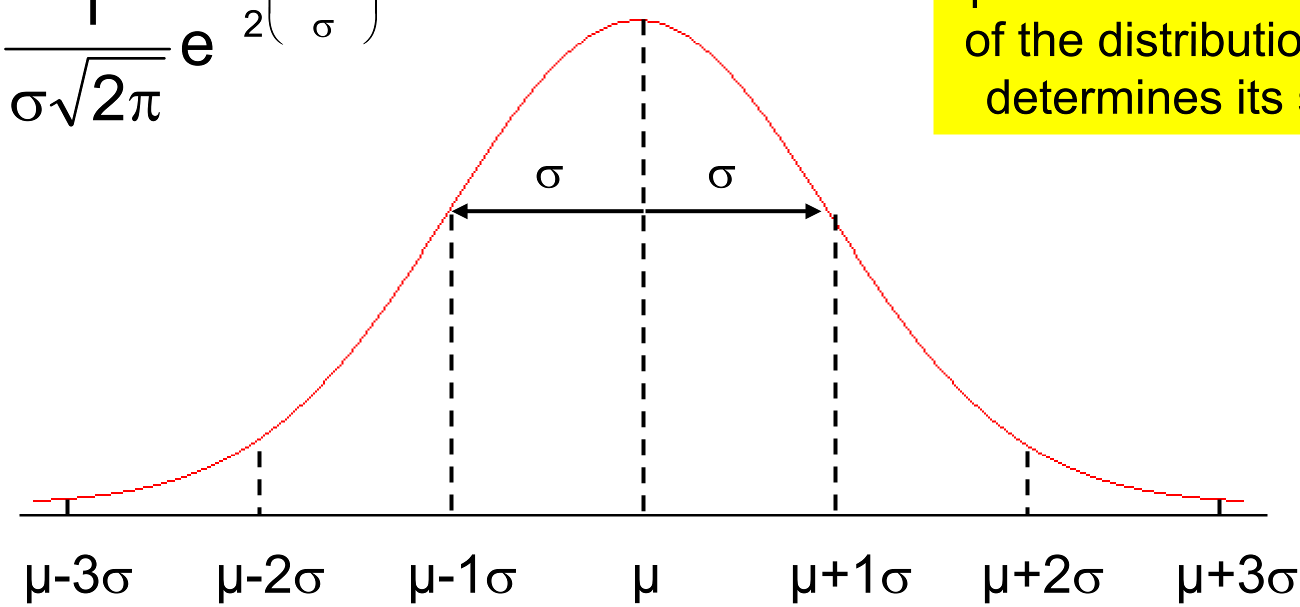
What are μ and σ for each distribution?

The Normal distribution

- The most important distribution for statistical inference!
  - Many real-world distributions are approximately normal.
- Also called "Gaussian distribution" or "bell curve".
- A symmetric, unimodal distribution $N(\mu, \sigma)$, determined by its mean μ and standard deviation σ:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ determines the center of the distribution, and σ determines its spread.



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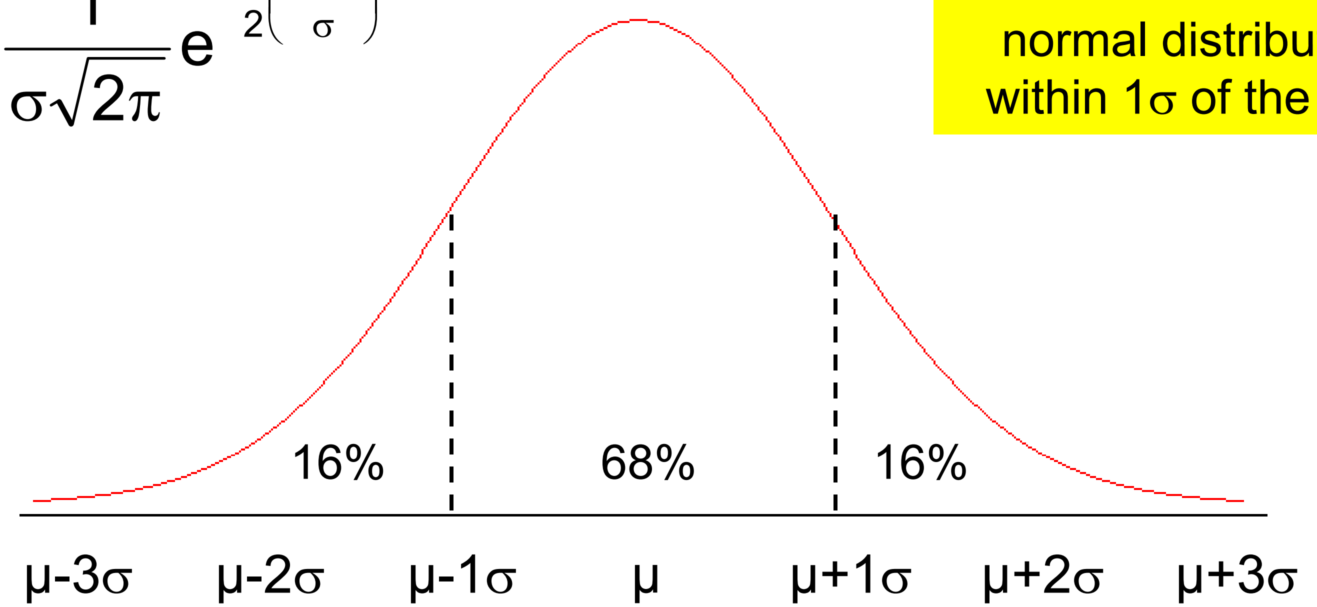
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

~68% of the area of the normal distribution is within 1σ of the mean.



