Fundamentals of Linear Algebra

Class 2 30 Aug 2012

Instructor: Bhiksha Raj

Administrivia

- Registration: Anyone on waitlist still?
- Homework 1: Will appear over weekend.
  - Linear algebra

Overview

- Vectors and matrices
- Basic vector/matrix operations
- Vector products
- Matrix products
- Various matrix types
- Projections

Book

- Fundamentals of Linear Algebra, Gilbert Strang
- Important to be very comfortable with linear algebra
  - Appears repeatedly in the form of Eigen analysis, SVD, Factor analysis
  - Appears through various properties of matrices that are used in machine learning, particularly when applied to images and sound
- Today's lecture: Definitions
  - Very small subset of all that's used
  - Important subset, intended to help you recollect

Incentive to use linear algebra

- Pretty notation!

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{y} \longleftrightarrow \sum_j y_j \sum_i x_i a_{ij}$$

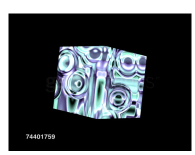
- Easier intuition
  - *Really convenient geometric interpretations*
  - Operations easy to describe verbally
- Easy code translation!

```
for i=1:n
  for j=1:m
    c(i)=c(i)+y(j)*x(i)*a(i,j)
  end
end
```

$\longleftrightarrow$

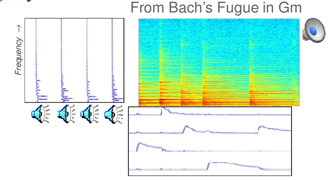
```
C=x*A*y
```


And other things you can do



Rotation + Projection + Scaling

From Bach's Fugue in Gm



Decomposition (NMF)

- Manipulate Images
- Manipulate Sounds

Scalars, vectors, matrices, …

- A *scalar* a is a number
  - a = 2, a = 3.14, a = -1000, etc.
- A *vector* **a** is a linear arrangement of a collection of scalars

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 3.14 \\ -32 \end{bmatrix}$$

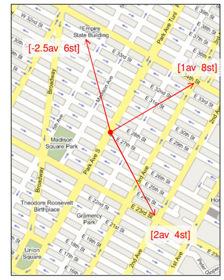
- A *matrix* **A** is a rectangular arrangement of a collection of scalars

$$\mathbf{A} = \begin{bmatrix} 3.12 & -10 \\ 10.0 & 2 \end{bmatrix}$$

- MATLAB syntax: a=[1 2 3], A=[1 2;3 4]

Vectors

- Vectors usually hold sets of numerical attributes
  - X, Y, Z coordinates
    - [1, 2, 0]
  - Earnings, losses, suicides
    - [$0 $1,000,000 3]
  - A location in Manhattan
    - [3av 33st]



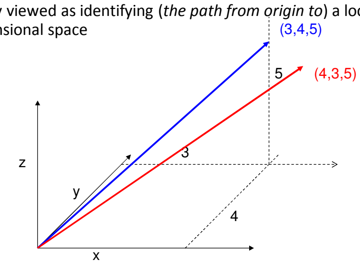
- Vectors are either column or row vectors

$$\mathbf{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} a & b & c \end{bmatrix}, \quad \mathbf{s} = [\ \ ]$$

  - A sound can be a vector, a series of daily temperatures can be a vector, etc …

Vectors in the abstract

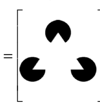
- Ordered collection of numbers
  - Examples: [3 4 5], [a b c d], ..
  - [3 4 5] != [4 3 5] → **Order is important**
- Typically viewed as identifying (*the path from origin to*) a location in an N-dimensional space



Matrices

- Matrices can be square or rectangular

$$\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \mathbf{M} =$$



  - Images can be a matrix, collections of sounds can be a matrix, etc.
  - A matrix can be vertical stacking of row vectors

$$\mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

  - Or a horizontal arrangement of column vectors

$$\mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

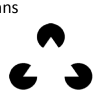
Dimensions of a matrix

- The matrix size is specified by the number of rows and columns

$$\mathbf{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} a & b & c \end{bmatrix}$$

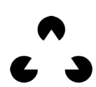
  - c = 3x1 matrix: 3 rows and 1 column
  - r = 1x3 matrix: 1 row and 3 columns

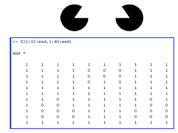
$$\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$



  - S = 2 x 2 matrix
  - R = 2 x 3 matrix
  - Pacman = 321 x 399 matrix

Representing an image as a matrix





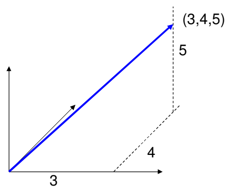
$$\begin{matrix} Y \\ X \\ v \end{matrix} \begin{bmatrix} 1 & 1 & . & 2 & . & 2 & 2 & . & 2 & . & 10 \\ 1 & 2 & . & 1 & . & 5 & 6 & . & 10 & . & 10 \\ 1 & 1 & . & 1 & . & 0 & 0 & . & 1 & . & 1 \end{bmatrix}$$

