CS 15-888 Computational Game Solving

Lecture 1

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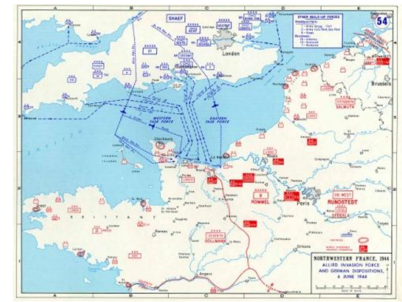
Main focus of the course:

Multi-step imperfect-information games

Why?

Most real-world games are incomplete-information games with sequential (& simultaneous) moves

- Negotiation
- Multi-stage auctions (e.g., FCC ascending, combinatorial auctions)
- Sequential auctions of multiple items
- A robot facing adversaries in uncertain, stochastic envt
- Card games, e.g., poker
- Currency attacks
- International (over-)fishing
- Political campaigns (e.g., TV spending in each region)
- Ownership games (polar regions, moons, planets)
- Allocating and timing troops/armaments to locations
  - US allocating troops in Afghanistan & Iraq
  - Military spending games, e.g., space vs ocean
  - Airport security, air marshals, coast guard, rail
  - Cybersecurity ...



So…

- Techniques for perfect-information games such as checkers, chess, and Go don’t apply

- because there are additional issues:
  - Private information
  - Need to understand signals and how other players will interpret signals
  - Need to understand deception
  - Need to deceive
  - …

Game representations,
game-theoretic solution concepts,
and complexity

The heart of the problem

- In a 1-agent setting, agent's expected utility maximizing strategy is well-defined

- But in a multiagent system, the outcome may depend on others' strategies also

  => the agent's best strategy may depend on what strategies the other agent(s) choose, and vice versa

Terminology

- ***Agent = player***
- ***Action = move*** **= choice that agent can make at a point in the game**
- ***Strategy*** **s_i = mapping from history (to the extent that the agent i can distinguish) to actions**
- ***Strategy set*** **S_i = strategies available to the agent**
- ***Strategy profile*** **$(s_1, s_2, ..., s_{|A|})$ = one strategy for each agent**
- **Agent's utility is determined after each agent (including nature that is used to model uncertainty) has chosen its strategy, and game has been played: $u_i = u_i(s_1, s_2, ..., s_{|A|})$**

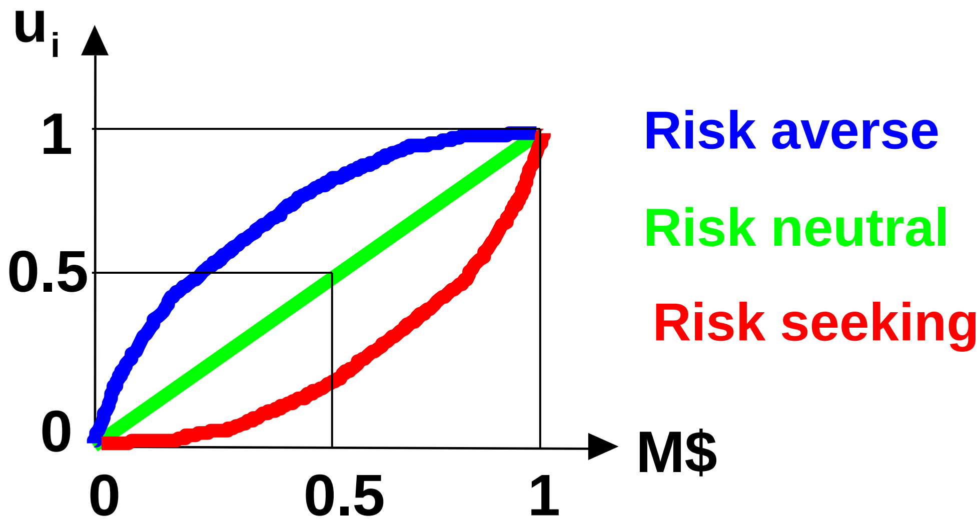
Agenthood

- Agent attempts to *maximize its expected utility*
- Utility function u_i of agent i is a mapping from outcomes to reals
  - Incorporates agent's risk attitude (allows quantitative tradeoffs)
    - E.g. outcomes over money