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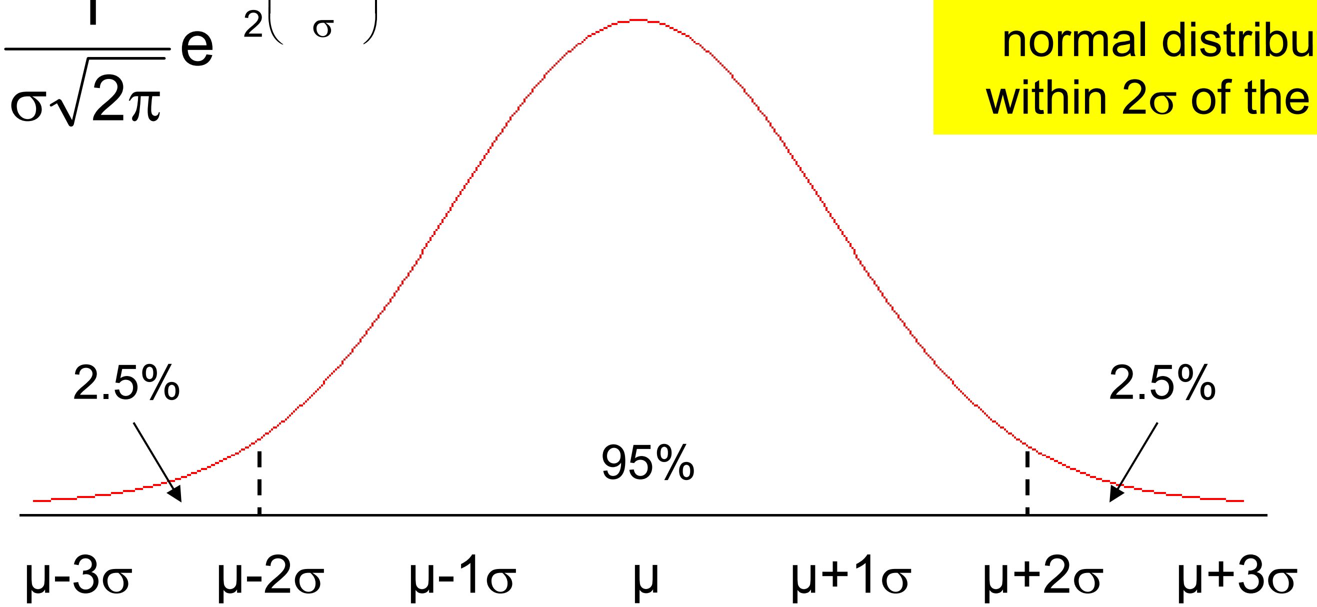
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

~95% of the area of the normal distribution is within 2σ of the mean.



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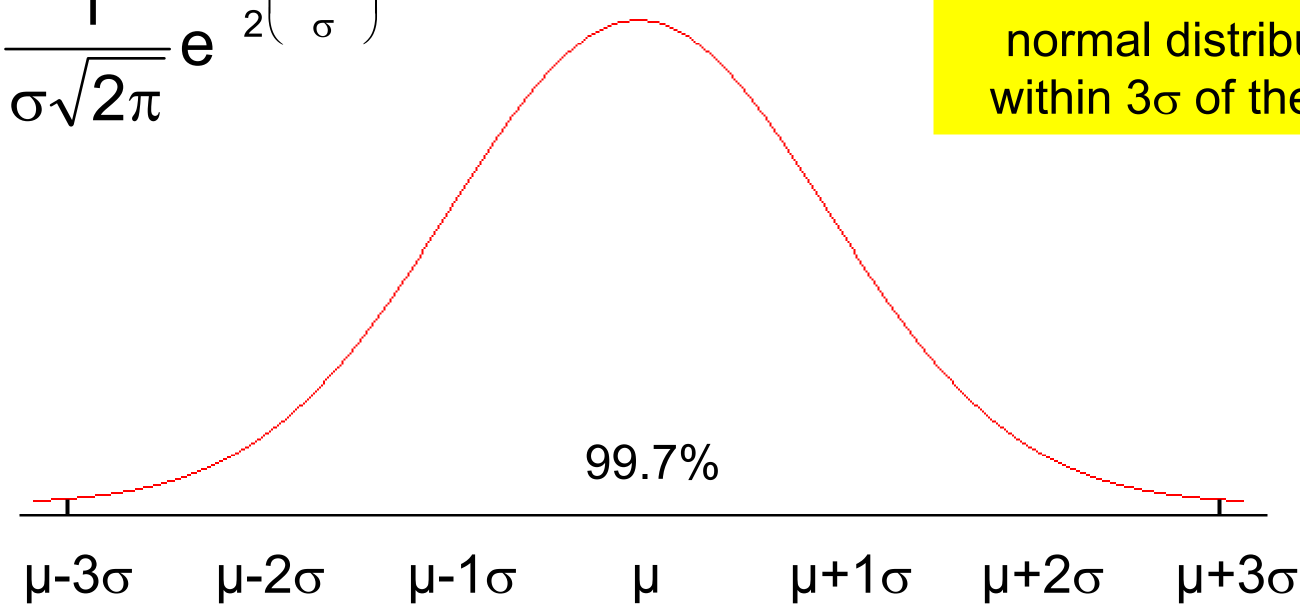
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