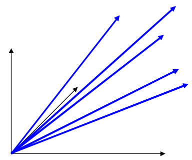
$$\begin{bmatrix} 1 & 1 & . & 1 & 1 & . & 0 & 0 & 0 & . & . & 1 \end{bmatrix}$$

Values only; X and Y are implicit

- 3 pacmen
- A 321 x 399 matrix
  - Row and Column = position
- A 3 x 128079 matrix
  - Triples of x,y and value
- A 1 x 128079 vector
  - "Unraveling" the matrix
- Note: All of these can be recast as the matrix that forms the image
  - Representations 2 and 4 are equivalent
    - The position is not represented

Vectors vs. Matrices





- A vector is a geometric notation for how to get from (0,0) to some location in the space
- A matrix is simply a collection of destinations!
  - Properties of matrices are *average* properties of the traveller's path to these destinations

Basic arithmetic operations

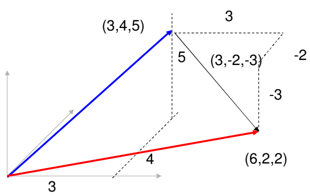
- Addition and subtraction
  - Element-wise operations

$$\mathbf{a}+\mathbf{b}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}+\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{bmatrix}\qquad \mathbf{a}-\mathbf{b}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}-\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}a_1-b_1\\a_2-b_2\\a_3-b_3\end{bmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}+\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}=\begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}\\a_{21}+b_{21}&a_{22}+b_{22}\end{bmatrix}$$

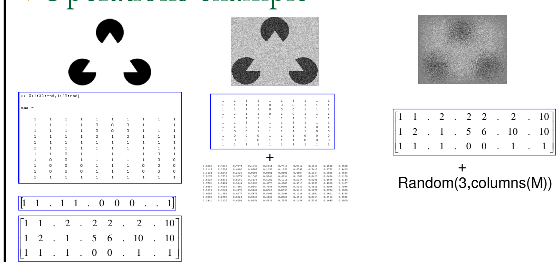
- MATLAB syntax: a+b and a-b

Vector Operations



- Operations tell us how to get from ({0}) to the result of the vector operations
  - (3,4,5) + (3,-2,-3) = (6,2,2)

Operations example



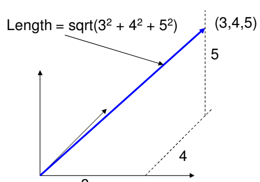
- Adding random values to different representations of the image

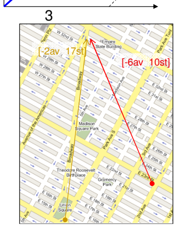
Vector norm

- Measure of how big a vector is:
  - Represented as $\|\mathbf{x}\|$

$$\|[a \quad b \quad \ldots]\| = \sqrt{a^2+b^2+\ldots^2}$$

- Geometrically the shortest distance to travel from the origin to the destination
  - As the crow flies
  - Assuming Euclidean Geometry
- MATLAB syntax: norm(x)





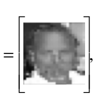
Transposition

- A transposed row vector becomes a column (and vice versa)

$$\mathbf{x}=\begin{bmatrix}a\\b\\c\end{bmatrix},\ \mathbf{x}^T=\begin{bmatrix}a & b & c\end{bmatrix}\qquad \mathbf{y}=\begin{bmatrix}a & b & c\end{bmatrix},\ \mathbf{y}^T=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

- A transposed matrix gets all its row (or column) vectors transposed in order

$$\mathbf{X}=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix},\ \mathbf{X}^T=\begin{bmatrix}a&d\\b&e\\c&f\end{bmatrix}$$

$\mathbf{M}=$ , $\mathbf{M}^T=$ 

- MATLAB syntax: a'

Vector multiplication

- Multiplication is not element-wise!
- Dot product, or inner product
  - Vectors must have the same number of elements
  - Row vector times column vector = scalar

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a \cdot d + b \cdot e + c \cdot f$$

- Outer product or vector direct product
  - Column vector times row vector = matrix

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} a \cdot d & a \cdot e & a \cdot f \\ b \cdot d & b \cdot e & b \cdot f \\ c \cdot d & c \cdot e & c \cdot f \end{bmatrix}$$

- MATLAB syntax: `a*b`

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Vector *dot product* in Manhattan

- Example:
  - Coordinates are yards, not ave/st
  - **a** = [200 1600],
    b = [770 300]
- The dot product of the two vectors relates to the length of a *projection*
  - How much of the first vector have we covered by following the second one?
  - Must normalize by the length of the "target" vector

$$\frac{\mathbf{a} \cdot \mathbf{b}^T}{\|\mathbf{a}\|} = \frac{\begin{bmatrix} 200 & 1600 \end{bmatrix} \cdot \begin{bmatrix} 770 \\ 300 \end{bmatrix}}{\left\| \begin{bmatrix} 200 & 1600 \end{bmatrix} \right\|} \approx 393\text{yd}$$

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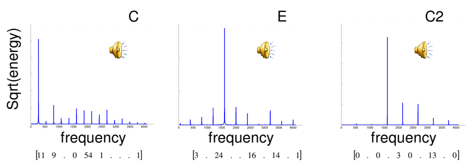
Vector dot product



