

Linear Programming and Vickrey Auctions*

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Abstract

The Vickrey sealed bid auction occupies a central place in auction theory because of its efficiency and incentive properties. Implementing the auction requires the auctioneer to solve $n + 1$ optimization problems, where n is the number of bidders. In this paper we survey various environments (some old and some new) where the payments bidders make under the Vickrey auction correspond to dual variables in certain linear programs. Thus, in these environments, at most two optimization problems must be solved to determine the Vickrey outcome. Furthermore, primal-dual algorithms for some of these linear programs suggest ascending auctions that implement the Vickrey outcome.

Keywords Vickrey auctions, multi-item auctions, combinatorial auctions, duality, primal-dual algorithm

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1 Introduction

The Vickrey sealed bid auction deservedly occupies an important place in the literature on mechanism design. It results in an economically efficient outcome, and it is a (weakly) dominant strategy for bidders to bid truthfully.

To describe the auction, let N be the set of n agents (or bidders), and F be the set of feasible allocations among them. Each agent $j \in N$ assigns a monetary benefit $v_j(a)$ to each allocation $a \in F$. This monetary benefit v_j is private information to agent j . The efficient outcome is the outcome a^* that solves $\max_{a \in F} \sum_{j \in N} v_j(a)$. Let

$$V(N) = \max_{a \in F} \sum_{j \in N} v_j(a) \text{ and}$$
$$V(N \setminus j) = \max_{a \in F} \sum_{i \in N \setminus j} v_i(a)$$

The sealed bid auction proceeds as follows. Agents submit their benefit functions $\{v_1, v_2, \dots, v_n\}$. For the moment, assume they do so truthfully; subsequently, we show that doing otherwise cannot improve any agent's payoff. The auctioneer computes a^* , $V(N)$ and $V(N \setminus j)$ for all $j \in N$. The allocation a^* is chosen, and bidder j pays

$$V(N \setminus j) - \sum_{i \neq j} v_i(a^*).$$

Thus, bidder j 's net benefit from participating is

$$v_j(a^*) - [V(N \setminus j) - \sum_{i \neq j} v_i(a^*)] = V(N) - V(N \setminus j).$$

This last term is sometimes called bidder j 's *marginal product*, and is clearly non-negative. It is the incremental benefit from including agent j in the auction.

Could any bidder benefit from misreporting his valuation function? No. To see why, suppose bidder j reports $u_j \neq v_j$ instead. The auctioneer then chooses an $a' \in F$ that solves $\max_{a \in F} [\sum_{i \neq j} v_i(a) + u_j(a)]$. Agent j 's payoff becomes

$$[\sum_{i \neq j} v_i(a') + v_j(a')] - V(N \setminus j) \leq V(N) - V(N \setminus j).$$

A closer inspection of the argument reveals more: No matter what the other agents report, no agent can benefit by misreporting his valuation function.

The Vickrey auction is not the only efficient auction with the property that it is a (weakly) dominant strategy for bidders to bid truthfully. The general class of such schemes is the class of Clark–Groves mechanisms.¹ The Vickrey auction is one member in this class and, in some sense, is focal. For instance, Krishna and Perry (1998) justify its use in terms of revenue maximization within a class of mechanisms.²

Computationally, the implementation of a Vickrey auction requires the solution of $n + 1$ optimization problems. In many environments of economic interest, each of these problems is a linear program. An agent makes its presence known in the choice of constraints (rows) or variables (columns) of the linear program. It is therefore tempting to think that the marginal product of an agent should be encoded in the optimal dual variables of the linear program. After all, it is these variables that inform us of the effect of changing the right hand side of a constraint, or modifying a coefficient in the objective function.

In this paper, we survey the instances where such a connection exists. We start in Section 2 with a discussion of the general problem, and various formulations of it, building on the notions of Bikhchandani and Ostroy (2000a). In the sections that follow, we examine various special cases of the general model which admit simpler formulations.

The Vickrey auction as we have just described is a sealed bid auction. In the auction of a single object, it has long been known that an open (ascending) auction exists that duplicates the outcomes of the Vickrey auction. This is the English auction, in which the auctioneer continuously raises the price for the object. An agent is to keep his hand raised until the price exceeds the amount the agent is willing to pay for the object. The auction terminates as soon as only one agent’s hand is left raised. That agent wins the object at that terminal price.

In such an auction, as long as every other agent is behaving in a way consistent with some valuation for the object, an agent cannot benefit by lowering his hand prematurely, or by keeping his hand raised beyond the price he is willing to pay. So, the ascending auction has desirable incentive properties comparable to its sealed bid counterpart, and a number of other

¹See Vickrey (1961), Clarke (1971), and Groves (1973).

²See also, Williams (1999). Moulin (1986) justifies its use in a different sense.

pleasant features that are listed by Ausubel (1997).

Here, we point out that such ascending auctions can be interpreted as particular implementations of a primal dual algorithm for the underlying optimization problem. Specific examples of this are provided by Ausubel (1997) and by Demange, Gale, and Sotomayor (1986). Such a connection should not be surprising; an ascending auction that implements the Vickrey outcome must solve the same optimization problem as an efficient algorithm, and the auction *is* an algorithm.

2 The general case

We begin with the general case of an auctioneer who must auction off a set of distinct heterogeneous objects, M . Let N be the set of bidders. For every set of objects $S \subseteq M$, let $v_j(S)$ be the value that agent $j \in N$ assigns to consuming S .³ An *allocation* is an assignment of objects to the agents. To formulate the problem of finding an *efficient* allocation let $y(S, j) = 1$ if the bundle $S \subseteq M$ is allocated to $j \in N$ and zero otherwise.⁴ The optimization problem, denoted **(P1)**, is to solve

$$\begin{aligned}
 V(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) \\
 \text{s.t. } &\sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \quad \forall i \in M \\
 &\sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\
 &y(S, j) = 0, 1 \quad \forall S \subseteq M, \forall j \in N
 \end{aligned}$$

The first constraint ensures that overlapping sets of goods are never assigned. The second ensures that no bidder receives more than one subset.

The linear relaxation of the above integer program admits fractional solutions, so it is natural to look for a stronger formulation. The way we do this

³A more general specification is to define agents' valuations over allocations, allowing for externalities: preferences affected by *others'* consumption. For simplicity, we do not specify such a general model.

⁴It is important to make clear the notation that, e.g., if agent j consumes the pair of distinct objects $i, i' \in M$, then $y(\{i, i'\}, j) = 1$, but $y(\{i\}, j) = 0$. An agent consumes exactly one set of objects.

is by using auxiliary variables. It is helpful to first consider the formulation (P2) below. This stronger formulation is not strong enough, but it suggests directions for other formulations; hence we introduce it.

Let Π be the set of all possible partitions of the objects in the set M . For any partition $\sigma \in \Pi$, we write $S \in \sigma$ to mean that the set $S \subset M$ is a part of the partition σ . Auxiliary variables are denoted z_σ , where $z_\sigma = 1$ if the partition σ is selected and $z_\sigma = 0$ otherwise. Using them we obtain formulation **(P2)**.

$$\begin{aligned}
V(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) \\
\text{s.t. } & \sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\
& \sum_{j \in N} y(S, j) \leq \sum_{\sigma \ni S} z_\sigma \quad \forall S \subset M \\
& \sum_{\sigma \in \Pi} z_\sigma \leq 1 \\
& y(S, j) = 0, 1 \quad \forall S \subseteq M, \forall j \in N
\end{aligned}$$

This formulation chooses a partition of M and then assigns the sets of the partition to bidders in such a way as to maximize total valuations. It is easy to see that formulation (P2) is stronger than (P1): Fix an $i \in M$ and add over all $S \ni i$ the inequalities

$$\sum_{j \in N} y(S, j) \leq \sum_{\sigma \ni S} z_\sigma \quad \forall S \subset M$$

to obtain

$$\sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \quad \forall i \in M$$

which are the inequalities that appear in the first formulation.

Even (P2) admits fractional solutions in its linear relaxation. The easiest

way to see this is to consider the following.

$$\begin{aligned}
F(z) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) \\
\text{s.t. } & \sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\
& \sum_{j \in N} y(S, j) \leq \sum_{\sigma \ni S} z_\sigma \quad \forall S \subseteq M \\
& y(S, j) \geq 0 \quad \forall S \subseteq M, \forall j \in N
\end{aligned}$$

$F(z)$ is a concave function of z . Hence

$$V(N) = \max\{F(z) : \text{s.t. } \sum_{\sigma \in \Pi} z_\sigma \leq 1\}.$$

Clearly, the optimal choice of z need not occur at an extreme point.

Bikhchandani and Ostroy (2000a) propose an even stronger formulation that is integral. Furthermore they identify a necessary and sufficient condition for the dual variables to correspond to the Vickrey payoffs. This formulation we consider next.

2.1 The Bikhchandani and Ostroy formulation

Bikhchandani and Ostroy (2000a) introduce a variable for every feasible integer solution. Let μ denote both a partition of the set of objects *and* an assignment of the elements of the partition to bidders. Thus μ and μ' can give rise to the same partition, but to different assignments of the parts to bidders. Let Γ denote the set of all such partition-assignment pairs. We will write $S^j \in \mu$ to mean that under μ , agent j receives the set S . Let $\delta_\mu = 1$ if the partition-assignment pair $\mu \in \Gamma$ is selected, and zero otherwise. Using these new variables the efficient allocation can be found by solving

$$\begin{aligned}
V(N) &= \max \sum_{\mu \in \Gamma} \left[\sum_{j \in N} \sum_{S^j \in \mu} v_j(S^j) \right] \delta_\mu \\
\text{s.t. } & \sum_{\mu \in \Gamma} \delta_\mu \leq 1 \\
& \delta_\mu \geq 0 \quad \forall \mu \in \Gamma
\end{aligned}$$

This formulation involves one constraint and so gives rise to one dual variable only. Hence it is not detailed enough to provide information about the marginal products of each of the bidders. What is needed is a disaggregate formulation that allows one to track the effect of removing or adding a bidder. Here is the one proposed by Bikhchandani and Ostroy (2000a), which we call **(P3)**.

$$\begin{aligned}
V(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) \\
\text{s.t. } y(S, j) &\leq \sum_{\mu \ni S^j} \delta_\mu \quad \forall j \in N, \forall S \subseteq M \\
\sum_{S \subseteq M} y(S, j) &\leq 1 \quad \forall j \in N \\
\sum_{\mu \in \Gamma} \delta_\mu &\leq 1 \\
y(S, j) &= 0, 1 \quad \forall S \subseteq M, \forall j \in N
\end{aligned}$$

A straightforward argument shows that the formulation has the integrality property.

