Wavelets

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Function Representations

- sequence of samples (time domain)
  - finite difference method
- pyramid (hierarchical)
- polynomial
- sinusoids of various frequency (frequency domain)
  - Fourier series
- piecewise polynomials (finite support)
  - finite element method, splines
- **wavelet** (hierarchical, finite support)
  - (time/frequency domain)

What Are Wavelets?

In general, a family of representations using:

- hierarchical (nested) basis functions
- finite ("compact") support
- basis functions often orthogonal
- fast transforms, often linear-time

Simple Example: Haar Wavelet

- Consider piecewise-constant functions over 2^j equal sub-intervals of [0,1]:

- j=2: four intervals

- A basis

Nested Function Spaces for Haar Basis

- Let V_j denote the space of all piecewise-constant functions represented over 2^j equal sub-intervals of [0,1]

- V_j has basis

Function Representations – Desirable Properties

- generality – approximate anything well
  - discontinuities, nonperiodicity, ...
- adaptable to application
  - audio, pictures, flow field, terrain data, ...
- compact – approximate function with few coefficients
  - facilitates compression, storage, transmission
- fast to compute with
  - differential/integral operators are sparse in this basis
  - Convert n-sample function to representation in O($n$log$n$) or O(n) time

Wavelet History, Part 1

- 1805 Fourier analysis developed
- 1965 Fast Fourier Transform (FFT) algorithm

  …
- 1980’s beginnings of wavelets in physics, vision, speech processing (ad hoc)
- … little theory … why/when do wavelets work?
- 1986 Mallat unified the above work
- 1985 Morlet & Grossman continuous wavelet transform … asking: how can you get perfect reconstruction without redundancy?

Wavelet History, Part 2

- 1985 Meyer tried to prove that no orthogonal wavelet other than Haar exists, found one by trial and error!
- 1987 Mallat developed multiresolution theory, DWT, wavelet construction techniques (but still noncompact)
- 1988 Daubechies added theory: found compact, orthogonal wavelets with arbitrary number of vanishing moments!
- 1990's: wavelets took off, attracting both theoreticians and engineers

Time-Frequency Analysis

- For many applications, you want to analyze a function in both time and frequency
- Analogous to a musical score

- Fourier transforms give you frequency information, smearing time.
- Samples of a function give you temporal information, smearing frequency.

- *Note: substitute "space" for "time" for pictures.*

Comparison to Fourier Analysis

- Fourier analysis
  - Basis is global
  - Sinusoids with frequencies in arithmetic progression
- Short-time Fourier Transform (& Gabor filters)
  - Basis is local
  - Sinusoid times Gaussian
  - Fixed-width Gaussian “window”
- Wavelet
  - Basis is local
  - Frequencies in geometric progression
  - Basis has constant shape independent of scale

Wavelet Applications

- Medical imaging
- Pictures less corrupted by patient motion than with Fourier methods
- Astrophysics
- Analyze clumping of galaxies to analyze structure at various scales, determine past & future of universe
- Analyze fractals, chaos, turbulence

Wavelets for Denoising

- White noise is independent random fluctuations at each sample of a function
- White noise distributes itself uniformly across all coefficients of a wavelet transform
- Wavelet transforms tend to concentrate most of the "energy" in a small number of coefficients

- Throw out the small coefficients and you've removed (most of) the noise
- Little knowledge about noise character required!

Wavelets, Vision, and Hearing

Human vision & hearing:

- The retina and brain have receptive fields (filters) sensitive to spots and edges at a variety of scales and translations
- Somewhat similar to wavelet multiresolution analysis
- Human hearing also uses approximately constant shape filters

Computer vision:

- Pyramid techniques popular and powerful for matching, tracking, recognition
- Gaussian & Laplacian pyramids – wavelet precursors

Speech processing:

- Quadrature Mirror Filters – wavelet precursors