- Vectors are spectra
  - Energy at a discrete set of frequencies
  - Actually 1 x 4096
  - X axis is the *index* of the number in the vector
    - Represents frequency
  - Y axis is the value of the number in the vector
    - Represents magnitude

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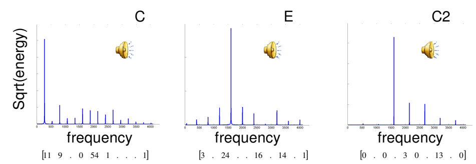
Vector dot product



- How much of C is also in E
  - How much can you fake a C by playing an E
  - C.E / |C||E| = 0.1
  - Not very much
- How much of C is in C2?
  - C.C2 / |C| /|C2| = 0.5
  - Not bad, you can fake it
- To do this, C, E, and C2 *must be the same size*

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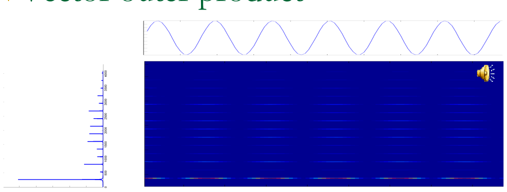
Vector outer product



- The column vector is the spectrum
- The row vector is an amplitude modulation
- The crossproduct is a spectrogram
  - Shows how the energy in each frequency varies with time
  - The pattern in each column is a scaled version of the spectrum
  - Each row is a scaled version of the modulation

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Multiplying a vector by a matrix

- Generalization of vector multiplication
  - **Left multiplication**: Dot product of each vector pair

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} \leftarrow & \mathbf{a}_1 & \rightarrow \\ \leftarrow & \mathbf{a}_2 & \rightarrow \end{bmatrix} \cdot \begin{bmatrix} \uparrow \\ \mathbf{b} \\ \downarrow \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b} \\ \mathbf{a}_2 \cdot \mathbf{b} \end{bmatrix}$$

  - Dimensions must match!!
    - No. of columns of matrix = size of vector
    - Result inherits the number of rows from the matrix
- MATLAB syntax: a*b

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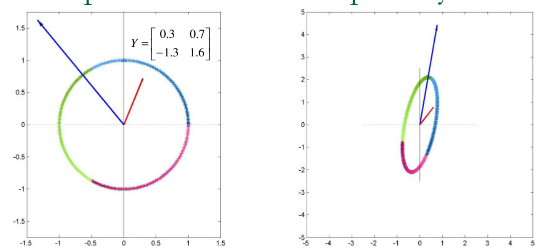
Multiplying a vector by a matrix

- Generalization of vector multiplication
  - **Right multiplication**: Dot product of each vector pair

$$\mathbf{A}\cdot\mathbf{B} = [\leftarrow\ \ \mathbf{a}\ \ \rightarrow]\cdot\begin{bmatrix}\uparrow & \uparrow\\ \mathbf{b}_1 & \mathbf{b}_2\\ \downarrow & \downarrow\end{bmatrix} = [\mathbf{a}.\mathbf{b}_1\ \ \ \mathbf{a}.\mathbf{b}_2]$$

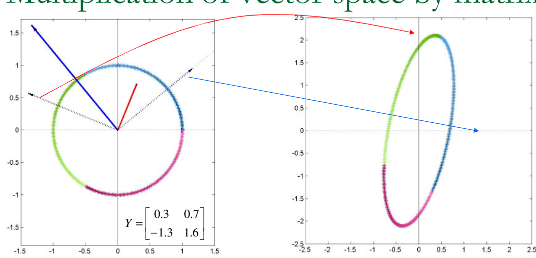
  - Dimensions must match!!
    - No. of rows of matrix = size of vector
    - Result inherits the number of columns from the matrix
- MATLAB syntax: a*b

Multiplication of vector space by matrix



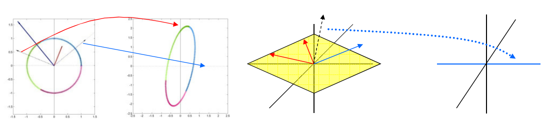
- The matrix rotates and scales the space
  - Including its own vectors

Multiplication of vector space by matrix



- The *normals* to the row vectors in the matrix become the new axes
  - X axis = normal to the *second* row vector
    - Scaled by the inverse of the length of the *first* row vector

Matrix Multiplication



- The k-th axis corresponds to the normal to the hyperplane represented by the 1..k-1,k+1..N-th row vectors in the matrix
  - Any set of K-1 vectors represent a hyperplane of dimension K-1 or less
- The distance along the new axis equals the length of the projection on the k-th row vector
  - Expressed in inverse-lengths of the vector

Matrix Multiplication: Column space

$$\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = x\begin{bmatrix}a\\ d\end{bmatrix} + y\begin{bmatrix}b\\ e\end{bmatrix} + z\begin{bmatrix}c\\ f\end{bmatrix}$$