The significance of this formulation is that the dual variable associated with the second constraint, $\sum_{S \subseteq M} y(S, j) \leq 1$, can be interpreted as agent j 's marginal product. To see why, for an agent $j \in N$, reduce the right hand side of the corresponding constraint to zero. This has the effect of removing agent j from the problem. The resulting change in optimal objective function value will be agent j 's marginal product.

The difficulty is to ensure that a dual solution exists such that, simultaneously, each dual variable associated with that type of constraint takes on the value of the corresponding agent's marginal product. Bikhchandani and Ostroy (2000a) derive the following necessary and sufficient condition for this to be realized.

$$V(N) - V(K) \geq \sum_{j \in N \setminus K} [V(N) - V(N \setminus j)] \quad \forall K \subseteq N. \quad (1)$$

They interpret this condition to mean “agents are substitutes.”

This condition formalizes the notion that the contribution (i.e., marginal product) of a group of agents is more than the sum of the contributions of the individual members of the group. Such a condition motivates, for example,

the idea that workers are better-off forming a union rather than bargaining separately with management.

To write down the dual, we associate with each constraint $y(S, j) \leq \sum_{\mu \ni S^j} \delta_\mu$ a variable $p_j(S) \geq 0$ which can be interpreted as the price that agent j pays for the set S . To each constraint $\sum_{S \subseteq M} y(S, j) \leq 1$ we associate a variable $\pi_j \geq 0$ which can be interpreted as agent j 's surplus. To the constraint $\sum_{\mu \in \Gamma} \delta_\mu \leq 1$ we associate the variable π^s which can be interpreted as the seller's surplus. The dual **(DP3)** becomes

$$\begin{aligned} \min \quad & \sum_{j \in N} \pi_j + \pi^s \\ \text{s.t.} \quad & p_j(S) + \pi_j \geq v_j(S) \quad \forall j \in N, \forall S \subseteq M \\ & - \sum_{S^j \in \mu} p_j(S) + \pi^s \geq 0 \quad \forall \mu \in \Gamma \\ & p_j(S) \geq 0 \quad \forall j \in N, \forall S \subseteq M \end{aligned}$$

Theorem 1 *Suppose the “agents are substitutes” condition (1) holds. Then there exists an optimal solution to (DP3), $((\pi_j), \pi^s)$, where $\pi_j = V(N) - V(N \setminus j)$ for all $j \in N$. That is, agent j 's surplus, π_j , is agent j 's marginal product. The converse is also true.*

Proof Set $\pi_j = V(N) - V(N \setminus j)$ for all $j \in N$. By setting

$$\begin{aligned} p_j(S) &= [v_j(S) - \pi_j]^+ \quad \forall S \subseteq M, \forall j \in N \\ \pi^s &= \max_{\mu \in \Gamma} \sum_{S^j \in \mu} p_j(S) \end{aligned}$$

we obtain a feasible solution to the dual. Let μ^* be the partition-assignment pair that defines π^s above. The objective function value of our feasible dual solution is

$$\sum_{j \in N} \pi_j + \sum_{S^j \in \mu^*} [v_j(S^j) - \pi_j]^+.$$

To obtain a contradiction suppose that

$$\sum_{j \in N} \pi_j + \sum_{S^j \in \mu^*} [v_j(S^j) - \pi_j]^+ > V(N).$$

Let $K = \{j : [v_j(S^j) - \pi_j]^+ > 0\}$. Then the previous inequality becomes

$$\sum_{j \in N} \pi_j - \sum_{j \in K} \pi_j + \sum_{j \in K} v_j(S^j) > V(N).$$

Since $\sum_{j \in K} v_j(S^j) \leq V(K)$, the last inequality implies

$$V(N) - V(K) < \sum_{j \in N \setminus K} [V(N) - V(N \setminus j)],$$

which is a violation of the substitutes condition. Running the argument in reverse yields the equivalence. \blacksquare

2.2 A new formulation

Here we give a new formulation that uses fewer variables than the Bikhchandani and Ostroy formulation. It consists of a collection of assignment problems linked together by two classes of constraints.

As before, let Π be the set of partitions of M . For any partition $\sigma \in \Pi$, $S \in \sigma$ means that $S \subset M$ is a part of the partition σ . Let $z_\sigma = 1$ if the partition σ is selected and zero otherwise. Set $y^\sigma(S, j) = 1$ to mean that partition σ is selected and $S \in \sigma$ is assigned to agent j , and set it equal to zero otherwise. We call this formulation **(P4)**.

$$\begin{aligned} V(N) &= \max \sum_{\sigma \in \Pi} \sum_{S \in \sigma} \sum_{j \in N} v_j(S) y^\sigma(S, j) \\ \text{s.t. } &\sum_{S \in \sigma} y^\sigma(S, j) \leq z_\sigma \quad \forall j \in N, \forall \sigma \in \Pi \\ &\sum_{j \in N} y^\sigma(S, j) \leq z_\sigma \quad \forall S \in \sigma, \forall \sigma \in \Pi \\ &\sum_{\sigma \in \Pi} z_\sigma \leq 1 \\ &\sum_{\sigma \in \Pi} \sum_{S \in \sigma} y^\sigma(S, j) \leq 1 \quad \forall j \in N \\ &y^\sigma(S, j) \geq 0 \quad \forall S \in \sigma, \forall j \in N \\ &z_\sigma \geq 0 \quad \forall \sigma \in \Pi \end{aligned}$$

The constraints $\sum_{\sigma \in \Pi} \sum_{S \in \sigma} y^\sigma(S, j) \leq 1$ are redundant. Their dual, however, corresponds to agent j 's marginal product.

Theorem 2 *Formulation (P4) has an optimal integral solution.*

Proof In what follows we ignore the redundant constraint. Let

$$\begin{aligned}
G(z_\sigma) &= \max \sum_{S \in \sigma} \sum_{j \in N} v_j(S) y^\sigma(S, j) \\
\text{s.t. } &\sum_{S \in \sigma} y^\sigma(S, j) \leq z_\sigma \quad \forall j \in N \\
&\sum_{j \in N} y^\sigma(S, j) \leq z_\sigma \quad \forall S \in \sigma \\
&y^\sigma(S, j) \geq 0 \quad \forall S \in \sigma, \forall j \in N
\end{aligned}$$

Notice that each $G(z_\sigma)$ is the optimal objective function value of an assignment problem with all right hand sides being some number z_σ . Then

$$\begin{aligned}
V(N) &= \max_{\sigma \in \Pi} G(z_\sigma) \\
\text{s.t. } &\sum_{\sigma \in \Pi} z_\sigma \leq 1 \\
&0 \leq z_\sigma \leq 1 \quad \forall \sigma \in \Pi
\end{aligned}$$

Observe that $G(z_\sigma)$ is linear in z_σ . Thus the optimal solution will have z_σ being integral. ■

We now formulate the dual to (P4) and call it **(DP4)**. Let $u_j^\sigma \geq 0$ be the dual variables associated with the constraints

$$\sum_{S \in \sigma} y^\sigma(S, j) \leq z_\sigma \quad \forall j \in N.$$

Let $w_\sigma^\sigma \geq 0$ be the dual variables associated with the constraints:

$$\sum_{j \in N} y^\sigma(S, j) \leq z_\sigma \quad \forall S \in \sigma.$$

Let $\pi^s \geq 0$ be the dual variable associated with the constraint

$$\sum_{\sigma \in \Pi} z_\sigma \leq 1.$$

Let $\pi_j \geq 0$ be the dual variable associated with the constraint

$$\sum_{\sigma \in \Pi} \sum_{S \in \sigma} y^\sigma(S, j) \leq 1 \quad \forall j \in N.$$

With these variables, (DP4) is

$$\begin{aligned}
V(N) &= \min \sum_{j \in N} \pi_j + \pi^s \\
\text{s.t. } \pi^s &\geq \sum_{S \in \sigma} w_S^\sigma + \sum_{j \in N} u_j^\sigma \quad \forall \sigma \in \Pi \\
\pi_j + u_j^\sigma + w_S^\sigma &\geq v_j(S) \quad \forall j \in N, \forall S \in \sigma \\
\pi_j, u_j^\sigma, w_S^\sigma, \pi^s &\geq 0 \quad \forall j \in N, \forall S \in \sigma, \forall \sigma \in \Pi
\end{aligned}$$

The analogous result to Theorem 1 is the following.

Theorem 3 *Suppose the “agents are substitutes” condition (1) holds. Then there exists an optimal solution to (DP4), $((\pi_j), \pi^s)$, where $\pi_j = V(N) - V(N \setminus j)$ for all $j \in N$. The converse is also true.*

Proof Set $\pi_j = V(N) - V(N \setminus j)$ for all $j \in N$, and for each $\sigma \in \Pi$, choose u^σ and w^σ so as to solve:

$$\begin{aligned}
\min \sum_{j \in N} u_j^\sigma + \sum_{S \in \sigma} w_S^\sigma \\
\text{s.t. } u_j^\sigma + w_S^\sigma &\geq [v_j(S) - \pi_j]^+ = v'_j(S)
\end{aligned}$$

This is just the dual to the following assignment problem:

$$\begin{aligned}
H(\sigma) &= \max \sum_{j \in N} \sum_{S \in \sigma} v'_j(S) y(S, j) \\
\text{s.t. } \sum_{S \in \sigma} y^\sigma(S, j) &\leq 1 \quad \forall j \in N \\
\sum_{j \in N} y^\sigma(S, j) &\leq 1 \quad \forall S \in \sigma \\
0 &\leq y^\sigma(S, j) \quad \forall S \in \sigma, \forall j \in N
\end{aligned}$$

Set $\pi^s = \max_{\sigma \in \Pi} H(\sigma)$. Let the maximum be attained on the partition σ^* . Then the objective function value of (DP4) is

$$\sum_{j \in N} \pi_j + H(\sigma^*).$$

For a contradiction, assume that

$$\sum_{j \in N} \pi_j + H(\sigma^*) > V(N).$$

In the partition σ^* let K be the set of agents who receive an element of σ^* as specified in $H(\sigma^*)$. Let S^j be the element assigned to agent j . Then

$$V(N) < \sum_{j \in N} \pi_j + \sum_{j \in K} v'_j(S^j) = \sum_{j \in N} \pi_j + \sum_{j \in K} v_j(S^j) - \sum_{j \in K} \pi_j \leq V(K) + \sum_{j \in N \setminus K} \pi_j,$$

which violates the substitutes condition.