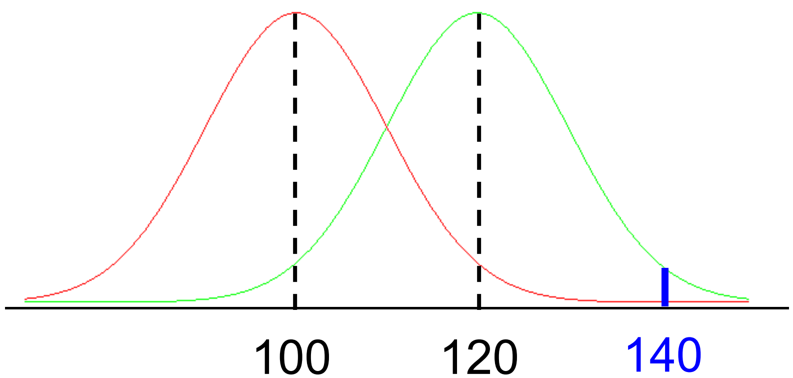
~99.7% of the area of the normal distribution is within 3σ of the mean.



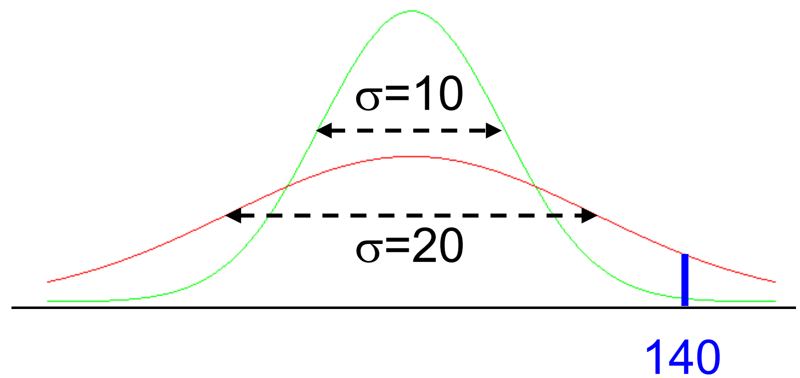
Computing normal probabilities

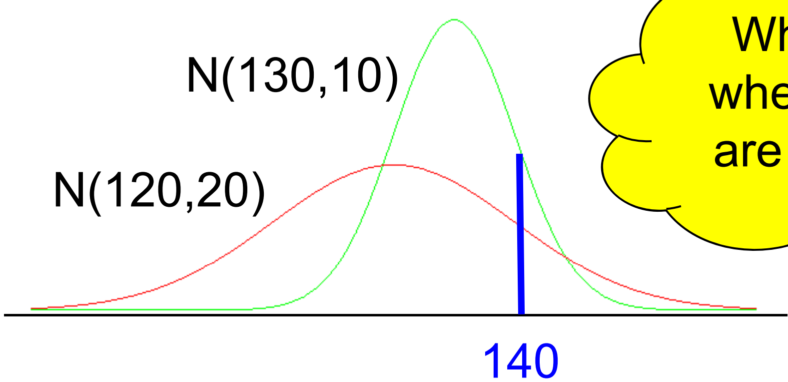
- Normal probabilities depend both on μ and σ.
  - <u>Example</u>: which has higher probability of $x > 140$?

Same σ=10, different μ



Same μ=110, different σ



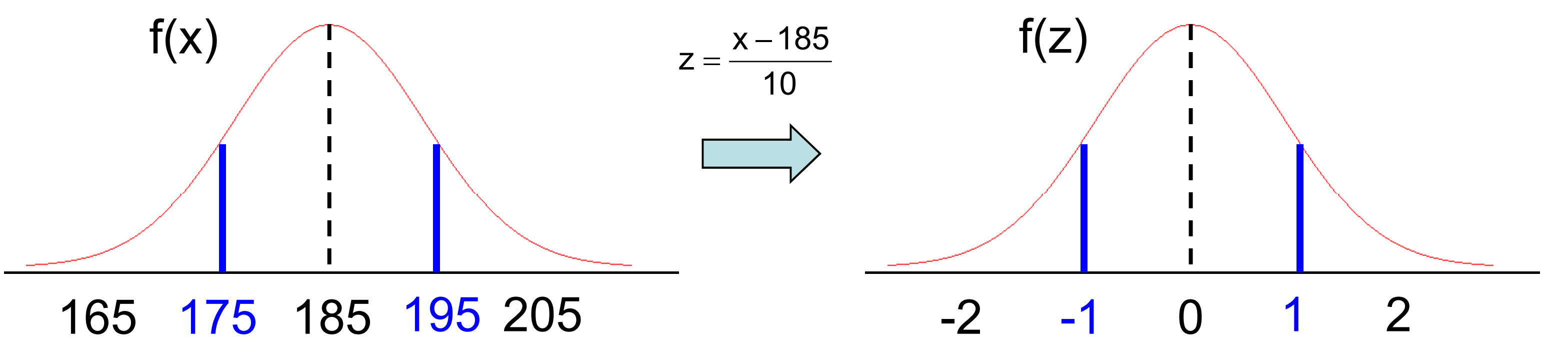


What about when μ <u>and</u> σ are different?