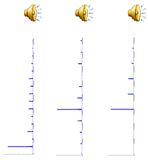
- So much for spaces .. what does multiplying a matrix by a vector really do?
- It *mixes* the column vectors of the matrix using the numbers in the vector
- The *column space* of the Matrix is the complete set of all vectors that can be formed by mixing its columns

Matrix Multiplication: Row space

$$[x\ \ y]\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix} = x[a\ \ b\ \ c] + y[d\ \ e\ \ f]$$

- Left multiplication mixes the *row vectors* of the matrix.
- The *row space* of the Matrix is the complete set of all vectors that can be formed by mixing its rows

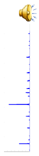
Matrix multiplication: Mixing vectors



X

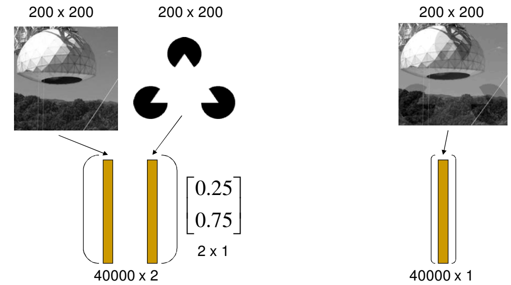
Y

$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ . \\ . \\ 2 \end{bmatrix}$$



- A physical example
  - The three column vectors of the matrix X are the spectra of three notes
  - The multiplying column vector Y is just a mixing vector
  - The result is a sound that is the mixture of the three notes

Matrix multiplication: Mixing vectors



- Mixing two images
  - The images are arranged as columns
    - position value not included
  - The result of the multiplication is rearranged as an image

Multiplying matrices

- Generalization of vector multiplication
  - **Outer product of dot products!!**

$$\mathbf{A}\cdot\mathbf{B} = \begin{bmatrix} \leftarrow & \mathbf{a}_1 & \rightarrow \\ \leftarrow & \mathbf{a}_2 & \rightarrow \end{bmatrix} \cdot \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{b}_1 & \mathbf{b}_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\cdot\mathbf{b}_1 & \mathbf{a}_1\cdot\mathbf{b}_2 \\ \mathbf{a}_2\cdot\mathbf{b}_1 & \mathbf{a}_2\cdot\mathbf{b}_2 \end{bmatrix}$$

  - Dimensions must match!!
    - Columns of first matrix = rows of second
    - Result inherits the number of rows from the first matrix and the number of columns from the second matrix
- MATLAB syntax: a*b

Matrix multiplication: another view

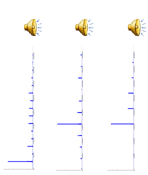
$$\mathbf{A}\cdot\mathbf{B} = \begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & . & b_{NK} \\ . & . & . \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} \sum_k a_{1k}b_{k1} & . & \sum_k a_{1k}b_{kK} \\ . & . & . \\ \sum_k a_{Mk}b_{k1} & . & \sum_k a_{Mk}b_{kK} \end{bmatrix}$$

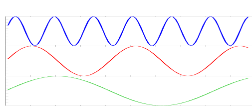
- What does this mean?

$$\begin{bmatrix} a_{11} & . & . & a_{1N} \\ a_{21} & . & . & a_{2N} \\ . & . & . & . \\ a_{M1} & . & . & a_{MN} \end{bmatrix} \begin{bmatrix} b_{11} & . & b_{NK} \\ . & . & . \\ . & . & . \\ b_{N1} & . & b_{NK} \end{bmatrix} = \begin{bmatrix} a_{11} \\ . \\ . \\ a_{M1} \end{bmatrix} \begin{bmatrix} b_{11} & . & b_{1K} \end{bmatrix} + \begin{bmatrix} a_{12} \\ . \\ . \\ a_{M2} \end{bmatrix} \begin{bmatrix} b_{21} & . & b_{2K} \end{bmatrix} + \ldots + \begin{bmatrix} a_{1N} \\ . \\ . \\ a_{MN} \end{bmatrix} \begin{bmatrix} b_{N1} & . & b_{NK} \end{bmatrix}$$

- The outer product of the first column of A and the first row of B + outer product of the second column of A and the second row of B + ...

Why is that useful?





$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

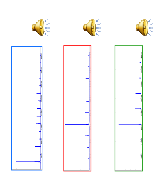
X

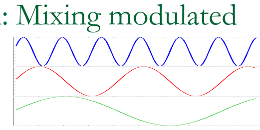
$$\begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \\ 1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & . & . & . & . & . & . \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & . & . & . & . & . \end{bmatrix}$$

Y

- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra





$$\begin{bmatrix} 1 & 3 & 0 \\ . & . & 0 \\ 9 & 24 & . \\ . & . & 1 \end{bmatrix}$$

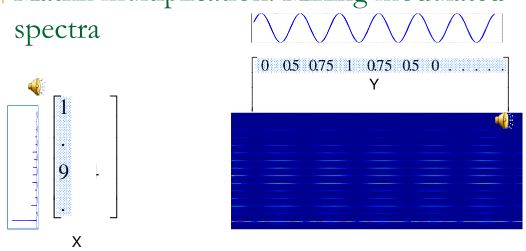
X

$$\begin{bmatrix} 0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & . & . & . & . & . \\ 1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & . & . & . & . & . & . \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & . & . & . & . & . \end{bmatrix}$$