Lottery 1: $0.5M w.p. 1

Lottery 2: $1M w.p. 0.5
$0 w.p. 0.5

Agent's strategy is the choice of lottery



Risk aversion => insurance companies

- Often in game theory we just talk about expected payoff or expected value (EV)

Utility functions are scale-invariant

- Agent i chooses a strategy that maximizes expected utility

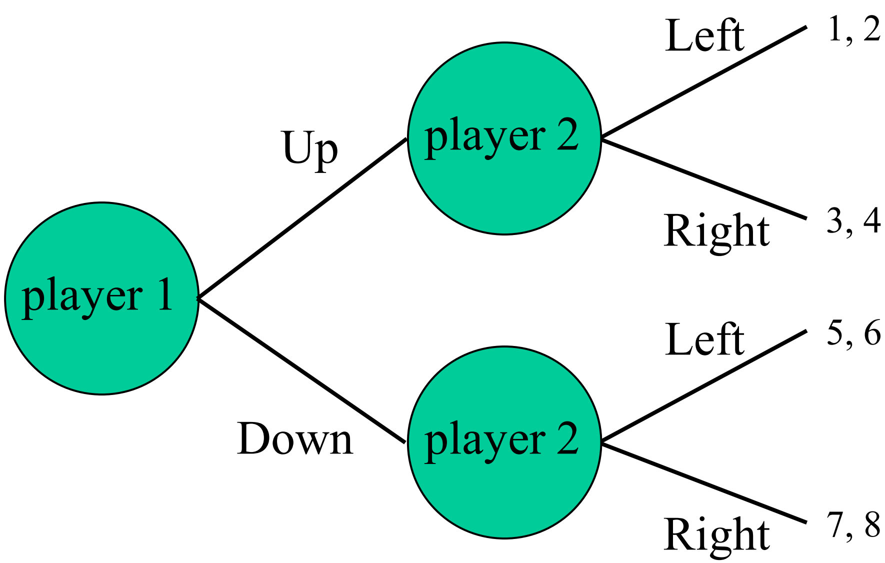
  $\max_{strategy} \Sigma_{outcome}\, p(outcome \mid strategy)\, u_i(outcome)$

- If $u_i'() = a\, u_i() + b$ for $a > 0$ then the agent will choose the same strategy under utility function u_i' as it would under u_i
  – (u_i has to be finite for each possible outcome; otherwise expected utility could be infinite for several strategies, so the strategies could not be compared.)

- Inter-agent utility comparison would be problematic

Game representations

Extensive form
(aka tree form)



Matrix form
(aka normal form
aka strategic form)

player 2's strategy

player 1's strategy		Left, Left	Left, Right	Right, Left	Right, Right
	Up	1, 2	1, 2	3, 4	3, 4
	Down	5, 6	7, 8	5, 6	7, 8

Potential combinatorial explosion →

Dominant strategy "equilibrium"

- ***Best response*** s_i^*: for all s_i', $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$
- ***Dominant strategy*** s_i^*: s_i^* is a best response for all s_{-i}
  - Does not always exist
  - Inferior strategies are called "dominated"
- ***Dominant strategy "equilibrium"*** is a strategy profile where each agent has picked its dominant strategy
  - Does not always exist
  - Requires no counterspeculation

E.g., Prisoners' Dilemma:

	cooperate	defect
cooperate	3, 3	0, 5
defect	5, 0	1, 1

Pareto optimal?

Social welfare maximizing?

Nash equilibrium

[Nash50]

- **Sometimes an agent's best response depends on others' strategies: a dominant strategy does not exist**
- **A strategy profile is a *Nash equilibrium* if no player has incentive to deviate from his strategy *given that others do not deviate*:**

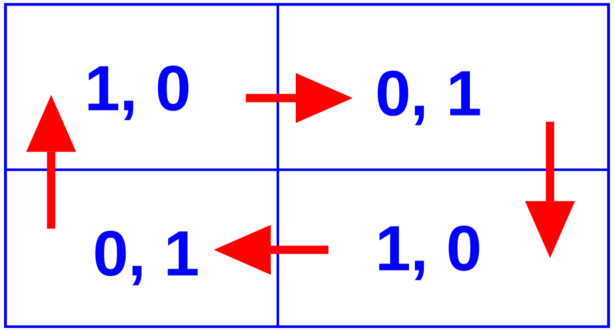
  for every agent i, $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$ for all s_i'
  - Dominant strategy equilibria are Nash equilibria but not vice versa
  - Defect-defect is the only Nash eq. in Prisoner's Dilemma
  - Battle of the Sexes game
    - Has no dominant strategy equilibria

