

# **CS 15-888 Computational Game Solving**

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### ***Lecture 1***

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**Main focus of the course:**

***Multi-step imperfect-information games***

**Why?**

# Most real-world games are incomplete-information games with sequential (& simultaneous) moves

- Negotiation
- Multi-stage auctions (e.g., FCC ascending, combinatorial auctions)
- Sequential auctions of multiple items
- A robot facing adversaries in uncertain, stochastic envt
- Card games, e.g., poker
- Currency attacks
- International (over-)fishing
- Political campaigns (e.g., TV spending in each region)
- Ownership games (polar regions, moons, planets)
- Allocating and timing troops/armaments to locations
- Military spending games, e.g., space vs ocean
- Airport security, air marshals, coast guard, rail
- Cybersecurity
- ...



# So...

- Techniques for perfect-information games such as checkers, chess, and Go don't apply
- ... because there are additional issues:
  - Private information
  - Need to understand signals and how other players will interpret signals
  - Need to understand deception
  - Need to deceive
  - ...

**Game representations,  
game-theoretic solution concepts,  
and complexity**

# The heart of the problem

- In a 1-agent setting, agent's expected utility maximizing strategy is well-defined
  - But in a multiagent system, the outcome may depend on others' strategies also
- ⇒ the agent's best strategy may depend on what strategies the other agent(s) choose, and vice versa

# Terminology

- **Agent = player**
- **Action = move** = choice that agent can make at a point in the game
- **Strategy**  $s_i$  = mapping from history (to the extent that the agent  $i$  can distinguish) to actions
- **Strategy set**  $S_i$  = strategies available to the agent
- **Strategy profile**  $(s_1, s_2, \dots, s_{|A|})$  = one strategy for each agent
- Agent's utility is determined after each agent (including **nature** that is used to model uncertainty) has chosen its strategy, and game has been played:  $u_i = u_i(s_1, s_2, \dots, s_{|A|})$

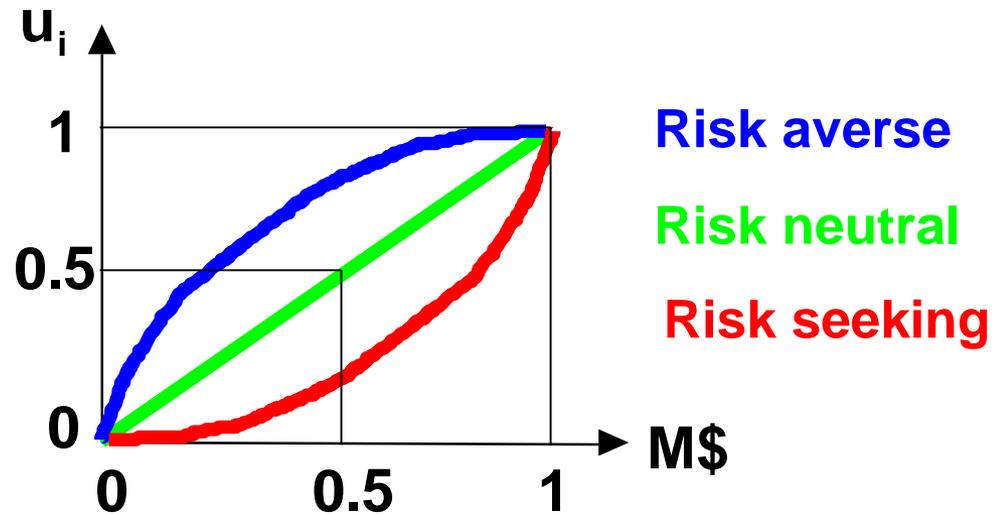
# Agenthood

- Agent attempts to *maximize its expected utility*
- Utility function  $u_i$  of agent  $i$  is a mapping from outcomes to reals
  - Incorporates agent's risk attitude (allows quantitative tradeoffs)
    - E.g. outcomes over money

Lottery 1: \$0.5M w.p. 1

Lottery 2: \$1M w.p. 0.5  
\$0 w.p. 0.5

Agent's strategy is the  
choice of lottery



Risk aversion => insurance companies

- Often in game theory we just talk about expected payoff or expected value (EV)