<u>Solution</u>: transform each distribution using the **z-score**!

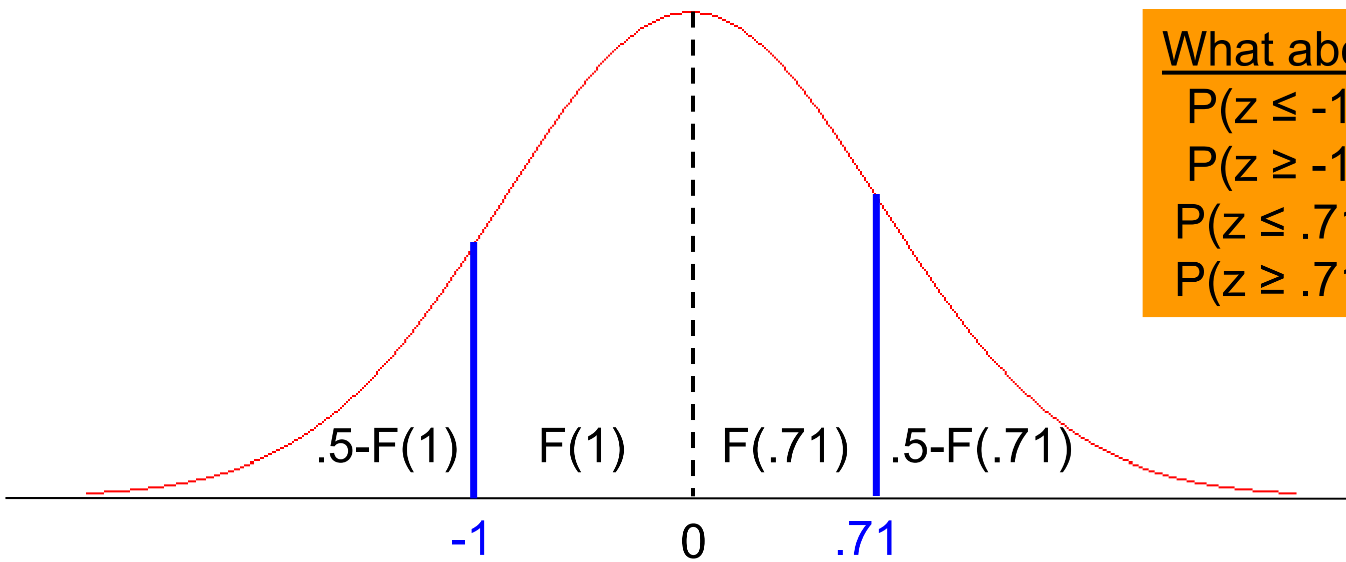
Computing z-scores

- If x is distributed according to N(μ, σ), then $z = \frac{x-\mu}{\sigma}$ will be distributed according to the **standard normal distribution**, N(0,1).
  - The **z-score** (z) is the number of standard deviations (σ) that the original measurement (x) is from the mean (μ).
  - Example: man's weight x ~ N(185,10).
    $P(175 \le x \le 195) = P(-1 \le z \le 1) \approx 68\%$.



Using a table of normal curve areas

- Once we have converted to z-scores, how do we compute more general probabilities, e.g. $P(-1 \le z \le .71)$?
- Answer: use a table of normal curve areas (or Minitab).
  - The table gives $F(z_0) = P(0 \le z \le |z_0|)$.
  - We can use these values to compute any desired probability.
- Example: $P(-1 \le z \le .71) = F(1) + F(.71) = .3413 + .2611 = .6024$



What about:
$P(z \le -1)$?
$P(z \ge -1)$?
$P(z \le .71)$?
$P(z \ge .71)$?

A practice problem

- Let us assume that men's weights are normally distributed with $\mu = 185$ and $\sigma = 20$, while women's weights are normally distributed with $\mu = 150$ and $\sigma = 10$. Are men or women more likely to have weight between 160 and 170?

<u>1st step: Convert to z-scores</u>

Men: $P(160 < x < 170) = P(-1.25 < z < -.75)$

Women: $P(160 < x < 170) = P(1 < z < 2)$

<u>2nd step: Compute probabilities</u>

Men: $P(-1.25 < z < -.75) = F(1.25) – F(.75) = .3944 – .2734 = .1210$

Women: $P(1 < z < 2) = F(2) – F(1) = .4772 – .3413 = .1359$

An "inverse" problem

- Large employers regularly use skill tests to evaluate potential employees. Suppose a test of programming proficiency has a mean score of 60% and standard deviation of 10%. If the employer only wants to hire the most proficient 20% of applicants, what is the minimum test score they should set?

<u>1st step: Compute the necessary range of z-scores</u>

$P(z > z_0) = 0.2$

$P(0 < z < z_0) = 0.5 - 0.2 = 0.3$

$z_0 = F^{-1}(0.3) \approx 0.84$

<u>2nd step: Compute the necessary range of values</u>

$z > 0.84$

$x > 60\% + 0.84(10\%) \rightarrow x > 68.4\%$

What if the employer wants to <u>avoid</u> hiring the bottom 20% of applicants?

Why the normal distribution?