E.g., Battle of the Sexes:

		Woman: boxing	Woman: ballet
Man	boxing	2, 1	0, 0
	ballet	0, 0	1, 2

Criticisms of Nash equilibrium

- Not unique in all games, e.g., Battle of the Sexes
  - Approaches for addressing this problem
    - Refinements (=strengthenings) of the equilibrium concept
      - Eliminate weakly dominated strategies first
      - Choose the Nash equilibrium with highest welfare
      - Subgame perfection …
    - Focal points
    - Mediation
    - Communication
    - Convention
    - Learning
- Does not exist in all games



Existence of (pure-strategy) Nash equilibria

- **Thrm.**
  - **Any finite game,**
  - **where each action node is alone in its information set**
    - (i.e., at every point in the game, the agent whose turn it is to move knows what moves have been played so far)
  - **is dominance solvable by backward induction (at least as long as ties are ruled out)**
- **Constructive proof: Multi-player minimax search**

Rock-scissors-paper game

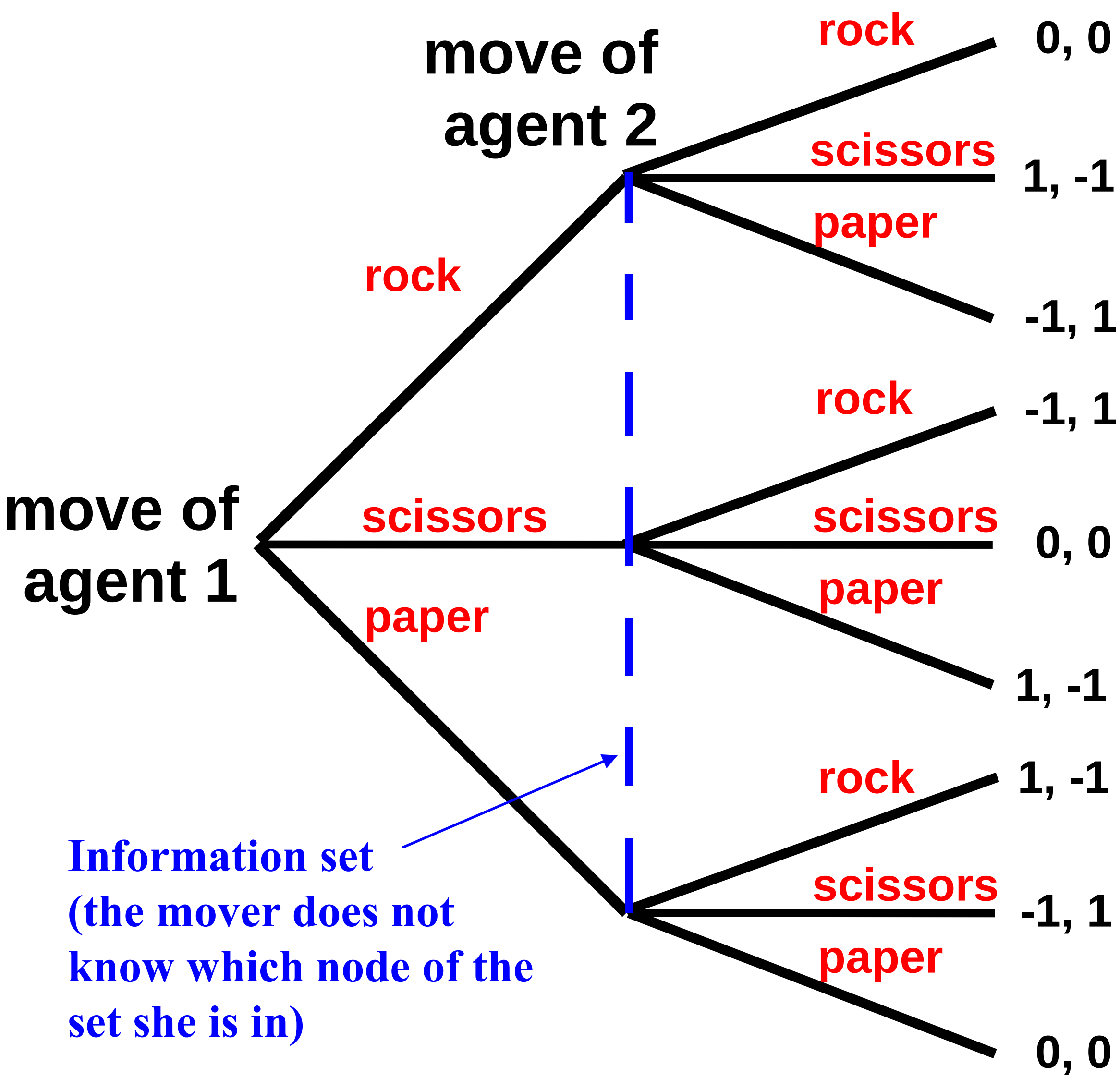
Sequential moves

Rock-scissors-paper game

Simultaneous moves

Mixed-strategy Nash equilibrium

Mixed strategy **= agent's chosen probability distribution over pure strategies from its strategy set**



(Bayes-)Nash equilibrium: Each agent uses a best-response strategy and has consistent beliefs

Rock-paper-scissors game has a symmetric *mixed-strategy* Nash equilibrium where each player plays each pure strategy with probability 1/3

Fact: In mixed-strategy equilibrium, each strategy that occurs in the mix of agent i has equal expected utility to i

Existence & complexity of mixed-strategy Nash equilibria

- **Every finite player, finite strategy game has at least one Nash equilibrium if we admit mixed-strategy equilibria as well as pure [Nash 50]**
  - **(Proof is based on Kakutani's fix point theorem)**

- **May be hard to compute**
  - **Complexity of finding a Nash equilibrium in a normal form game:**