To prove the converse, if the partition-assignment pair μ contains the partition σ , we write $\mu \in \sigma$. Let $p_j^\sigma(S) \equiv u_j^\sigma + w_S^\sigma$, and $P^\sigma \equiv \sum_{S^j \in \mu, \mu \in \sigma} p_j^\sigma(S)$. Thus, we may write (DP4) as:

$$\begin{aligned} V(N) &= \min \sum_{j \in N} \pi_j + \pi^s \\ \text{s.t. } &\pi^s - P^\sigma \geq 0 \quad \forall \sigma \in \Pi \\ &\pi_j + p_j^\sigma(S) \geq v_j(S) \quad \forall j \in N, \forall S \in \sigma, \forall \sigma \in \Pi \\ &\pi_j, u_j^\sigma, w_S^\sigma \geq 0 \quad \forall j \in N, \forall S \in \sigma, \forall \sigma \in \Pi \end{aligned}$$

Let $(\pi_j, \pi^s, p_j^\sigma(S))$ be a feasible solution to (DP4), and define $p_j(S) \equiv \min_{\sigma} p_j^\sigma(S)$. Clearly, $(\pi_j, \pi^s, p_j(S))$ satisfies the buyer maximization constraints in (DP3); the satisfaction of the seller maximization constraints in (DP3) follows from the fact that for any $\mu \in \sigma$, $P^\sigma \geq \sum_{S^j \in \mu} p_j(S)$. Hence the feasible region of (DP4) is a subset of the feasible region of (DP3).

Thus by Theorem 1, if agents are not substitutes then there does not exist a (DP4) optimal solution which gives each buyer his marginal product. ■

It is interesting to observe that the dual variables in (DP3) can be related to the ones in (DP4) by setting

$$p_j(S) = \min_{\sigma \in \Pi} (u_j^\sigma + w_S^\sigma).$$

This allows us to break the agents' payments into a non-anonymous component u_j^σ and a non-linear component w_S^σ .

2.3 On accelerated computation

In order to compute the marginal products of agents using (P4), we would have to solve two optimization problems. The first is (P4) itself to determine $V(N)$. The second is

$$\begin{aligned}
& \max \sum_{j \in N} \pi_j \\
& \text{s.t. } V(N) = \sum_{j \in N} \pi_j + \pi^s \\
& \quad \pi^s \geq \sum_{S \in \sigma} w_S^\sigma + \sum_{j \in N} u_j^\sigma \quad \forall \sigma \in \Pi \\
& \quad \pi_j + u_j^\sigma + w_S^\sigma \geq v_j(S) \quad \forall j \in S, \forall S \in \sigma \\
& \quad \pi_j, u_j^\sigma, w_S^\sigma, \pi^s \geq 0 \quad \forall j \in N, \forall S \in \sigma, \forall \sigma \in \Pi
\end{aligned}$$

In words, among all optimal solutions to (DP4) find the one that maximizes $\sum_{j \in N} \pi_j$.

With some prior knowledge about the range of valuations, one can reduce the computation to a single optimization problem. Standard arguments from linear programming show that there must be $\epsilon > 0$ sufficiently small such that the optimal solution to the previous linear program coincides with the following one.

$$\begin{aligned}
& \min \sum_{j \in N} \pi_j + (1 + \epsilon)\pi^s \\
& \text{s.t. } \pi^s \geq \sum_{S \in \sigma} w_S^\sigma + \sum_{j \in N} u_j^\sigma \quad \forall \sigma \in \Pi \\
& \quad \pi_j + u_j^\sigma + w_S^\sigma \geq v_j(S) \quad \forall j \in S, \forall S \in \sigma \\
& \quad \pi_j, u_j^\sigma, w_S^\sigma, \pi^s \geq 0 \quad \forall j \in N, \forall S \in \sigma, \forall \sigma \in \Pi
\end{aligned}$$

3 Heterogeneous goods, unit demand

The first special case we consider is when each agent is interested in at most one object. It is well known that in this case the “agents are substitutes” condition (1) holds. Following formulation (P4), this means that there is only one partition of the set M of objects that needs to be considered. (P4) thus reduces to an assignment problem.

To fix notation, let $v_{ij} \geq 0$ be the value that agent $j \in N$ assigns to object $i \in M$. By adding dummy objects of zero value to all agents, we can always ensure that $|M| \geq |N|$. To formulate the problem of finding an efficient allocation let $x_{ij} = 1$ if agent j is allocated object i and zero otherwise.

$$\begin{aligned} \max \quad & \sum_{j \in N} \sum_{i \in M} v_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j \in N} x_{ij} \leq 1 \quad \forall i \in M \\ & \sum_{i \in M} x_{ij} \leq 1 \quad \forall j \in N \\ & x_{ij} \geq 0 \end{aligned}$$

This, of course, is the well known assignment problem in which the linear program has all integral extreme points. The dual to it is:

$$\begin{aligned} \min \quad & \sum_{j \in N} u_j + \sum_{i \in M} w_i \\ \text{s.t.} \quad & u_j + w_i \geq v_{ij} \quad \forall j \in N, \forall i \in M \\ & u_j, w_i \geq 0 \quad \forall j \in N, \forall i \in M \end{aligned}$$

In Leonard (1983) it is shown that among all optimal dual solutions, the one that maximizes $\sum_{j \in N} u_j$ yields the Vickrey payments. Specifically u_j is the marginal product of agent j . If one thinks about the definition of dual variables as the change in objective function value from a small change in the right hand side of the relevant constraint, then this should come as no surprise in the unit demand setting. Outside this setting, it need not be true. If we interpret w_i as the price of object i , we see that the price an agent pays for the object is anonymous; that is, it depends on the object alone. Given the form of (DP4), this is to be expected.

More interesting is that a particular implementation of the primal-dual algorithm for solving the assignment problem—one which finds, among optimal dual solutions, the one that maximizes total bidder surplus—produces an ascending auction that duplicates the outcome of the sealed bid Vickrey auction. Further, it is a Nash equilibrium for players to bid truthfully in each round of the auction. Such an auction was first proposed by Crawford and Knoer (1981) under the assumption that bidders have strict preferences (that is, no bidder is indifferent between any two objects). When the strict

preference condition is dropped, the Crawford and Knoer auction/algorithm can be made to produce an outcome arbitrarily close to the optimal solution of the assignment by choosing a tie-breaking parameter to be sufficiently small. Here we describe a version of the Crawford–Knoer algorithm due to Demange, Gale and Sotomayor (1986) that sidesteps the tie-breaking issues.

First we review the primal-dual algorithm for the assignment problem. Fix nonnegative $(w_i)_{i \in M} \geq 0$ and choose u so that $u_j = \max_i [v_{ij} - w_i]^+$ for $j \in N$. The resulting choice produces a feasible dual solution. We can interpret w as the price vector of objects and u as the vector of maximum profits that bidders achieve at the current prices. For each agent j let $D_j(w) = \{i : [v_{ij} - w_i]^+ = u_j\}$. Thus $D_j(w)$ is the set of objects that maximize agent j 's payoff at the current prices w .

Assuming the current feasible dual solution were optimal, we could determine a primal solution by enforcing the complementary slackness conditions. Thus an agent j and an object i such that $i \notin D_j(w)$, i.e., $u_j + w_i > v_{ij}$ would require that $x_{ij} = 0$. Hence the only variables, x_{ij} in the primal that can be set to 1 are those for which $u_j + w_i = v_{ij}$, i.e., $i \in D_j(w)$. The next step is to use only these variables to assign as many objects to agents so that no agent gets more than one object and no object is assigned to more than one agent. This auxiliary optimization problem, sometimes called the restricted primal (**RP**), is:

$$\begin{aligned} \max \quad & \sum_{j \in N} \sum_{i \in D_j(w)} x_{ij} \\ \text{s.t.} \quad & \sum_{j \in N : i \in D_j(w)} x_{ij} \leq 1 \quad \forall i \in M \\ & \sum_{i \in D_j(w)} x_{ij} \leq 1 \quad \forall j \in N \\ & x_{ij} \geq 0 \end{aligned}$$

The optimal solution to the restricted primal will either find an assignment in which all agents receive exactly one object or prove that no such assignment exists. In the first case, stop; the optimal solution has been found. In the second case, when no such assignment exists (assuming the appropriate solution algorithm is used), a set S of objects such that $|S| < |\{j : D_j(w) \subseteq S\}|$ will be found. The set S is called an **overdemanded set**. The existence of such a set is proven by Demange, Gale, and

Sotomayor (1986) via Hall’s Marriage Theorem. It can be derived from the dual of the restricted primal.

The optimal dual solution of the restricted primal determines the “direction” in which the current dual solution should be adjusted. If (u, w) is the current dual solution and (u', w') is the optimal solution to the dual of the restricted primal, then the adjusted dual solution for the next iteration is $(u, w) + \theta(u', w')$, where $\theta > 0$ is a user-specified step size. In the case of the assignment problem, each $w_i \in S$ is increased by $\theta > 0$ (suitably chosen) and each u_j is then recomputed.

Different primal-dual algorithms differ by the choice of the overdemanded set used and how the dual variables are adjusted. Since we want the dual variables to yield the Vickrey payoffs, we need to ensure that a particular optimal dual solution is produced. We need the optimal dual solution with the largest $\sum_{j \in N} u_j$ or, equivalently, smallest $\sum_{i \in M} w_i$. This can be done by choosing, at each iteration, a *minimal* overdemanded set.

Since such algorithms will appear in subsequent sections, it will be useful to provide a high level description. Algorithm 1 is from Papadimitriou and Steiglitz (1982).

We now describe the auction. Suppose that bidders valuations are integral.⁵ The auction proceeds in rounds. At the start of round t , let p_i^t be the current price of object i . We set $p_i^0 = 0$ for all $i \in M$. At the beginning of the round each bidder is asked to name all their ‘favorite objects’, i.e., the set $\arg \max_i [v_{ij} - p_i^t]$. Bidders whose favorite objects yield negative payoff leave the auction.

Let B^t be a bipartite graph with a node for each remaining bidder and one for each object. Insert edges between a bidder and an object that is her favorite. If the graph has a matching such that every bidder is assigned to exactly one object and no object is assigned to more than one bidder, stop. The assignment is made at the price p^t . If no such matching exists, there is, by Hall’s Marriage theorem, a subset B of bidders such that $|B| > |N(B)|$ where $N(B)$ is the set of neighbors of B . The set $N(B)$ is called an overdemanded set of objects. From among all overdemanded sets, choose one that is minimal. Raise the price on all objects in that set by \$1, terminating round t of the auction.

It is clear that if bidders bid truthfully in each round, the auction will mimic the primal-dual algorithm for the assignment problem. In fact, it is

⁵A standard scaling argument will take care of the non-integral case.