- Central Limit Theorem: averages are approximately normally distributed.
  - More samples = closer to a normal distribution.
  - More samples = lower variance.
- Other probability distributions (e.g. binomial) can be expressed as a sum, and thus are also approximately normally distributed.
- These properties will be very useful for inference (confidence intervals and hypothesis testing), as we will discuss in Module II.

Parameters and sample statistics

- If we know the probability distribution of a random variable, we can compute its mean μ, standard deviation σ, and associated probabilities.
  - "The average response time in minutes for a network outage is normally distributed with $\mu = 47$, $\sigma = 18$."
- What if we <u>don't</u> know the distribution, but only have samples from this distribution?
  - "For the last 5 network outages, response times were 43, 79, 21, 71, and 51 minutes ($\bar{x} = 53$, $s \approx 23$)."

What can we conclude about **population parameters** μ and σ, using the **sample statistics** $\bar{x}$ and s?

Parameters and sample statistics

- If we know the probability distribution of a random variable, we can compute its mean μ, standard

The sample mean $\bar{x}$ can be used as an estimate of the population mean μ. But how good an estimate is it?

Intuitively, $\bar{x}$ will be a good estimate if the number of samples is large, and a poor estimate if the number of samples is small.

•

  – "For the last 5 network outages, response times were 43, 79, 21, 71, and 51 minutes ($\bar{x}$ = 53, s ≈ 23)."

What can we conclude about **population parameters** μ and σ, using the **sample statistics** $\bar{x}$ and s?

Sampling distributions

- A parameter such as μ or σ describes some characteristic of a population. It is a fixed quantity that is calculated from all observations in the population.
- A sample statistic such as $\bar{x}$ or s describes some characteristic of a sample. It is calculated only from those members of the population that are included in the sample.
- Since the value of a sample statistic will be different for each sample, a sample statistic is a random variable.
  - The probability distribution of this random variable is called its sampling distribution.

Sampling distributions

- <u>Example</u>: You want to know the proportions of children and adults in a room.
- You observe only two of the five people in the room: let x be the proportion of children in ths sample.
- If there are actually four adults and one child, what is the sampling distribution of x?

$p(0) = 6/10$ $\{A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4\}$

$p(1/2) = 4/10$ $\{A_1C, A_2C, A_3C, A_4C\}$

$\mu_x = 1/5$

$\sigma_x \approx .24$

The sample statistic x is an <u>unbiased estimate</u> of the proportion of children in the population.

Sampling distributions

- Example: You want to know the proportions of children and adults in a room.
- You observe only **four** of the five people in the room: let x be the proportion of children in ths sample.
- If there are actually four adults and one child, what is the sampling distribution of x?

$p(0) = 1/5$ $\{A_1A_2A_3A_4\}$

$p(1/4) = 4/5$ $\{A_1A_2A_3C, A_1A_2A_4C, A_1A_3A_4C, A_2A_3A_4C\}$

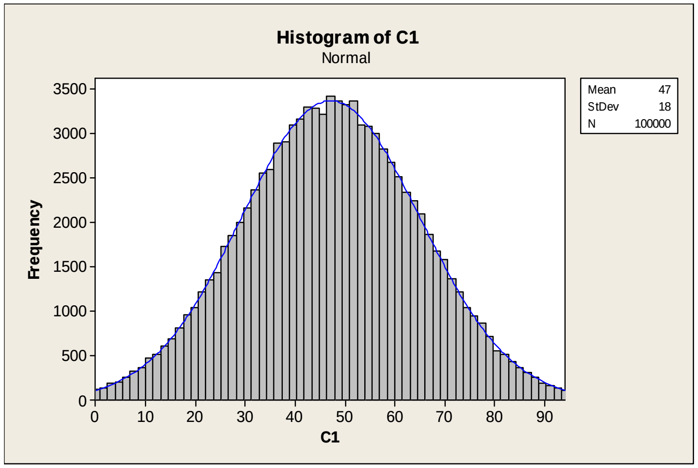
$\mu_x = 1/5$

$\sigma_x = \mathbf{.10}$ — Larger sample size leads to a lower variance of the sampling distribution, i.e. better estimates!

Using $\overline{x}$ to estimate μ

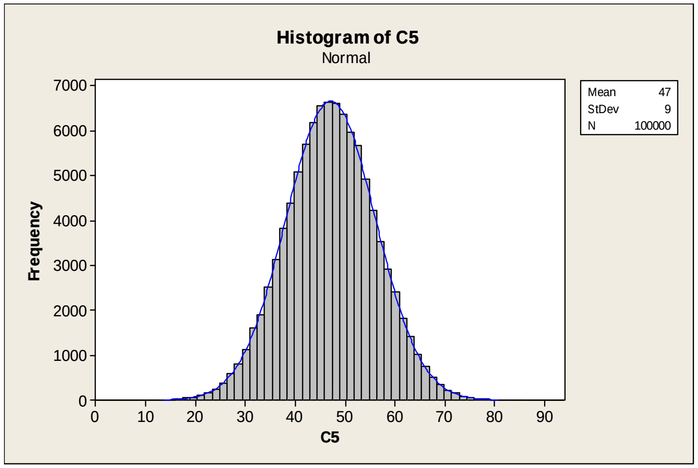
Let us assume that the population is normally distributed with μ = 47, σ = 18.

Here is a histogram of 100,000 samples drawn from the population.



Now consider drawing N = 4 samples from the population and taking their mean, $\overline{x}$.