# Utility functions are scale-invariant

- Agent  $i$  chooses a strategy that maximizes expected utility

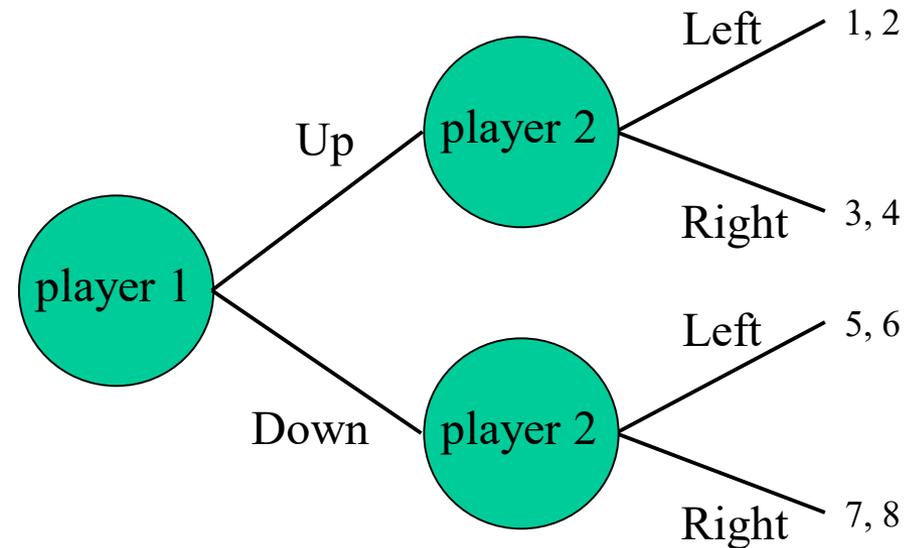
$$\max_{\text{strategy}} \sum_{\text{outcome}} p(\text{outcome} \mid \text{strategy}) u_i(\text{outcome})$$

- If  $u_i'() = a u_i() + b$  for  $a > 0$  then the agent will choose the same strategy under utility function  $u_i'$  as it would under  $u_i$ 
  - ( $u_i$  has to be finite for each possible outcome; otherwise expected utility could be infinite for several strategies, so the strategies could not be compared.)
- Inter-agent utility comparison would be problematic

# Game representations

Extensive form  
(aka tree form)

Matrix form  
(aka normal form  
aka strategic form)



player 2's strategy

		player 2's strategy			
		Left, Left	Left, Right	Right, Left	Right, Right
player 1's strategy	Up	1, 2	1, 2	3, 4	3, 4
	Down	5, 6	7, 8	5, 6	7, 8

**Potential combinatorial explosion**



# Dominant strategy “equilibrium”

- **Best response**  $s_i^*$ : for all  $s_i'$ ,  $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$
- **Dominant strategy**  $s_i^*$ :  $s_i^*$  is a best response for all  $s_{-i}$ 
  - Does not always exist
  - Inferior strategies are called “dominated”
- **Dominant strategy equilibrium** is a strategy profile where each agent has picked its dominant strategy
  - Does not always exist
  - Requires no counterspeculation
  - E.g., Prisoners’ Dilemma:

	cooperate	defect
cooperate	3, 3	0, 5
defect	5, 0	1, 1

Pareto optimal?

Social welfare  
maximizing?

# Nash equilibrium

[Nash50]



- Sometimes an agent's best response depends on others' strategies: a dominant strategy does not exist
- A strategy profile is a **Nash equilibrium** if no player has incentive to deviate from his strategy given that others do not deviate: for every agent  $i$ ,  $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$  for all  $s_i'$ 
  - Dominant strategy equilibria are Nash equilibria but not vice versa
  - Defect-defect is the only Nash eq. in Prisoner's Dilemma
  - Battle of the Sexes game
    - Has no dominant strategy equilibria

		Woman	
		boxing	ballet
Man	boxing	2, 1	0, 0
	ballet	0, 0	1, 2

Red arrows point from the (2,1) and (1,2) cells towards the (0,0) cell, indicating that (0,0) is a Nash equilibrium.