Algorithm 1 The Primal-Dual Algorithm (Papadimitriou & Steiglitz)

```
1: infeasible ← 'no'
2: opt ← 'no'
3: start with a dual feasible solution
4: while infeasible = 'no' and opt = 'no' do
5:   J ← indices of dual inequalities fulfilled with equality
6:   solve the restricted primal (RP)
7:   if opt of (RP) = 0 then
8:     opt ← 'yes'
9:   else
10:    solve the dual of the restricted primal
11:    if the product of the solution with each column of the primal constraint matrix is nonpositive then
12:      infeasible ← 'yes'
13:    else
14:      update the dual solution
15:    end if
16:  end if
17: end while
```

a Nash equilibrium for the bidders to bid truthfully in each round. The intuition for this fact can be described as follows. Since all other bidders are bidding truthfully, the only thing a bidder can do in each round is to prevent prices from increasing. Prices rise only on objects in a minimally overdemanded set. Thus, to affect a price rise, an agent's favorite objects (at the current price) must lie in this overdemanded set. The only way the agent can prevent prices rising on objects in this set is by reducing his own demand for objects in this set, thereby making the agent worse-off.

4 Heterogeneous goods, gross substitutes

Gul and Stacchetti (2000) have proposed an algorithm that generalizes the auction procedure of Demange, Gale, and Sotomayor (1986) to environments in which bidders have preferences over subsets of items. It is a primal-dual algorithm, but does not share the incentive properties of the Demange, Gale, and Sotomayor algorithm. It also is a variation of the Kelso and Crawford

(1982) algorithm that sidesteps tie-breaking issues when agents are indifferent between utility maximizing bundles of objects.

In this set-up, agents have preferences over subsets of heterogeneous objects. The value that bidder j assigns to the set $S \subset M$ of objects is $v_j(S)$. An efficient allocation is a partition σ of the objects among the agents that maximizes the sum of values, i.e., problem (P1).

As in Kelso and Crawford (1982), Gul and Stacchetti (2000) impose two conditions on agent's value functions to ensure that the linear relaxation of (P1) has an optimal integer solution. They then provide a primal-dual algorithm for the optimization problem and give it an auction interpretation. To describe these two properties we need some notation.

Given a vector of prices p on objects let the collection of subsets of objects that maximize bidder j 's utility be denoted $D_j(p)$, defined as follows.

$$D_j(p) = \{S \subset M : v_j(S) - \sum_{i \in S} p_i \geq v_j(T) - \sum_{i \in T} p_i \quad \forall T \subset M\}.$$

Monotonicity: For all $j \in N$ and all $A \subset B \subset M$, $v_j(A) \leq v_j(B)$.

Single improvement: For all $A \notin D_j(p)$ there exists $B \subset M$ such that $v_j(B) - \sum_{i \in B} p_i > v_j(A) - \sum_{i \in A} p_i$ and $|A \setminus B|, |B \setminus A| \leq 1$.

Together, these two conditions imply that v_j is submodular.

While monotonicity seems to be economically plausible, the single improvement property does not. In Gul and Stacchetti (1999), however, it is shown that the single improvement property (under monotonicity) is equivalent to a property called *gross substitutes*, which does have economic content. The gross substitutes property, introduced by Kelso and Crawford (1982), requires that a demanded object still be demanded if the prices of all other objects increase.

Gross substitutes: For all price vectors p, p' such that $p' \geq p$, and all $A \in D_j(p)$, there exists $B \in D_j(p')$ such that $\{i \in A : p_i = p'_i\} \subset B$.

These two conditions impose two useful properties on the sets $D_j(p)$. The first is that

$$r_j(T, p) = \min_{S \in D_j(p)} |S \cap T|$$

is the dual rank function of a matroid. From this it follows that for any $T \subset M$ and pair of price vectors $p' \geq p$ such that $p_i = p'_i$ for all $i \in M \setminus T$, we have

$$r_j(T, p) \leq r_j(T, p').$$

Proofs of both statements can be found in Gul and Stacchetti (2000).

Recall formulation (P1):

$$\begin{aligned} V(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) \\ \text{s.t.} \quad &\sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \quad \forall i \in M \\ &\sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\ &y(S, j) = 0, 1 \quad \forall S \subseteq M, \forall j \in N \end{aligned}$$

Relaxing the integrality constraint in (P1) gives

$$\begin{aligned} V_{LP}(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) \\ \text{s.t.} \quad &\sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \quad \forall i \in M \\ &\sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\ &y(S, j) \geq 0 \quad \forall S \subseteq M, \forall j \in N \end{aligned}$$

Theorem 4 *When each agent's value function satisfies monotonicity and the single improvement property, $V(N) = V_{LP}(N)$.*

Proof Let p^* be an optimal dual solution to the linear relaxation of (P1). Let

$$\begin{aligned} V_p(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} v_j(S) y(S, j) - \sum_{i \in M} p_i \left[\sum_{S \ni i} \sum_{j \in N} y(S, j) \right] + \sum_{i \in M} p_i \\ \text{s.t.} \quad &\sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\ &y(S, j) \geq 0 \quad \forall S \subseteq M, \forall j \in N \end{aligned}$$

Rewriting the objective function,

$$\begin{aligned}
V_p(N) &= \max \sum_{j \in N} \sum_{S \subseteq M} \left[v_j(S) - \sum_{i \in S} p_i \right] y(S, j) + \sum_{i \in M} p_i \\
\text{s.t. } &\sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in N \\
&y(S, j) \geq 0 \quad \forall S \subseteq M, \forall j \in N
\end{aligned}$$

In this problem, if $y(S, j) = 1$ is part of an optimal solution, then $S \in D_j(p)$. By the duality theorem of linear programming, $V_{p^*}(N) = V_{LP}(N)$. If there is a partition of M so that each agent j receives at most one element of $D_j(p)$, it would follow that (P1) has an optimal integer solution. Suppose no such partition exists. Given that each r_j is a dual rank function of a matroid, it follows from the matroid partition theorem (Edmonds and Fulkerson (1965)) that there is a $T \subset M$ such that

$$\sum_{j \in N} r_j(T, p^*) > |T|.$$

Choose a new dual solution p such that $p_i = p_i^*$ for all $i \in M \setminus T$ and $p_i = p_i^* + \epsilon$ for all $i \in T$ for sufficiently small $\epsilon > 0$. For each $j \in N$, choose $B^j \subset M$ such that $r_j(T, p^*) = |B^j \cap T|$. Then

$$\begin{aligned}
V_p(N) &= \sum_{j \in N} \left[v_j(B^j) - \sum_{i \in M} p_i \right] + \sum_{i \in M} p_i^* + \epsilon |T| \\
&= \sum_{j \in N} \left[v_j(B^j) - \sum_{i \in M} p_i^* \right] - \epsilon \sum_{j \in N} |B^j \cap T| + \sum_{i \in M} p_i^* + \epsilon |T| \\
&= V_{p^*}(N) + \epsilon |T| - \epsilon \sum_{j \in N} r_j(T, p^*) \\
&= V_{p^*} - \epsilon \left(\sum_{j \in N} r_j(T, p^*) - |T| \right) < V_{p^*}(N)
\end{aligned}$$

which contradicts the optimality of p^* . ■

The proof suggests a primal-dual algorithm. Fix a price vector p . At those prices, determine a partition of M that gives to each agent an element

of $D_j(p)$. If no such partition exists, find an overdemanded set T , i.e., a set T such that

$$\sum_{j \in N} r_j(T, p^*) > |T|.$$

Particularly, choose a minimal one, according to the algorithm proposed by Gul and Stacchetti (2000), and raise the price on each object in that set by $\epsilon > 0$. Assuming sincere bidding at each round, the auction produces, at termination, *minimal* Walrasian prices.⁶ As noted by Gul and Stacchetti, the auction is not guaranteed to produce Vickrey prices. We now provide an example to demonstrate.

The example consists of three agents, each of whom may consume at most two of four given objects. Their valuations for such subsets are additive. Let $N = \{1, 2, 3\}$ and $M = \{a, b, c, d\}$. The agents value single-object subsets as shown in Table 1. For $|S| = 2$, where $\{i, i'\} = S$, let $v_j(S) = v_j(i) + v_j(i')$.

Table 1: $v_j(S)$ when $|S| = 1$.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1	10	10	10	10
2	4	7	9	8
3	2	2	2	4

For $|S| > 2$, let $v_j(S) = \max\{v_j(S') : S' \subset S, |S'| = 2\}$.

The efficient allocation is to give objects *a* and *b* to agent 1, and the other two objects to agent 2, i.e., $y(\{a, b\}, 1) = 1$ and $y(\{c, d\}, 2) = 1$. For (p_a, p_b, p_c, p_d) to be a *Walrasian price vector* supporting this allocation, each agent must be consuming his most preferred bundle of objects, given these prices.

This requires many inequalities to hold. In particular, since agent 1 is consuming *b* and not *c*, this requires $p_b \leq p_c$. Since agent 2 consumes *d* and not *b*, this requires $p_d \leq p_b + 1$. Finally, since agent 3 consumes no object, this requires $p_a \geq 2$ and $p_d \geq 4$.

⁶Prices are Walrasian for an allocation of objects if, given those prices, each agent is consuming a set of objects that maximizes his payoff (net of prices). See any standard microeconomics textbook for a formal definition, and see the example below.

Therefore, the minimal Walrasian price vector⁷ is $(2, 3, 3, 4)$. However, agent 2's Vickrey payment is $6 < p_c + p_d = 7$. Therefore, in this example, the agents' Vickrey payments cannot be additively decomposed into a vector of Walrasian prices for the objects.

Recently, Parkes and Ungar (2000) have proposed a primal-dual algorithm with an auction interpretation for (P3). This allows them to drop the gross substitutes assumption and determine non-linear, non-anonymous prices that support an efficient allocation. These are not, however, Vickrey prices. Ausubel (2000) proposes an ascending auction for this case that does implement the Vickrey outcome. This auction does not appear to have an interpretation as a primal-dual algorithm for some optimization problem.

4.1 Agents are substitutes

Here we show that when all agents' value functions are monotonic and satisfy the single improvement property, the substitutes condition (1) holds. This condition enables us to invoke formulation (P4) to derive the Vickrey prices. From (DP4) we see that the Vickrey prices cannot be anonymous.

For every subset K of agents, let

$$\begin{aligned} V_p(K) &= \max \sum_{j \in K} \sum_{S \subseteq M} (v_j(S) - \sum_{i \in S} p_i) y(S, j) + \sum_{i \in M} p_i \\ \text{s.t. } &\sum_{S \subseteq M} y(S, j) \leq 1 \quad \forall j \in K \\ &y(S, j) \geq 0 \quad \forall S \subseteq M, \forall j \in K. \end{aligned}$$

$V_p(K)$ can be expressed more succinctly as:

$$V_p(K) = \sum_{j \in K} \max_{S \subseteq M} [v_j(S) - \sum_{i \in S} p_i] + \sum_{i \in M} p_i$$

By the duality theorem of linear programming:

$$V(K) = \min_{p \geq 0} V_p(K).$$

Amongst all $p \in \arg \min_{p \geq 0} V_p(K)$ let p^K denote the minimal one. From Gul and Stacchetti (1999) we know that the set $\arg \min_{p \geq 0} V_p(K)$ forms a

⁷The vector is minimal in the sense that every other Walrasian price vector weakly dominates this one.

complete lattice and a minimal element exists. It is also easy to see that if $K \subset T$ then $p^K \leq p^T$. Associated with each $V(K)$ is a partition of M . Denote by S_j^K the subset of objects that agent j receives in this partition, i.e., $y(S_j^K, j) = 1$.