We repeat this experiment 100,000 times and form a histogram of the values of $\overline{x}$.

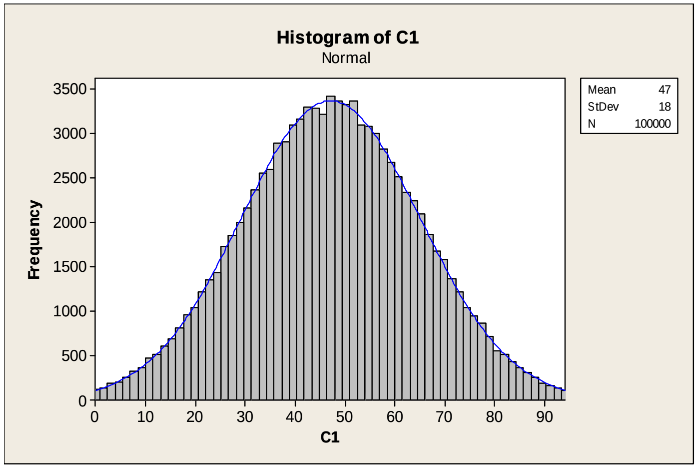


The sampling distribution of $\overline{x}$ is normal, with mean $\mu_{\overline{x}}$ = 47 and standard deviation $\sigma_{\overline{x}}$ = 9. Notice that the sample mean $\overline{x}$ is an unbiased estimator of the population mean μ. Additionally, the sample mean will be between 38 and 56 about 68% of the time.

Using $\bar{x}$ to estimate μ

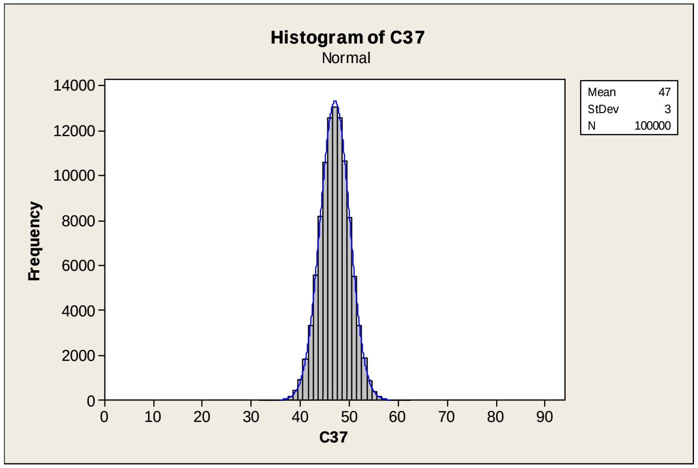
Let us assume that the population is normally distributed with $\mu = 47$, $\sigma = 18$.

Here is a histogram of 100,000 samples drawn from the population.



Now consider drawing **N = 36** samples from the population and taking their mean, $\bar{x}$.

We repeat this experiment 100,000 times and form a histogram of the values of $\bar{x}$.

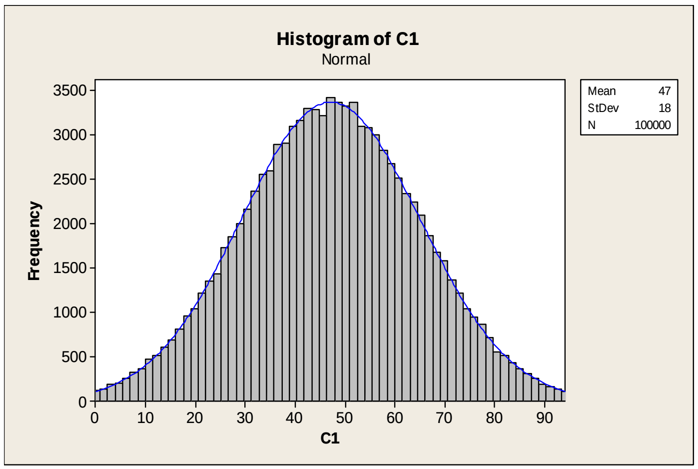


The sampling distribution of $\bar{x}$ is normal, with mean $\mu_{\bar{x}} = 47$ and standard deviation $\sigma_{\bar{x}} = $ **3**. Notice that the sample mean $\bar{x}$ is an unbiased estimator of the population mean μ. Additionally, the sample mean will be between **44** and **50** about 68% of the time.

Using $\bar{x}$ to estimate μ

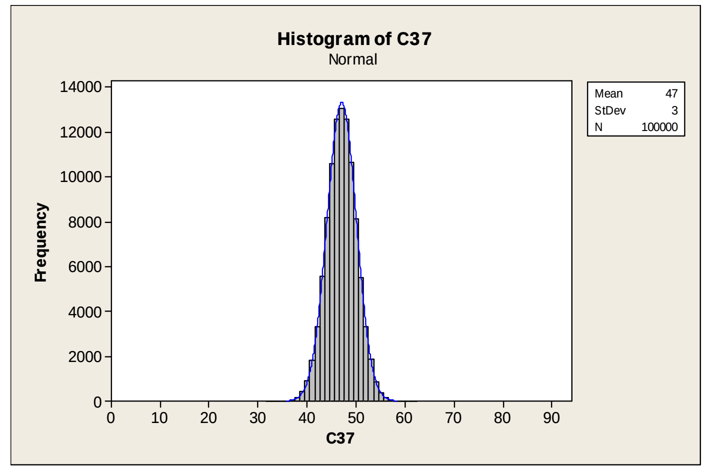
Let us assume that the population is normally distributed with μ = 47, σ = 18.