Y

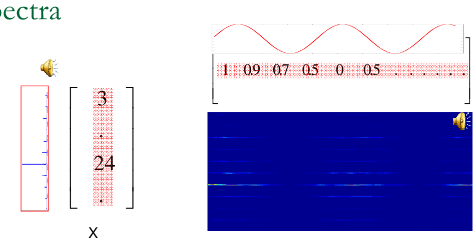
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



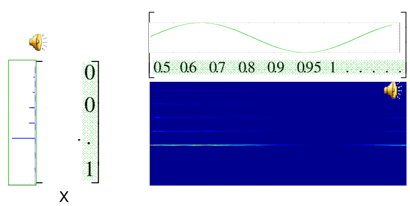
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



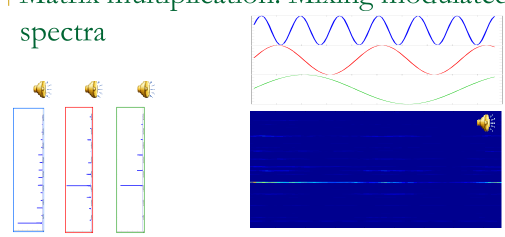
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



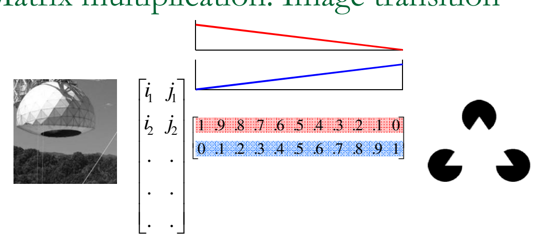
- Sounds: Three notes modulated independently

Matrix multiplication: Mixing modulated spectra



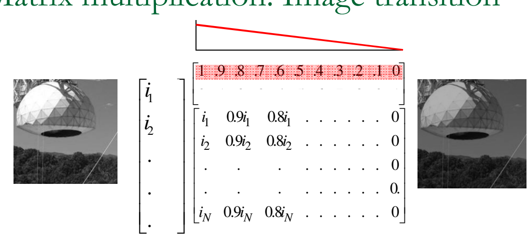
- Sounds: Three notes modulated independently

Matrix multiplication: Image transition



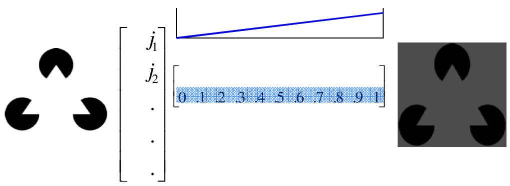
- Image1 fades out linearly
- Image 2 fades in linearly

Matrix multiplication: Image transition



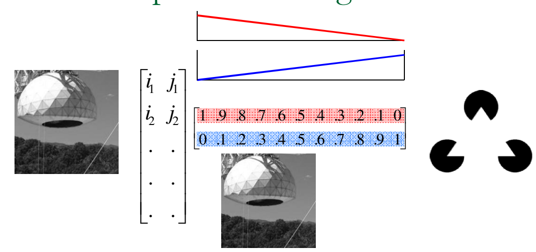
- Each column is one image
  - The columns represent a sequence of images of decreasing intensity
- Image1 fades out linearly

Matrix multiplication: Image transition



- Image 2 fades in linearly

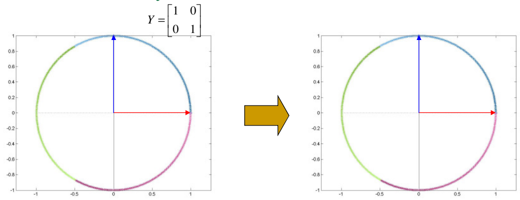
Matrix multiplication: Image transition



- Image1 fades out linearly
- Image 2 fades in linearly

The Identity Matrix

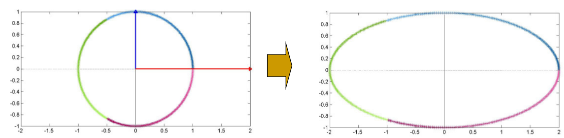
$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



- An identity matrix is a square matrix where
  - All diagonal elements are 1.0
  - All off-diagonal elements are 0.0
- Multiplication by an identity matrix does not change vectors

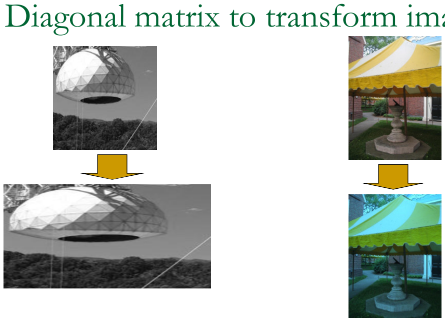
Diagonal Matrix

$$Y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



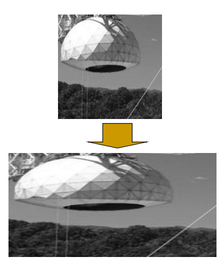
- All off-diagonal elements are zero
- Diagonal elements are non-zero
- Scales the axes
  - May flip axes

Diagonal matrix to transform images



- How?