For each $k \in N$ we have

$$\begin{aligned} V(N \setminus k) = V_{p^{N \setminus k}}(N \setminus k) &\geq \sum_{j \in N \setminus k} [v_j(S_j^N) - \sum_{i \in S_j^N} p_i^{N \setminus k}] + \sum_{i \in M} p_i^{N \setminus k} \\ &= \sum_{j \in N \setminus k} v_j(S_j^N) + \sum_{i \in S_k^N} p_i^{N \setminus k} \end{aligned}$$

Hence

$$V(N) - V(N \setminus k) \leq \sum_{j \in N} v_j(S_j^N) - \sum_{j \in N \setminus k} v_j(S_j^N) - \sum_{i \in S_k^N} p_i^{N \setminus k} = v_k(S_k^N) - \sum_{i \in S_k^N} p_i^{N \setminus k}.$$

Let p' be a price vector such that

$$p'_i = p_i^N \quad \forall i \in \bigcup_{j \in K} S_j^N$$

and

$$p'_i = p_i^{N \setminus k} \quad \forall i \in S_k^N, \forall k \in N \setminus K.$$

Then

$$\begin{aligned} V(K) = V_{p^K}(K) \leq V_{p'}(K) &= \sum_{j \in K} \max_{S \subseteq M} [v_j(S) - \sum_{i \in S} p_i^N] + \sum_{i \in M} p'_i \\ &= \sum_{j \in K} v_j(S_j^N) + \sum_{k \in N \setminus K} \sum_{i \in S_k^N} p_i^{N \setminus k} \end{aligned}$$

Therefore,

$$V(N) - V(K) \geq \sum_{j \in N \setminus K} v_j(S_j^N) - \sum_{k \in N \setminus K} \sum_{i \in S_k^N} p_i^{N \setminus k} \geq \sum_{k \in N \setminus K} [V(N) - V(N \setminus k)].$$

5 Spanning trees

Let $G = (V, E)$ be a complete graph with vertex set V and edge set E . Each edge of the graph is owned by some agent. Denote by E_j the set of edges

owned by agent $j \in N$, and let $E_K = \bigcup_{j \in K} E_j$. The cost of using an edge $e \in E$ is c_e . An efficient allocation is a set of edges that span the graph with minimal total cost.

Let T denote the minimum spanning tree (MST) and let L denote its length. For any subset $E' \subset E$ of edges, let $T^{-E'} \subset E \setminus E'$ be the MST that does not use any edge in E' ; denote its length by $L^{-E'}$, which by convention is equal to ∞ if there is no spanning tree after deleting edges in E' .

For the remainder of our discussion on spanning trees, we assume that no agent owns a cut; that is, for each agent $i \in N$, we assume that $L^{-E_i} < \infty$. We show that with this assumption in this model, agents are substitutes.

Theorem 5 *In the minimum spanning tree problem, the “agents are substitutes” condition (1) holds (when no agent owns a cut): For all $K \subset N$,*

$$L^{-E_K} - L \geq \sum_{i \in K} [L^{-E_i} - L].$$

Proof A sufficient condition for the “agents are substitutes” condition to hold is that an agent’s marginal contribution to coalitions is monotonically decreasing as the coalition increases. In other words, for the MST problem, it is sufficient to show that for all $K \subset K' \subseteq N$ and all $i \notin K'$,

$$L^{-E_{K \cup i}} - L^{-E_K} \leq L^{-E_{K' \cup i}} - L^{-E_{K'}}.$$

By applying this monotonicity condition to a telescoping sum, the substitutes condition is proven.

In turn, the following condition, which we prove, implies the above monotonicity condition.

$$\forall a \neq e \in E, \quad L^{-a,e} - L^{-e} \geq L^{-a} - L$$

This condition says that as other edges are removed from a graph, the marginal value of a given edge increases.

If either $L^{-a} = L$ or $L^{-e} = L$, we are done. Otherwise, there exist $b, b', f, f' \in E$ such that

$$\begin{aligned} T^{-a} &= (T \setminus a) \cup b & T^{-e} &= (T \setminus e) \cup f \\ T^{-ae} &= (T^{-e} \setminus a) \cup b' & T^{-ae} &= (T^{-a} \setminus e) \cup f' \end{aligned}$$

Note that $\{b, b', f, f'\} \subset T^{-ae}$. Therefore, if $b \neq f$, then $T^{-ae} = T \setminus \{a, e\} \cup \{b, f\}$, and $L^{-a,e} - L^{-e} = L^{-a} - L$. Otherwise, if $b = f$, then $b \in T^{-e}$, so $b' \neq b$. Similarly, $f' \neq f$. Therefore $b' = f'$.

What we are trying to show from the above inequality is

$$(L^{-a} - c_e + c_{f'}) - (L - c_e + c_f) \geq L^{-a} - L$$

which is equivalent to $c_{f'} \geq c_f$. Since $c_b = c_f$ and $c_{b'} = c_{f'}$, it is clear that $c_{f'} \geq c_f$, otherwise either T^{-a} or T^{-e} would not have been minimal. ■

5.1 Agents own one edge

There are many ways to formulate the problem of finding a minimum cost spanning tree as a linear program. In fact we could use formulations (P3) or (P4) directly. Instead we use a standard “covering” formulation as in Bertsimas and Teo (1998) for the case when $|E_j| = 1$. This allows us to extend the arguments made here to the more general case of matroids.

Let Π be the set of partitions of V . For any partition $\sigma \in \Pi$, $S \in \sigma$ means that $S \subset V$ is a part of the partition σ and $|\sigma|$ refers to the number of parts of this partition. For each partition of vertices $\sigma \in \Pi$, let $\delta(\sigma) \subset E$ be the set of edges with endpoints in different parts of σ . Set $x_e = 1$ if edge $e \in E$ is selected and zero otherwise.

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(\sigma)} x_e \geq |\sigma| - 1 \quad \forall \sigma \in \Pi \\ & \sum_{e \in E} x_e = |V| - 1 \\ & 0 \leq x_e \leq 1 \quad \forall e \in E \end{aligned}$$

Actually, the constraint $x_e \leq 1$ is redundant, but its dual will yield agent e 's marginal product. It is well known that this formulation is integral.

Let λ_e be the dual variable associated with the constraint $x_e \leq 1$, and μ_σ the dual variable associated with $\sum_{e \in \delta(\sigma)} x_e \geq |\sigma| - 1$. Let μ be the dual

variable associated with $\sum_{e \in E} x_e = |V| - 1$. The dual problem is

$$\begin{aligned}
& \max - \sum_{e \in E} \lambda_e + (|V| - 1)\mu + \sum_{\sigma \in \Pi} (|\sigma| - 1)\mu_\sigma \\
& \text{s.t. } -\lambda_e + \mu + \sum_{\delta(\sigma) \ni e} \mu_\sigma \leq c_e \quad \forall e \in E \\
& \quad \lambda_e \geq 0 \quad \forall e \in E \\
& \quad \mu_\sigma \geq 0 \quad \forall \sigma \in \Pi
\end{aligned}$$

As before let T be the minimum spanning tree. If $e \notin T$ set $\lambda_e = 0$. If $e \in T$ set $\lambda_e = c_{f_e} - c_e$ where f_e is the shortest edge such that $T \cup f_e$ contains a cycle through e . Clearly, each λ_e is agent e 's marginal product. For this choice of λ_e we must show that we can choose μ and μ_σ to be dual feasible and to yield an objective function value equal to the length of the minimum spanning tree.

Substituting in the value of λ_e we have specified into the dual and dropping the part $-\sum_{e \in E} \lambda_e$ from the objective function yields

$$\begin{aligned}
& \max (|V| - 1)\mu + \sum_{\sigma \in \Pi} (|\sigma| - 1)\mu_\sigma \\
& \text{s.t. } \mu + \sum_{\delta(\sigma) \ni e} \mu_\sigma \leq c_e \quad \forall e \notin T \\
& \quad \mu + \sum_{\delta(\sigma) \ni e} \mu_\sigma \leq c_{f_e} \quad \forall e \in T \\
& \quad \mu_\sigma \geq 0 \quad \forall \sigma \in \Pi
\end{aligned}$$

This is the dual to the problem of finding a minimum spanning tree in the original graph whose edge weights have been modified. Each edge $e \in T$ has had its weight increased to c_{f_e} . The length of the minimum spanning tree in this modified graph is of course $\sum_{e \in T} c_{f_e}$.

Hence the optimal values of μ and μ_σ from the second dual program combined with our choice of λ_e form a feasible solution to the original dual problem. The value of this dual solution is:

$$- \sum_{e \in T} (c_{f_e} - c_e) + \sum_{e \in T} c_{f_e} = \sum_{e \in T} c_e.$$

That is, it coincides with the optimal value of the primal.

A primal-dual algorithm for this formulation requires that at each iteration, prices on $\delta(\sigma)$ be announced. These can be interpreted as prices for inserting an edge connecting different parts of σ . Roughly speaking, the auctioneer runs a second price auction on each partition of vertices. This observation allows us to derive from the primal-dual algorithm a descending auction (since we are in a procurement setting) to implement the Vickrey outcome. We omit a description of the primal-dual algorithm and proceed directly to the auction.

To run the auction, we first set the offered price on each edge at some large number P . Each agent is to announce whether they are willing to offer their edge at that price. Now, we begin to drop the price. As the price drops, agents withdraw their edges when they become unprofitable at the given price, reducing the connectivity of the graph. When the connectivity of the graph drops to 1, the auction pauses, and the agent who owns the critical edge is given the price at which the auction has paused for that edge. That edge is not removed for the duration of the auction, and is committed to being a part of the final tree. Now restart the auction, pausing at the next critical edge whose removal would disconnect the graph, and so on, making a payment and committing an edge at each pause. The auction ends when a tree has been completed.