    - **2-player 0-sum games can be solved in polytime with LP**
    - **2-player games are**
      - **PPAD-complete (even with 0/1 payoffs) [Chen, Deng & Teng JACM-09; Abbott, Kane & Valiant FOCS-05; Daskalakis, Goldberg & Papadimitriou STOC-06],**
      - **NP-complete to find an even approximately *good* Nash equilibrium [Conitzer & Sandholm GEB-08]**
    - **3-player games are FIXP-complete [Etessami & Yannakakis FOCS-07]**

Properties of 2-player 0-sum games

- **Swappability:** if (x,y) and (x',y') are equilibria, then so are (x',y) and (x,y')
  - => no equilibrium selection problem: player is safe playing any one of her equilibrium strategies
- A player's equilibrium strategies form a bounded convex polytope
- Any convex combination of a player's equilibrium strategies is an equilibrium strategy
- The set of Nash equilibria are exactly the set of solutions to the minmax problem $\max_x \min_y u_1(x,y)$
- Minmax theorem [von Neumann 1928]:

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be compact convex sets. If $f : X \times Y \to \mathbb{R}$ is a continuous function that is concave-convex, i.e.

$f(\cdot, y) : X \to \mathbb{R}$ is concave for fixed y, and
$f(x, \cdot) : Y \to \mathbb{R}$ is convex for fixed x.

Then we have that

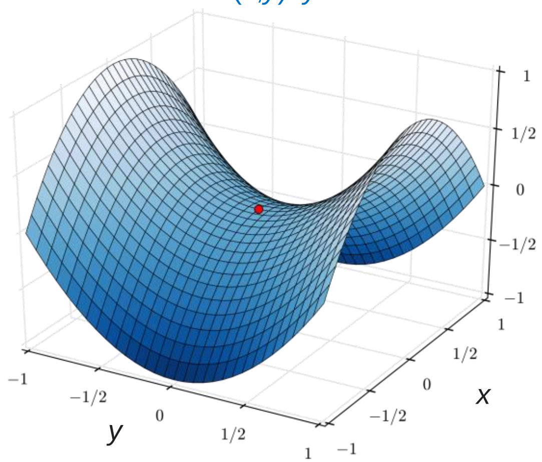
$$\max_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \max_{x \in X} f(x,y).$$

Example

If $f(x,y) = x^T A y$ for a finite matrix $A \in \mathbb{R}^{n \times m}$, we have:

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y.$$

The function $f(x,y)=y^2-x^2$ is concave-convex.



- Amazing in multi-step imperfect-information games:
  - By playing a non-equilibrium strategy, our opponent can cause our beliefs to be wrong, but not by so much that the opponent's expected value increases!
- Solvable in polynomial time in the size of the game tree using LP
  - But what if the tree has 10^{165} nodes?