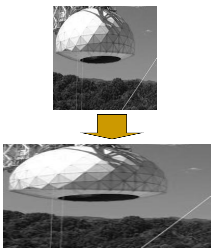
Stretching



$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & . & 2 & . & 2 & 2 & . & 2 & . & 10 \\ 1 & 2 & . & 1 & . & 5 & 6 & . & 10 & . & 10 \\ 1 & 1 & . & 1 & . & 0 & 0 & . & 1 & . & 1 \end{bmatrix}$$

- Location-based representation
- Scaling matrix – only scales the X axis
  - The Y axis and pixel value are scaled by identity
- Not a good way of scaling.

Stretching



D =

1	1	1	1	1	1	1	1	1	1
1	1	1	1	0	0	0	1	1	1
1	1	1	1	0	0	0	1	1	1
1	1	1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	0	1
1	0	0	1	1	1	1	1	0	0
1	0	0	0	1	1	1	0	0	0
1	0	0	0	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1

$$A = \begin{bmatrix} 1 & .5 & 0 & 0 & . \\ 0 & .5 & 1 & .5 & . \\ 0 & 0 & 0 & .5 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{bmatrix} (N\text{x}2N)$$

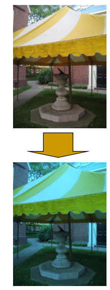
$$\text{Newpic} = EA$$

- Better way

Modifying color

$$P = \begin{bmatrix} R & G & B \end{bmatrix}$$

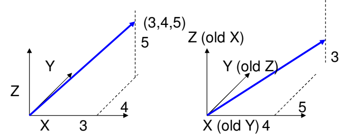
$$Newpic = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Scale only Green

Permutation Matrix

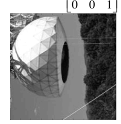
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$



- A permutation matrix simply rearranges the axes
  - The row entries are axis vectors in a different order
  - The result is a combination of rotations and reflections
- The permutation matrix effectively *permutes* the arrangement of the elements in a vector

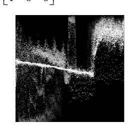
Permutation Matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & . & 2 & . & 2 & 2 & . & 2 & . & 10 \\ 1 & 2 & . & 1 & . & 5 & 6 & . & 10 & . & 10 \\ 1 & 1 & . & 1 & . & 0 & 0 & . & 1 & . & 1 \end{bmatrix}$$

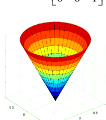
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

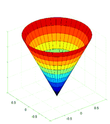


- Reflections and 90 degree rotations of images and objects

Permutation Matrix

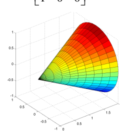
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





$$\begin{bmatrix} x_1 & x_2 & . & . & x_N \\ y_1 & y_2 & . & . & y_N \\ z_1 & z_2 & . & . & z_N \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



- Reflections and 90 degree rotations of images and objects
  - Object represented as a matrix of 3-Dimensional "position" vectors
  - Positions identify each point on the surface

Rotation Matrix

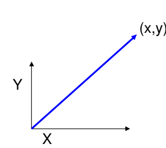
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

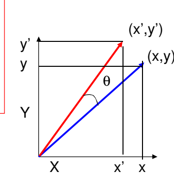
$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X_{new} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

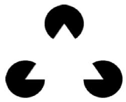
$$R_\theta X = X_{new}$$



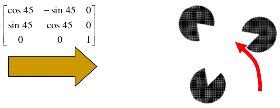


- A rotation matrix *rotates* the vector by some angle q
- Alternately viewed, it rotates the axes
  - The new axes are at an angle θ to the old one

Rotating a picture



$$R = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

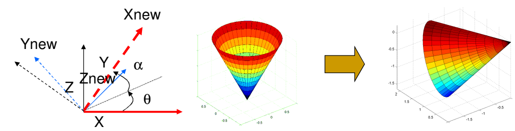


$$\begin{bmatrix} 1 & 1 & . & 2 & . & 2 & 2 & . & 2 & . & . \\ 1 & 2 & . & 1 & . & 5 & 6 & . & 10 & . & . \\ 1 & 1 & . & 1 & . & 0 & 0 & . & 1 & . & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\sqrt{2} & . & \sqrt{2} & . & -3\sqrt{2} & -4\sqrt{2} & . & -8\sqrt{2} & . & . \\ \sqrt{2} & 3\sqrt{2} & . & 3\sqrt{2} & . & 7\sqrt{2} & 8\sqrt{2} & . & 12\sqrt{2} & . & . \\ 1 & 1 & . & 1 & . & 0 & 0 & . & 1 & . & 1 \end{bmatrix}$$

- Note the representation: 3-row matrix
  - Rotation only applies on the "coordinate" rows
  - The value does not change
  - Why is pacman grainy?