5.2 Agents own multiple edges

We now give a formulation for the case in which $|E_j| \geq 1$. For each $F \subseteq E_j$, set $c_j(F) = \sum_{e \in F} c_e$. For each $j \in N$ and $F \subseteq E_j$, set $y_j(F) = 1$ if F is selected to be in a MST and zero otherwise. The formulation, which we

denote **(TP)**, is

$$\begin{aligned}
& \min \sum_{j \in N} \sum_{F \subseteq E_j} c_j(F) y_j(F) \\
& \text{s.t.} \sum_{j \in N} \sum_{F \subseteq E_j : \delta(\sigma) \cap F \neq \emptyset} y_j(F) \geq |\sigma| - 1 \quad \forall \sigma \in \Pi \\
& \sum_{j \in N} \sum_{F \subseteq E_j} |F| y_j(F) = |V| - 1 \\
& - \sum_{F \subseteq E_j} y_j(F) \geq -1 \quad \forall j \in N \\
& y_j(F) \geq 0 \quad \forall j \in N, \forall F \subseteq E_j
\end{aligned}$$

The first constraint ensures that the tree spans the graph; the second says that exactly $|V| - 1$ edges are used (so this could be weakened to an inequality); the third says that each agent supplies at most one subset of edges, F .

We will show that this formulation has an optimal integral solution. In addition, the dual variable corresponding to the constraint $-\sum_{j \in N} y_j(F) \geq -1$ yields agent j 's marginal product.

Theorem 6 *Formulation (TP) has an optimal integral solution.*

Proof Let y denote an optimal extreme (possibly fractional) solution to (TP). For each agent j and edge $e \in E_j$, set

$$z_e = \sum_{F \subseteq E_j} y_j(F).$$

We will show that z is a feasible solution to the following system:

$$\begin{aligned}
& \sum_{e \in \delta(\sigma)} x_e \geq |\sigma| - 1 \quad \forall \sigma \in \Pi \\
& \sum_{e \in E} x_e = |V| - 1 \\
& 0 \leq x_e \quad \forall e \in E
\end{aligned}$$

which we know to have integral extreme points. Furthermore, since we will prove that each vertex of the latter formulation has an integral preimage

in the previous formulation, we conclude that the latter formulation is a projection of the earlier formulation. Given an integral solution x^* to this system, we can construct an integral solution y^* to (TP). Let $G_j = \{e \in E_j : x_e^* = 1\}$. For all $j \in N$, set $y_j^*(G_j) = 1$ and $y_j^*(F) = 0$ for $F \neq G_j$. Observe that $\sum_e c_e x_e^* = \sum_{j \in N} c_j(G_j) y_j^*(G_j)$, i.e. both solutions have the same objective function value.

Clearly $z_e \geq 0$. Also,

$$\sum_e z_e = \sum_{j \in N} \sum_{e \in E_j} \sum_{F \subseteq E_j: e \in F} y_j(F) = \sum_{j \in N} \sum_{F \subseteq E_j} |F| y_j(F) = |V| - 1.$$

Next,

$$\begin{aligned} \sum_{e \in \delta(\sigma)} z_e &= \sum_{j \in N} \sum_{e \in E_j \cap \delta(\sigma)} \sum_{F \subseteq E_j: e \in F} y_j(F) \\ &= \sum_{j \in N} \sum_{F \subseteq E_j: F \cap \delta(\sigma) \neq \emptyset} y_j(F) \geq |\sigma| - 1. \end{aligned}$$

Notice also that

$$\sum_{j \in N} \sum_{F \subseteq E_j} c_j(F) y_j(F) = \sum_e c_e z_e.$$

Hence, if z_e is integral we are done. If not, we can express z_e as a convex combination of integral points of the second formulation. But each of these corresponds in turn to an integral point of (TP). Thus, y can be expressed as a convex combination of extreme points of (TP), a contradiction. ■

The dual to the formulation (TP) is

$$\begin{aligned} \max \quad & \sum_{S \subset V} \mu_S + (|V| - 1)\mu - \sum_{j \in N} \lambda_j \\ \text{s.t.} \quad & \sum_{F \subseteq E_j: \delta(S) \cap F \neq \emptyset} |\delta(S) \cap F| \mu_S + |F| \mu - \lambda_j \leq c_j(F) \quad \forall F \subseteq E_j \\ & \mu_S \geq 0 \quad \forall S \subset V \\ & \lambda_j \geq 0 \quad \forall j \in N \end{aligned}$$

As before, T denotes the MST, and T^{-E_j} is the MST subject to the constraint that no edges in E_j are used. Let $F^j = T \cap E_j$ be the set of edges

owned by agent j in a MST. Observe that $|F^j| = |T^{-E_j} \setminus T|$, i.e., when edges are deleted from possible use, they are replaced on a one-for-one basis. Let

$$\lambda_j = \left(\sum_{e \in T^{-E_j} \setminus T} c_e \right) - c_j(F^j).$$

Notice this is agent j 's marginal product.

Order the edges in F^j by increasing cost: e_1, e_2, \dots, e_t for some t . Do the same with edges in $T^{-E_j} \setminus T$: e'_1, e'_2, \dots, e'_t . It is easy to check that $c_{e_i} \leq c_{e'_i}$ for $1 \leq i \leq t$. Also $T \cup e'_i$ has a cycle that goes through e_i and e'_i . Suppose we modify the costs of the edges by increasing the cost of each e_i to $c_{e'_i}$. Then T will still be a minimum spanning tree with respect to the modified weights. Let c' denote the vector of modified costs.

Substituting the value for λ into the dual yields

$$\begin{aligned} & - \sum_{j \in N} \lambda_j + \max_{\sigma \in \Pi} \sum_{\sigma \in \Pi} \mu_\sigma + (|V| - 1)\mu \\ & \sum_{\sigma: \delta(\sigma) \cap F \neq \emptyset} |\delta(\sigma) \cap F| \mu_\sigma + |F| \mu \leq \lambda_j + c_j(F) \quad \forall F \subseteq E_j \\ & \mu_\sigma \geq 0 \quad \forall \sigma \in \Pi \end{aligned}$$

It suffices to show that we can find a feasible solution to the dual with objective function value $\sum_{e \in T} c_e$. By the previous theorem this dual solution must be optimal.

Notice that any feasible solution to

$$\begin{aligned} & \sum_{\sigma: \delta(\sigma) \cap F \neq \emptyset} |\delta(\sigma) \cap F| \mu_{\sigma} + |F| \mu \leq c'_j(F) \quad \forall F \subseteq E_j \\ & \mu_\sigma \geq 0 \quad \forall \sigma \in \Pi \end{aligned}$$

is a feasible solution to the dual because

$$\begin{aligned} \lambda_j + c_j(F) &= \left(\sum_{e \in T^{-E_j} \setminus T} c_e \right) - c_j(F^j) + c_j(F) \\ &= \left(\sum_{e \in T^{-E_j} \setminus T} c_e \right) - c_j(F^j \setminus F) + c_j(F \setminus F^j) \geq c'(F). \end{aligned}$$

However, this second system is the dual to the minimum spanning tree problem with respect to the cost vector c' . Thus this second system admits a feasible solution with objective function value $\sum_{e \in T} c'_e = \sum_{j \in N} \sum_{e \in F^{-j}} c_e$. Substituting these variables into the original dual system gives us a solution with objective function value

$$-\sum_{j \in N} \lambda_j + \sum_{j \in N} \sum_{e \in F^{-j}} c_e = \sum_{e \in T} c_e.$$

By the previous theorem this dual solution must be optimal.

5.3 Matroids

The problem of finding a minimum weight spanning tree is a special case of the problem of finding a minimum/maximum weight basis in a matroid.⁸ Here we outline how the descending auction can easily be adapted to the matroid case.

Let E be the ground set and E^r the subset of E owned by agent r . For each $e \in E$ let w_e be the weight of e known only to the agent who owns that element. Let I be the family of independent subsets of E . The pair (E, I) forms a matroid.

Now we describe the auction. Start with a high ‘offer’ price for all elements. Agents announce which elements they are prepared to offer at the current price. These are called active elements. Now drop the price. As the price drops, agents withdraw some elements. Meanwhile, the auctioneer checks that for any agent, a basis is contained in the set of active elements not belonging to that agent.

Consider the instant this condition is violated. Let j be the agent, p the current price and E_a^j the currently active set of elements belonging to agent j . Let M be any minimal subset of E_a^j whose removal from the set of all active edges would eliminate the existence of a basis. From the matroid property, all such sets have the same cardinality. The auctioneer pays $|M|p$ to agent j , and agent j chooses which subset of size $|M|$ from E_a^j to proffer. The price on these elements is fixed at p for the remainder of the auction, and the auction continues.

If the goal is to find a maximum weight basis, then start with a low price and raise it. The rest of the auction works as before.

⁸This section assumes familiarity with matroid theory. Gul and Stacchetti (2000) also make an appeal to matroid theory.

Consider now an auction where K identical units must be auctioned off to n bidders. The value that bidder j assigns to their i^{th} unit is v_j^i . We assume that $v_j^i \geq v_j^{i+1}$, indicating that bidders have diminishing marginal utilities. The problem of finding an efficient allocation can be formulated as the problem of finding a maximum weight basis.

For each bidder j we introduce K elements, j^1, j^2, \dots, j^K with weights $v_j^1, v_j^2, \dots, v_j^K$. Let E be the collection of all such elements. Let I be the collection of all subsets of size at most K . It is easy to see that (E, I) is a matroid. In this case the ascending auction coincides with Ausubel's (1997) ascending auction for homogeneous products.

6 Shortest paths

Let $G = (V, E)$ be a directed graph with vertex set V and edge set E . Specify a source node $s \in V$ and sink node $t \in V$. Each edge $e \in E$ has a cost c_e . In what follows, we sometimes refer to an edge e as a pair of vertices (i, j) where e is directed from i to j . The efficient allocation is the s - t path of minimal total cost, i.e., the shortest path.

Using the results of Hoffman and Markowitz (1963), we can convert the shortest path problem into an assignment problem. Replace each vertex $i \in V$ by two vertices i and i' . Replace each directed edge (i, j) by two directed edges: (i, j') with cost c_{ij} , and (i, i') with cost zero. Delete vertices s' and t from this extended network. An optimal assignment in this new graph corresponds to a shortest path in the original one, and vice versa.

Given this correspondence, it is tempting to think that one can apply the methods of Demange, Gale, and Sotomayor (1986) to this problem; unfortunately one cannot. Even if each agent is associated with only a single edge, each agent appears on both sides of this assignment problem.

To be precise about the ownership structure, each agent j owns a set of edges E^j . The interesting cases are those in which the following no-monopoly condition holds, which we now assume: no agent owns a cut that disconnects s from t .

In this model, the substitutes condition (1) need not hold. To illustrate the complementarities of agents, we give an example even for the case in which each s - t path uses at most one edge belonging to any particular agent. The example involves three edges and three agents. To each directed edge in Figure 1 is associated an agent-cost pair (e.g., agent A's edge has a cost of

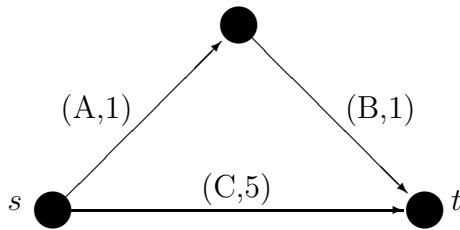


Figure 1: A shortest path problem.