# Criticisms of Nash equilibrium

- Not unique in all games, e.g., Battle of the Sexes
  - Approaches for addressing this problem
    - Refinements (=strengthenings) of the equilibrium concept
      - Eliminate weakly dominated strategies first
      - Choose the Nash equilibrium with highest welfare
      - Subgame perfection ...
    - Focal points
    - Mediation
    - Communication
    - Convention
    - Learning
- Does not exist in all games

1, 0	0, 1
0, 1	1, 0

# Rock-scissors-paper game

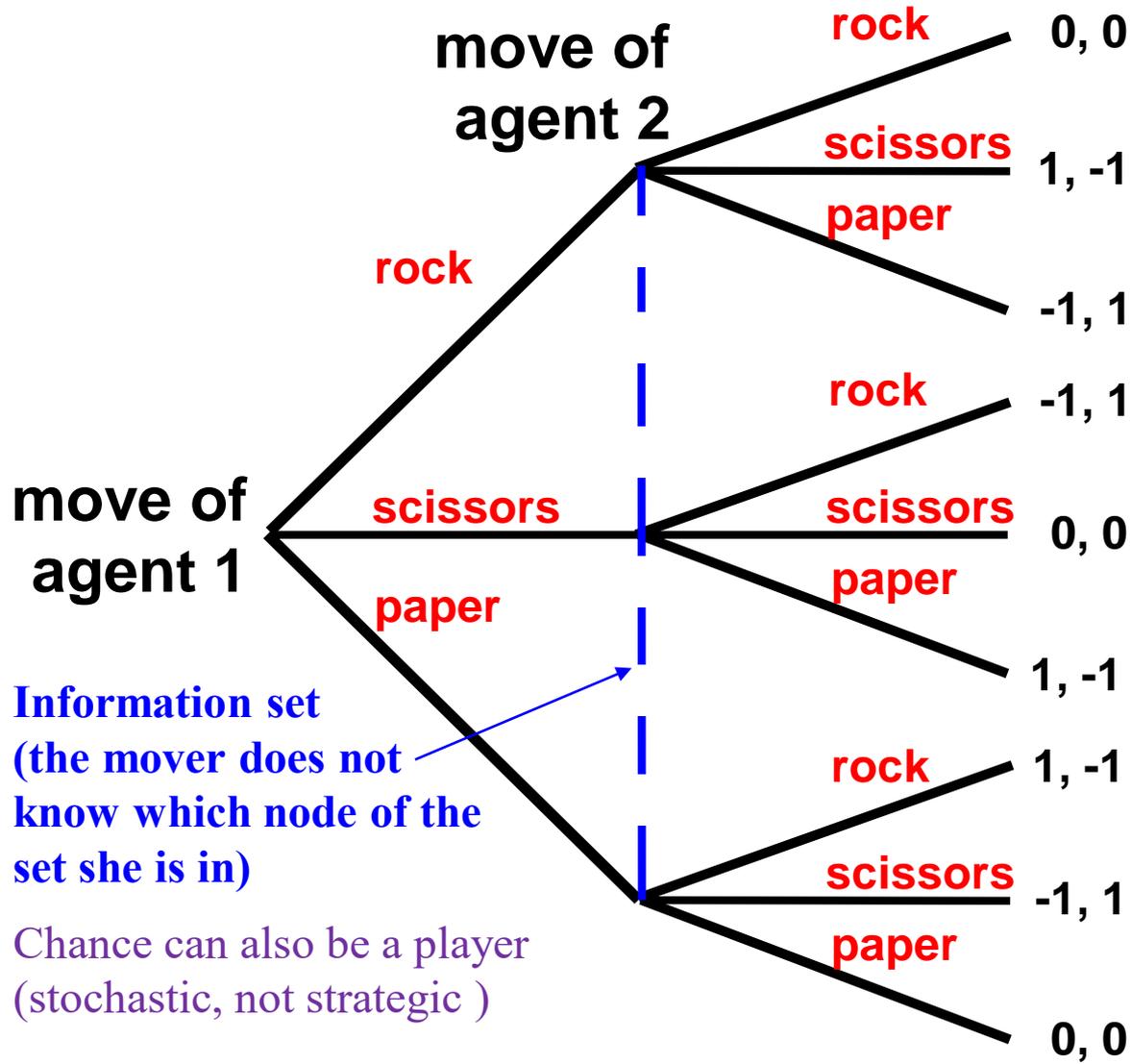
Sequential moves

# Rock-scissors-paper game

*Simultaneous* moves

# Imperfect-information extensive-form games

**Mixed strategy** = agent's chosen probability distribution over pure strategies from its strategy set



(Bayes-)Nash equilibrium: Each agent uses a best-response strategy and has consistent beliefs