3-D Rotation

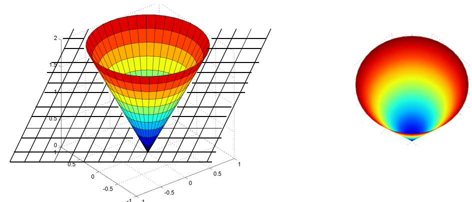


- 2 degrees of freedom
  - 2 separate angles
- What will the rotation matrix be?

Matrix Operations: Properties

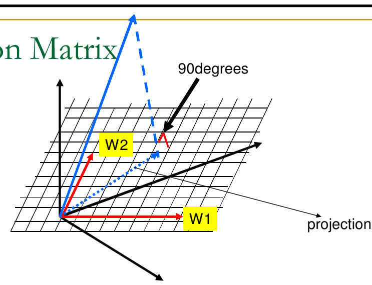
- A+B = B+A
- AB != BA

Projections



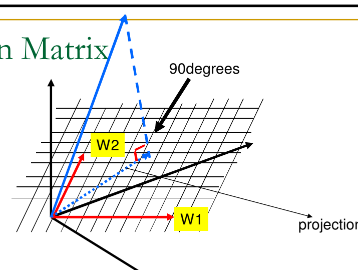
- What would we see if the cone to the left were transparent if we looked at it from above the plane shown by the grid?
  - Normal to the plane
  - Answer: the figure to the right
- How do we get this? Projection

Projection Matrix



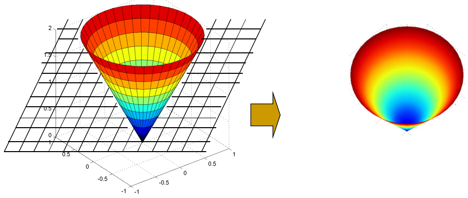
- Consider any plane specified by a set of vectors W_1, W_2..
  - Or matrix [W_1 W_2 ..]
  - Any vector can be projected onto this plane
  - The matrix A that rotates and scales the vector so that it becomes its projection is a projection matrix

Projection Matrix



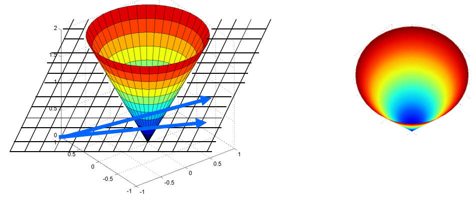
- Given a set of vectors W1, W2, which form a matrix W = [W1 W2..]
- The projection matrix that transforms any vector X to its projection on the plane is
  - $P = W (W^T W)^{-1} W^T$
    - We will visit matrix inversion shortly
- Magic – any set of vectors from the same plane that are expressed as a matrix will give you the same projection matrix
  - $P = V (V^T V)^{-1} V^T$

Projections



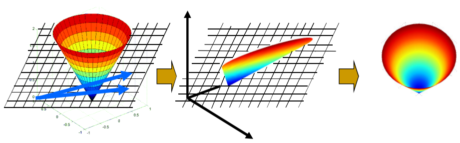
- HOW?

Projections



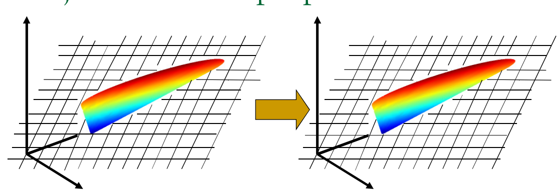
- Draw any two vectors W1 and W2 that lie on the plane
  - ***ANY two* so long as they have different angles**
- Compose a matrix W = [W1 W2]
- Compose the projection matrix $P = W (W^TW)^{-1} W^T$
- Multiply every point on the cone by P to get its projection
- View it ☺
  - I'm missing a step here – what is it?

Projections



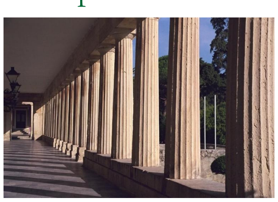
- The projection actually projects it onto the plane, but you're still seeing the plane in 3D
  - The result of the projection is a 3-D vector
  - $P = W (W^TW)^{-1} W^T = 3x3$, $P*Vector = 3x1$
  - The image must be rotated till the plane is in the plane of the paper
    - The Z axis in this case will always be zero and can be ignored
    - How will you rotate it? (remember you know W1 and W2)

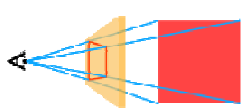
Projection matrix properties



- The projection of any vector that is already on the plane is the vector itself
  - Px = x if x is on the plane
  - If the object is already on the plane, there is no further projection to be performed
- The projection of a projection is the projection
  - P (Px) = Px
  - That is because Px is already on the plane
- Projection matrices are *idempotent*
  - $P^2 = P$
    - Follows from the above

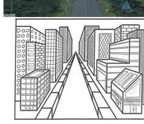
Perspective

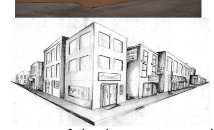




- The picture is the equivalent of "painting" the viewed scenery on a glass window
- Feature: The lines connecting any point in the scenery and its projection on the window merge at a common point
  - The eye

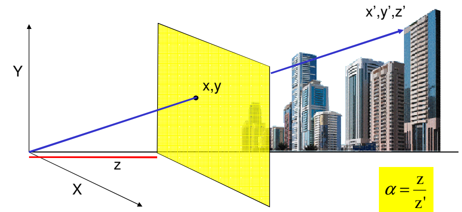
An aside on Perspective..