1).

Let P^* be the shortest $s-t$ path and denote its length by $L(P^*)$. Let P^{-S} be the shortest $s-t$ path that does not use any edge belonging to an agent in $S \subset N$, and let $L(P^{-S})$ be its length. The substitutes condition (1) is the requirement

$$L(P^{-S}) - L(P^*) \geq \sum_{j \in S} [L(P^{-j}) - L(P^*)] \quad \forall S \subset N.$$

For this example $L(P^*) = 2$. The marginal product of agent A is $5 - 2 = 3$, which is the same as that of agent B. But for the coalition $S = \{1, 2\}$, the marginal product is also $5 - 2 = 3$. Since $3 < 3 + 3$, the substitutes condition does not hold. In this shortest path problem, agents A and B are complements; neither can be on an $s-t$ path without the other.

6.1 Dual variables

While the substitutes condition need not hold in general in the shortest path environment, there are special cases where it does. One such example is in Schummer and Vohra (2001). When the substitutes condition holds, one can ask if there is a linear programming formulation whose dual variables give rise to agents' marginal products.⁹ Here, we describe one such formulation.

To be consistent with the typical notation for this problem, for the remainder of this section we let A denote the set of agents. We assume that any $s-t$ path uses at most one edge belonging to a particular agent $r \in A$.

To formulate the shortest path problem as a linear program, let N be the node-arc incidence matrix of the graph; the column of N corresponding with

⁹Hershberger and Suri (2001) provide an algorithm for finding Vickrey payments when valuations are given, even when the substitutes condition does not hold.

the edge from vertices i to j contains a -1 in the row corresponding to i , a 1 in the row for j , and a 0 in the other rows. Let $x_e = 1$ if edge e is used, and zero otherwise. Then the problem is

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & Nx = b \\ & x_e \geq 0 \quad \forall e \in E \\ & \sum_{e \in E^r} x_e \leq 1 \quad \forall r \in A \end{aligned}$$

where E^r is the set of edges belonging to agent $r \in A$.

Here, b is a column vector defined as $b_s = -1$, $b_t = 1$, and $b_i = 0$ for all $j \in V \setminus \{s, t\}$. The last set of constraints is redundant given the assumption that any s - t path uses at most one edge belonging to a particular agent. While redundant, some of them are binding in an optimal solution. This degeneracy in the primal implies multiple optimal dual solutions. Let P^* be the shortest path (and without loss of generality, assume that it is unique).

We claim that among all optimal dual solutions, there is one where for each $r \in A$, the dual variable associated with the constraint

$$\sum_{e \in E^r} x_e \leq 1$$

is agent r 's marginal product.

To demonstrate this, let $q \in A$ be an agent who owns an edge in the shortest s - t path. To determine q 's marginal product, we need to compute the length of the shortest s - t path that does not use any edge in E^q . Call this path P^{-q} . Hence agent q 's marginal product is $L(P^{-q}) - L(P^*)$.

Let

$$\begin{aligned} Z(\mu) = \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & Nx = b \\ & x_e \geq 0 \quad \forall e \in E \\ & \sum_{e \in E^r} x_e \leq 1 \quad \forall r \in A \setminus q \\ & \sum_{e \in E^q} x_e \leq \mu \end{aligned}$$

Now, $Z(t)$ is piecewise linear in μ . It is easy to see that $\mu = 1$ is a breakpoint. For $\mu > 1$, the dual variable must be zero.

For $\mu < 1$, consider $\mu = 1 - \theta$ for $0 < \theta < 1$. The optimal solution involves a possibly fractional flow of one unit from s to t . By path-flow decomposition we can write this flow as a sum of flows along s - t paths. It is then easy to see that $1 - \theta$ units must flow along P^* (the shortest path for $\mu = 1$), and the remaining θ units can flow along path P^{-q} . The remaining θ units cannot flow on any other path that intersects E^q because this would violate the last constraint.

Hence,

$$Z(\mu) = Z(1) - \theta \left[\sum_{e \in P^*} c_e - \sum_{e \in P^{-q}} c_e \right] = Z(1) + \theta [L(P^{-q}) - L(P^*)].$$

So the change in objective function value is exactly θ times agent q 's marginal product, demonstrating the claim.

To write down the dual to the shortest path formulation, we associate a variable ν_i with each row of $Nx = b$ and λ_r with each constraint $\sum_{e \in E^r} x_e \leq 1$. The dual is

$$\begin{aligned} \max \quad & \nu_t - \nu_s - \sum_{r \in A} \lambda_r \\ \text{s.t.} \quad & \nu_j - \nu_i - \lambda_r \leq c_{ij} \quad \forall (i, j) \in E^r \\ & \lambda_r \geq 0 \quad \forall r \in A \end{aligned}$$

We prove that when agents are substitutes, there is an optimal solution to the dual where $\lambda_r = L(P^{-r}) - L(P^*)$ for all $r \in A$. First, observe that for any choice of $\lambda \geq 0$ we can always find ν to satisfy the dual constraints. In fact, for each choice of λ , let $\nu(\lambda)$ be the optimal solution to:

$$\begin{aligned} \max \quad & \nu_t - \nu_s \\ \text{s.t.} \quad & \nu_j - \nu_i \leq c_{ij} + \lambda_r \quad \forall (i, j) \in E^r \end{aligned}$$

From now on, fix $\lambda_r = L(P^{-r}) - L(P^*)$ for all $r \in A$.

For any s - t path P , let A_P denote the set of agents whose edges are traversed in the path P . Then, for any path P , we have, by summing up the dual constraints $\nu_j - \nu_i - \lambda_r \leq c_{ij}$ for every edge on this path:

$$\nu_t(\lambda) - \nu_s(\lambda) \leq L(P) + \sum_{r \in A_P} \lambda_r.$$

In fact, because the linear program for $\nu(\lambda)$ corresponds to the dual of a shortest path problem with costs $c'_{ij} = c_{ij} + \lambda_r$ for $(i, j) \in E^r$, we know that there is a shortest path realizing $\nu_t(\lambda) - \nu_s(\lambda)$. Hence, we conclude that

$$\nu_t(\lambda) - \nu_s(\lambda) = \min_P \{L(P) + \sum_{r \in A_P} \lambda_r\}.$$

Weak duality implies that

$$L(P^*) \geq \nu_t(\lambda) - \nu_s(\lambda) - \sum_{r \in A} \lambda_r = \nu_t(\lambda) - \nu_s(\lambda) - \sum_{r \in A_{P^*}} \lambda_r.$$

Thus

$$L(P^*) \geq \min_P \{L(P) + \sum_{r \in A_P} \lambda_r\} - \sum_{r \in A_{P^*}} \lambda_r.$$

Let $S = A \setminus A_{P^*}$.

$$\min_P \{L(P) + \sum_{r \in A_P} \lambda_r\} = L(P^{-S}) + \sum_{r \in A_{P-S}} \lambda_r.$$

Then

$$L(P^*) \geq L(P^{-S}) + \sum_{r \in A_{P-S}} \lambda_r - \sum_{r \in A_{P^*}} \lambda_r = L(P^{-S}) - \sum_{r \in S} \lambda_r \geq L(P^*)$$

where the last inequality follows from the substitutes condition. Hence, $(\nu(\lambda), \lambda)$ constitutes an optimal dual solution.

7 Homogeneous goods

We now turn to the simplest case of K identical indivisible units of a product. We will consider two cases. In Section 7.1, each bidder wants at most one unit. In Section 7.2, each bidder may want multiple units.

7.1 Unit demand

Let v_j denote bidder j 's value for one unit, and suppose it to be integral. Given such valuations, we can find the efficient allocation of the K units by solving a linear program.

Let $x_j = 1$ if an object is assigned to bidder j , and 0 otherwise.

$$\begin{aligned} V(N) &= \max \sum_{j \in N} v_j x_j \\ \text{s.t. } &\sum_{j \in N} x_j \leq K \\ &0 \leq x_j \leq 1 \quad \forall j \in N \end{aligned}$$

Notice that this LP has all integral extreme points, so the optimal solution/allocation is integral. This formulation is clearly a special case of (P4).

Without loss of generality, assume that values are in decreasing order so that $v_j \geq v_{j+1}$. The optimal solution is $x_j = 1$ for $1 \leq j \leq K$ and $x_j = 0$ for $j \geq K + 1$.

Let y be the dual variable associated with the first constraint above. The dual to this LP is:

$$\begin{aligned} \min &\sum_{j \in N} \mu_j + Ky \\ \text{s.t. } &y + \mu_j \geq v_j \quad \forall j \in N \\ &y, \mu_j \geq 0 \end{aligned}$$

Complementary slackness requires that $y + \mu_j = v_j$ for $j = 1, \dots, K$. Given this, it is easy to see that one set of optimal dual variables is $y = v_{K+1}$, $\mu_j = v_j - y$ for $1 \leq j \leq K$, and $\mu_j = 0$ for $j \geq K + 1$.

Now consider the following auction. Bidders submit bids. Compute the optimal solution to the primal problem above, and make the assignment implied by it. Charge each bidder who gets an object the value of y found above.

The value of y corresponds exactly to the payment prescribed by the Vickrey auction. To see why, observe that in this case, $V(N) = \sum_{i=1}^K v_i$. Furthermore, $V(N \setminus j) = \sum_{i=1}^K v_i - v_j + v_{K+1} = V(N) - v_j + v_{K+1}$ for $j \leq K$, and $V(N \setminus j) = V(N)$ otherwise. Hence,

$$V(N) - V(N \setminus j) = [v_j - v_{K+1}]^+ = [v_j - y]^+.$$

In the dual, we can interpret each binding constraint $y + \mu_j = v_j$ as saying that if bidder j gets an object, then her surplus is $\mu_j = v_j - y$. The particular

dual solution we have chosen is the one that maximizes the total surplus to the bidders:

$$\begin{aligned}
& \min y \\
& \text{s.t. } \sum_{j \in N} \mu_j + Ky = z_{\text{lp}} \\
& \quad y + \mu_j \geq v_j \quad \forall j \in N \\
& \quad y, \mu_j \geq 0
\end{aligned}$$

The ascending auction that generates the same outcome is well known, and it is an equilibrium for everyone to bid sincerely in each round of that auction. It will be instructive to show how it arises naturally from a primal-dual algorithm for the problem.

7.1.1 The primal-dual algorithm for this setting

Here we show how the ascending Vickrey auction arises naturally from a primal-dual algorithm (see Algorithm 1). Most of the difficulty in describing the algorithm stems from the possibility of ties, i.e., different agents having identical valuations.