Rock-paper-scissors game has a symmetric mixed-strategy Nash equilibrium where each player plays each pure strategy with probability  $1/3$

Fact: In mixed-strategy equilibrium, each strategy that occurs in the mix of agent  $i$  has equal expected utility to  $i$

Information set (the mover does not know which node of the set she is in)

Chance can also be a player (stochastic, not strategic)

# Behavioral strategy

- Agent has a probability distribution over her actions at each of her information sets
- **Kuhn's theorem:** If an agent has perfect recall, for every mixed strategy there is a behavioral strategy that has an equivalent payoff (i.e., the strategies are equivalent)
  - Applies also to infinite games

# Existence of pure-strategy Nash equilibria

- **Thrm.**
  - Any finite game,
  - where **each action node is alone in its information set**
    - (i.e., at every point in the game, the agent whose turn it is to move knows what moves have been played so far)
  - is dominance solvable by backward induction (at least as long as ties are ruled out)
- **Constructive proof: Multi-player minimax search**
- **Lots of interesting work has been done on computer chess and Go to tackle the computational complexity**

# Existence & complexity of mixed-strategy Nash equilibria

- **Every finite player, finite strategy game has at least one Nash equilibrium if we admit mixed-strategy equilibria as well as pure**  
[Nash 50]
  - (Proof is based on Kakutani's fix point theorem)
- **May be hard to compute**
  - **Complexity of finding a Nash equilibrium in a normal-form game:**
    - 2-player 0-sum games can be solved in polytime with LP
    - 2-player games are
      - PPAD-complete (even with 0/1 payoffs) [Chen, Deng & Teng JACM-09; Abbott, Kane & Valiant FOCS-05; Daskalakis, Goldberg & Papadimitriou STOC-06],
      - NP-complete to find an even approximately good Nash equilibrium [Conitzer & Sandholm GEB-08]
    - 3-player games are FIXP-complete [Etessami & Yannakakis FOCS-07]

# Properties of 2-player 0-sum games

- **Swappability:** if  $(x,y)$  and  $(x',y')$  are equilibria, then so are  $(x',y)$  and  $(x,y')$ 
  - $\Rightarrow$  no equilibrium selection problem: player is safe playing any one of her equilibrium strategies
- A player's equilibrium strategies form a bounded convex polytope
- Any convex combination of a player's equilibrium strategies is an equilibrium strategy
- The set of Nash equilibria are exactly the set of solutions to the minmax problem  $\max_x \min_y u_1(x,y)$
- Minmax theorem [von Neumann 1928]:

Let  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^m$  be compact convex sets. If  $f : X \times Y \rightarrow \mathbb{R}$  is a continuous function that is concave-convex, i.e.

$f(\cdot, y) : X \rightarrow \mathbb{R}$  is concave for fixed  $y$ , and

$f(x, \cdot) : Y \rightarrow \mathbb{R}$  is convex for fixed  $x$ .

Then we have that

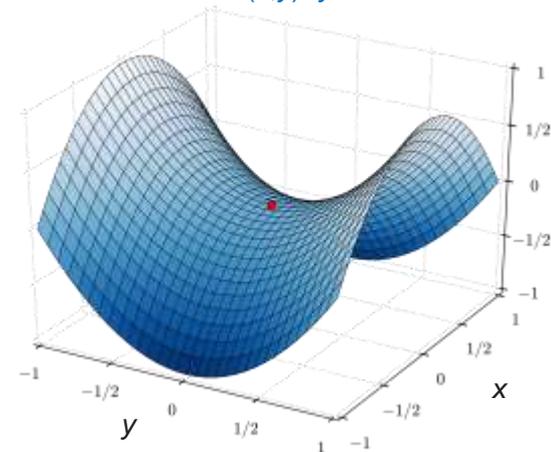
$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

## Example

If  $f(x, y) = x^T A y$  for a finite matrix  $A \in \mathbb{R}^{n \times m}$ , we have:

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y.$$

The function  $f(x,y)=y^2-x^2$  is concave-convex.



- Amazing in multi-step imperfect-information games:
  - By playing a non-equilibrium strategy, our opponent can cause our beliefs to be wrong, but not by so much that the opponent's expected value increases!
- Solvable in polynomial time in the size of the game tree using LP
  - But what if the tree has  $10^{165}$  nodes?