Here is a histogram of 100,000 samples drawn from the population.



Now consider drawing **N = 36** samples from the population and taking their mean, $\bar{x}$.

We repeat this experiment 100,000 times and form a histogram of the values of $\bar{x}$.

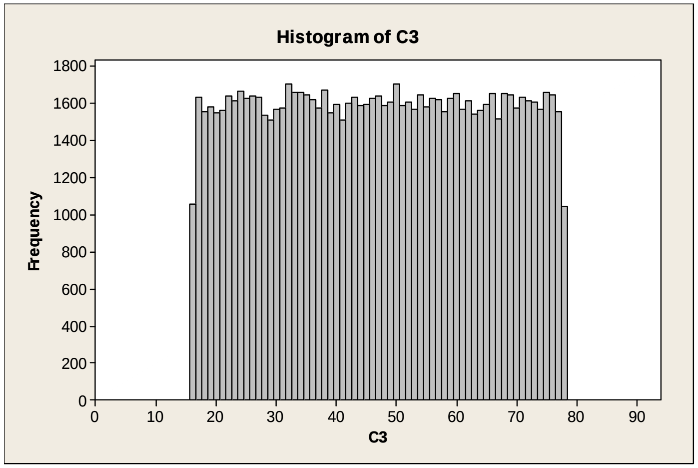


If the population is normally distributed with mean μ and standard deviation σ, then the sample mean $\bar{x}$ is also normally distributed, with mean μ and standard deviation $\sigma / \sqrt{N}$.

Using $\bar{x}$ to estimate μ

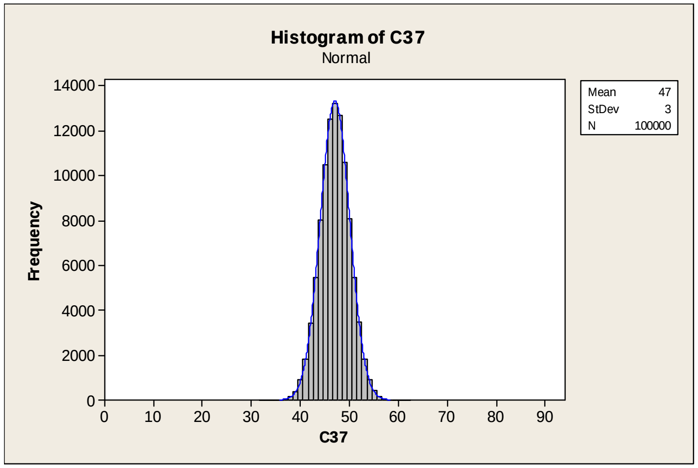
Let us assume that the population is **uniformly** distributed with $\mu = 47$, $\sigma = 18$.

Here is a histogram of 100,000 samples drawn from the population.



Now consider drawing N = 36 samples from the population and taking their mean, $\bar{x}$.

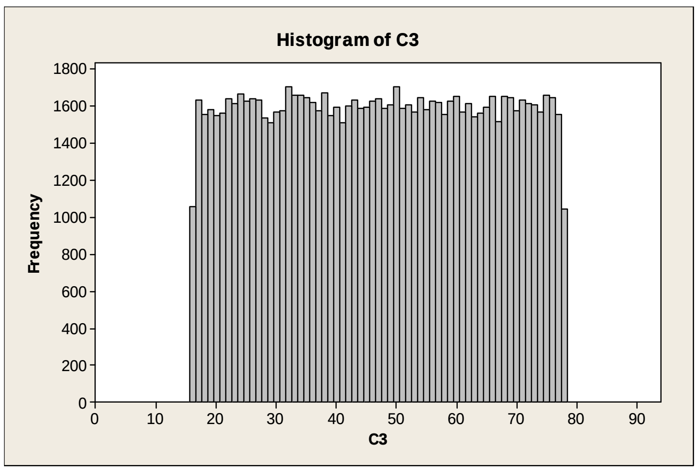
We repeat this experiment 100,000 times and form a histogram of the values of $\bar{x}$.



Using $\bar{x}$ to estimate μ

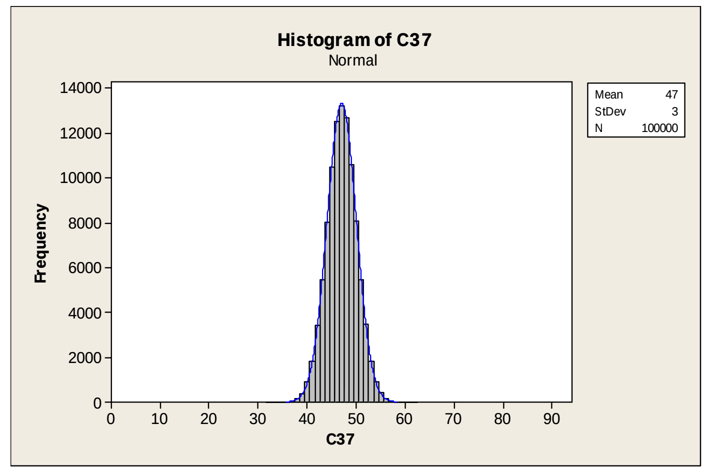
Let us assume that the population is **uniformly** distributed with $\mu = 47$, $\sigma = 18$.