- Perspective is the result of convergence of the image to a point
- Convergence can be to multiple points
  - Top Left: One-point perspective
  - Top Right: Two-point perspective
  - Right: Three-point perspective

Central Projection



$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}$ Property of a line through origin

$\alpha = \frac{z}{z'}$
$x = \alpha x'$
$y = \alpha y'$

- The positions on the "window" are scaled along the line
- To compute (x,y) position on the window, we need z (distance of window from eye), and (x',y',z') (location being projected)

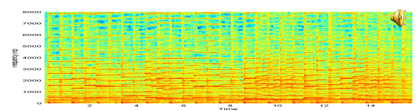
Representing Perspective

- Perspective was not always understood.
- Carefully represented perspective can create illusions..

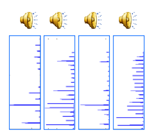
Projections: A more physical meaning

- Let W_1, W_2 .. W_k be "bases"
- We want to explain our data in terms of these "bases"
  - We often cannot do so
  - But we can explain a significant portion of it
- The portion of the data that can be expressed in terms of our vectors W_1, W_2, .. W_k, is the projection of the data on the W_1 .. W_k (hyper) plane
  - In our previous example, the "data" were all the points on a cone, and the bases were vectors on the plane

Projection : an example with sounds

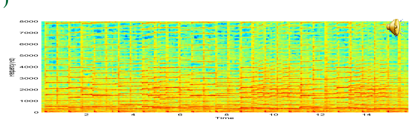


- The spectrogram (matrix) of a piece of music

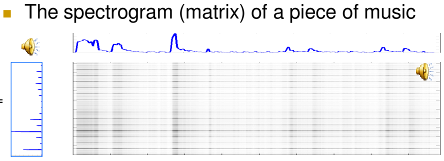


- How much of the above music was composed of the above notes
  - I.e. how much can it be explained by the notes

Projection: one note

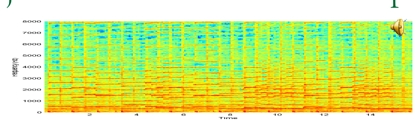
M = 

- The spectrogram (matrix) of a piece of music

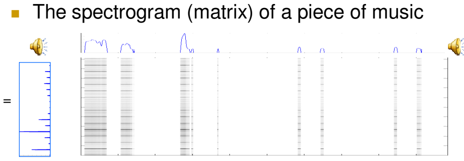
W = 

- M = spectrogram; W = note
- $P = W (W^TW)^{-1} W^T$
- Projected Spectrogram = P * M

Projection: one note – cleaned up

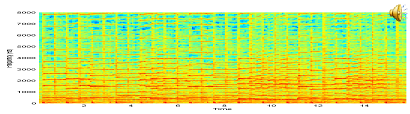
M = 

- The spectrogram (matrix) of a piece of music

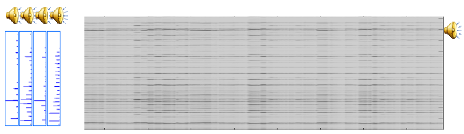
W = 

- Floored all matrix values below a threshold to zero

Projection: multiple notes

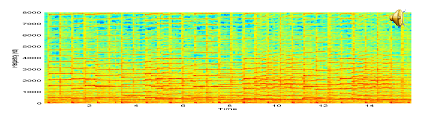
M = 

- The spectrogram (matrix) of a piece of music

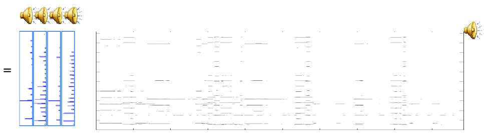
W = 

- $P = W (W^T W)^{-1} W^T$
- Projected Spectrogram = P * M

Projection: multiple notes, cleaned up

M = 

- The spectrogram (matrix) of a piece of music

W = 

- $P = W (W^T W)^{-1} W^T$
- Projected Spectrogram = P * M

Projection and Least Squares

- Projection actually computes a *least squared error* estimate
- For each vector V in the music spectrogram matrix
  - Approximation: V_{approx} = a*note1 + b*note2 + c*note3..

$$V_{approx} = \begin{bmatrix} \text{note1} & \text{note2} & \text{note3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

  - Error vector E = $V - V_{approx}$
  - Squared error energy for V $e(V) = norm(E)^2$
  - Total error = sum over all V { e(V) } = Σ_V e(V)
- Projection computes V_{approx} for all vectors such that Total error is minimized
  - It does not give you "a", "b", "c".. Though
    - That needs a different operation – the inverse / pseudo inverse