Here is a histogram of 100,000 samples drawn from the population.



Now consider drawing N = 36 samples from the population and taking their mean, $\bar{x}$.

We repeat this experiment 100,000 times and form a histogram of the values of $\bar{x}$.



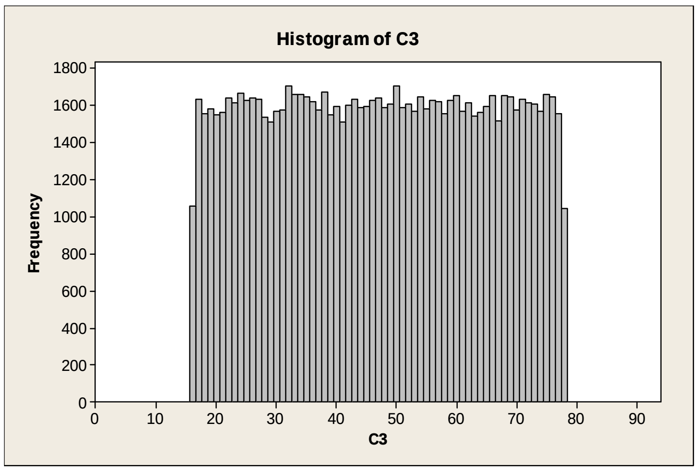
If the population has <u>any distribution</u> with mean μ and standard deviation σ, **and if $N \geq 30$**, then the sample mean $\bar{x}$ is normally distributed, with mean μ and standard deviation $\sigma / \sqrt{N}$.

This rule is called the <u>Central Limit Theorem</u>.

What if N is too small?

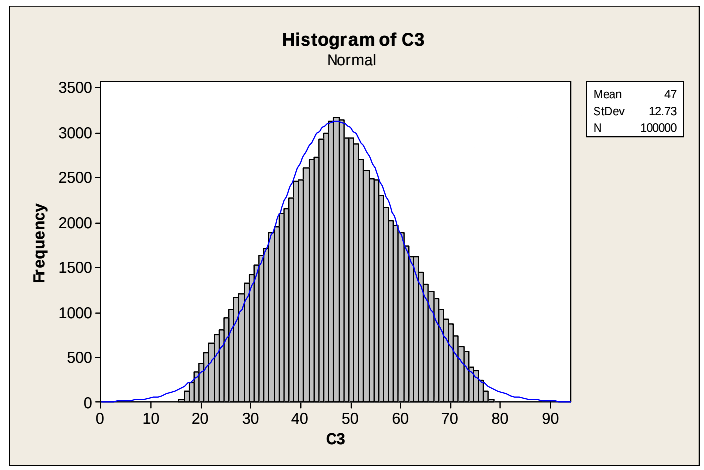
Let us assume that the population is uniformly distributed with $\mu = 47$, $\sigma = 18$.

Here is a histogram of 100,000 samples drawn from the population.



Now consider drawing **N = 2** samples from the population and taking their mean, $\bar{x}$.

We repeat this experiment 100,000 times and form a histogram of the values of $\bar{x}$.



In general, the sample mean $\bar{x}$ has mean μ and standard deviation $\sigma/\sqrt{N}$, but it is only approximately normal for large N.

The Central Limit Theorem

If the population has <u>any distribution</u> with mean μ and standard deviation σ, **and if N ≥ 30**, then the sample mean $\bar{x}$ is normally distributed, with mean μ and standard deviation $\sigma / \sqrt{N}$.

<u>Example problem</u>: if the daily number of hits for your website follows some distribution with $\mu = 1000$ and $\sigma = 300$, what is the probability that you will receive more than 39,600 hits in the next 36 days?

Given $\mu = 1000$, $\sigma = 300$, and $N = 36$, we know that the sample mean $\bar{x}$ is normally distributed with $\mu_{\bar{x}} = 1000$ and $\sigma_{\bar{x}} = 300 / \sqrt{36} = 50$.

Then $\Pr(\bar{x} > \frac{39{,}600}{36}) = \Pr(\bar{x} > 1100) = \Pr(z > \frac{1100 - 1000}{50}) = \Pr(z > 2)$.
Using the table of normal curve areas, we obtain .5 - .4772 = .0228.

Given μ and σ, the Central Limit Theorem lets you reason about $\bar{x}$.

The Central Limit Theorem

Example problem #2: An analyst for an internet consulting company is charged with collecting data on the performance of file sharing networks. A network is rated "satisfactory" if the average number of retries needed to gain entry is at most 1.

The analyst tests a site by attempting to gain entry 100 times. She finds a mean of 1.5 retries and a standard deviation of 1. Can she reliably conclude that the performance of the site is unsatisfactory?

Let us assume that $\sigma \approx s = 1$. Does a sample mean of $\bar{x} = 1.5$, computed from N =100 trials, seem consistent with the assumption that the population mean μ is equal to 1?

If the population had $\mu = 1$ and $\sigma = 1$, we would expect $\bar{x}$ to be normally distributed with mean 1 and std. deviation $1 / \sqrt{100} = 0.1$.

Then $\Pr(\bar{x} \geq 1.5) = \Pr(z \geq 5) \approx 0$.

Given $\bar{x}$ and s, the Central Limit Theorem lets you reason about μ.