Without loss of generality, we assume that each bidder has a strictly positive value for the good. The dual problem is:

$$\begin{aligned}
& \min \sum_{j \in N} \mu_j + Ky \\
& \text{s.t. } y + \mu_j \geq v_j \quad \forall j \in N \\
& \quad y, \mu_j \geq 0
\end{aligned}$$

Given a value \hat{y} (initially chosen to equal 0) for y , a feasible dual solution can be constructed by setting $\hat{\mu}_j = [v_j - \hat{y}]^+$. Let $I = \{j : \hat{\mu}_j = 0\}$ and $J = \{j : \hat{\mu}_j + \hat{y} = v_j\}$. The elements of J correspond to bidders with nonnegative surplus. Note that $J \cup I = N$.

Bidders in $J \setminus I$ have strictly positive surplus, i.e., they strictly want to buy an object at the current price \hat{y} . Bidders in $J \cap I$ are indifferent between buying at the current price \hat{y} (with zero surplus) and not buying. Bidders in $I \setminus J$ value the items at less than \hat{y} ; they strictly prefer not to buy. In the auction environment, we would not know the sets I and J , and would have to rely on the reports of the bidders.

If this dual solution were in fact optimal, we know, by complementary slackness, that for every primal optimal solution \bar{x} ,

1. $j \in I \Rightarrow \bar{x}_j \leq 1$,
2. $j \notin I \Rightarrow \bar{x}_j = 1$,
3. $j \in J \Rightarrow \bar{x}_j \geq 0$,
4. $j \notin J \Rightarrow \bar{x}_j = 0$ and
5. $y > 0 \Rightarrow \sum_{j \in N} \bar{x}_j = K$.

A primal solution satisfying these conditions exists if and only if both $|N \setminus I| \leq K$ and $|J| \geq K$. The restricted primal simply formulates, as a linear program, the problem of deciding if these inequalities holds. To ensure that the restricted primal is feasible, we introduce artificial variables x_j^a . The restricted primal (**RP**) is

$$\begin{aligned}
& \max -x_0^a - \sum_{j \in N \setminus I} x_j^a \\
& \text{s.t. } \sum_{j \in N} \bar{x}_j + x_0^a = K \\
& \quad \bar{x}_j + x_j^a = 1 \quad \forall j \in N \setminus I \\
& \quad 0 \leq \bar{x}_j \quad \forall j \in N \setminus I \\
& \quad \bar{x}_j = 0 \quad \forall j \in N \setminus J \\
& \quad \bar{x}_j \leq 1 \quad \forall j \in I \cap J \\
& \quad 0 \leq \bar{x}_j \quad \forall j \in I \cap J \\
& \quad 0 \leq x_j^a \quad \forall j \in N \cup \{0\}
\end{aligned}$$

For $\hat{y} = 0$ the objective function would be $\max -\sum_{j \in N \setminus I} x_j^a - \sum_{j \in N \setminus J} x_j^a$.

If both $|N \setminus I| \leq K$ and $|J| \geq K$, then the optimal objective function value of (RP) is zero and we have found a primal solution that satisfies all complementary slackness conditions.

We show that the case $|J| < K$ (where there is too little demand) can be ignored, because the auction will end before it occurs. The first time (round) at which $|J| < K$ occurs, there must have been “overdemand” in the previous round, i.e., $K < |N \setminus I'|$ (assuming non-negative values v_j , where the prime signifies “previous round”) otherwise the auction would have ended. Since

$J' \cup I' = N$, this implies $|J' \setminus I'| > |J|$, which is a contradiction, since, from one round to the next, we must have $J' \setminus I' \subset J$. In other words, from one round to the next, an index $j \in N$ can move from $J \setminus I$ to $I \cap J$ or from $I \cap J$ to $I \setminus J$; moving from $J \setminus I$ to $I \setminus J$ requires at least two rounds.

If $K - |N \setminus I| < 0$, then the optimal objective function value is $K - |N \setminus I| < 0$, so there is no primal solution that is complementary to the current dual solution. To determine how to update the dual variables we consider the dual to the restricted primal (**DRP**):

$$\begin{aligned}
\min \quad & \sum_{j \in J} \mu_j + Ky \\
\text{s.t.} \quad & y + \mu_j \geq 0 \quad \forall j \in J \\
& y \geq 0 \\
& \mu_j \geq 0 \quad \forall j \in (N \setminus I) \cup (N \setminus J) \\
& \mu_j = 0 \quad \forall j \in N \setminus J \\
& \mu_j \geq -1 \quad \forall j \in N \setminus I \\
& \mu_j \geq 0 \quad \forall j \in I \cap J \\
& y \geq -1 \text{ (or } y \geq 0 \text{ if } \hat{y} = 0)
\end{aligned}$$

(The symbol \geq is used to mean that the variable is unrestricted.) This problem can be simplified to

$$\begin{aligned}
\min \quad & \sum_{j \in J} \mu_j + Ky \\
\text{s.t.} \quad & y + \mu_j \geq 0 \quad \forall j \in J \\
& \mu_j \geq -1 \quad \forall j \in N \setminus I \\
& \mu_j \geq 0 \quad \forall j \in I \cap J \\
& y \geq -1 \text{ (or } y \geq 0 \text{ if } \hat{y} = 0)
\end{aligned}$$

Denote the optimum to this problem with $(\bar{y}, \bar{\mu}^T)^T$. It is obvious, because $K < |N \setminus I| \leq |J|$, that $\bar{y} = 1$, $\bar{\mu}_j = -1$ for $j \in N \setminus I$ is the optimum (since it is feasible and its value of $K - |N \setminus I|$ matches the value of the (RP)). This corresponds to raising the price by one, decreasing the surplus of bidders with strictly positive surplus by one, and removing the bidders in $I \cap J$ from the auction.

We need to update the dual prices to: $(\hat{y}, \hat{\mu})^T \leftarrow (\hat{y}, \hat{\mu})^T + \theta(\bar{y}, \bar{\mu})^T$ for suitable $\theta > 0$. Notice that, as the optimum of (DRP) is strictly negative,

this corresponds to raising the price by θ and forcing the bidders with strictly positive surplus to decrease their surplus by θ .

Clearly, θ has to be chosen to maintain feasibility of $(\hat{y}, \hat{\mu})^T + \theta(\bar{y}, \bar{\mu})^T$ in the dual. For $j \in J$ the constraints $y + \mu_j \geq v_j$ are certainly fulfilled.

For $j \in N \setminus J$ those constraints require $\hat{y} + \theta\bar{y} + \hat{\mu}_j + \theta\bar{\mu}_j \geq v_j$. If $\bar{y} + \bar{\mu}_j \geq 0$, the constraint is fulfilled for all $\theta > 0$. The case that $\bar{y} + \bar{\mu}_j < 0$ is by our choice of the dual solution impossible.

Similarly, we have to ensure the nonnegativity of the μ_j 's. Again, this is a non-issue for $i \in I$ (because they are unchanged). For $i \in N \setminus I$ the condition is $\hat{\mu}_j + \theta\bar{\mu}_j \geq 0$ and this reduces to $\theta \leq -\hat{\mu}_j/\bar{\mu}_j$. So we could choose (if we know the bidder's μ_j 's) a step size by

$$\theta = \min_{j \in N \setminus I} \frac{-\hat{\mu}_j}{\bar{\mu}_j}$$

By examination, we see that (for integral v_j), $\theta \geq 1$ unless the procedure is over.

7.2 Homogeneous goods, multi-unit demand

As before, there are K indivisible units to be auctioned off. In this case, though, bidders may want more than one object. Let v_i^t be the value that bidder i assigns to their t^{th} unit (conditional on having already received $t - 1$ units). We assume that bidders have diminishing marginal values for units: $v_i^t \geq v_i^{t+1}$ for all $i \in N$ and $1 \leq t \leq K$.

7.2.1 A natural formulation

A natural formulation for the problem defines $x_{i,t} = 1$ if bidder i receives a t^{th} object.

$$\begin{aligned} \max \quad & \sum_{i \in N} \sum_{t=1}^K v_i^t x_{i,t} \\ \text{s.t.} \quad & \sum_{i \in N} \sum_{t=1}^K x_{i,t} \leq K \\ & x_{i,t+1} \leq x_{i,t} \quad \forall i \in N, \forall t < K \\ & x_{i,1} \leq 1 \quad \forall i \in N \\ & x_{i,K} \geq 0 \quad \forall i \in N \end{aligned}$$

The dual variable on $x_{i,1}$ gives the surplus of agent i .

To the resource-constraint we associate a dual variable y , to each inequality $x_{i,t+1} - x_{i,t} \leq 0$ for $t < K$ a variable $\mu_{i,t+1}$, and with each inequality $x_{i,t} \leq 1$ a variable $\lambda_{i,t}$.

Now we can write down the dual formulation.

$$\begin{aligned}
\min \quad & Ky + \sum_{i \in N} \sum_{t=1}^K \lambda_{i,t} \\
\text{s.t.} \quad & \lambda_{i,1} - \mu_{i,2} + y \geq v_i^1 \quad \forall i \in N \\
& \lambda_{i,t} - \mu_{i,t+1} + \mu_{i,t} + y \geq v_i^t \quad \forall i \in N, \forall t < K \\
& \lambda_{i,K} + \mu_{i,K} + y \geq v_i^K \quad \forall i \in N \\
& y \geq 0 \\
& \lambda_{i,t} \geq 0 \quad \forall i \in N, \forall t \leq K \\
& \mu_{i,t} \geq 0 \quad \forall i \in N, \forall t > 1
\end{aligned}$$

The primal-dual algorithm for this formulation does not, however, implement the Vickrey outcome. In fact, it leads to the “uniform price” auction, in which all winning bidders pay the same price per unit: the highest losing bid.

7.2.2 Ausubel’s ascending auction

For the case of homogeneous goods and multi-unit demand, Ausubel (1997) was the first to propose an ascending auction to implement the Vickrey outcome. Bikhchandani and Ostroy (2000b) were the first to show that Ausubel’s auction was an instance of the primal-dual algorithm for formulation (P3). We have mentioned elsewhere in this paper that Ausubel’s auction can be interpreted as a primal-dual algorithm for a matroid optimization problem. Here we outline how Ausubel’s auction can be derived from a primal-dual algorithm for the longest path problem in an acyclic network. To do this, we refine a transformation from knapsack problems to longest paths problems; see Ahuja et al. (1997).

Let $V_r^t \equiv \sum_{\tau=1}^t v_r^\tau$ be agent r ’s total value for obtaining t units. We will formulate the problem of finding the efficient allocation as a longest path problem in an acyclic directed graph G .

Each vertex in G is identified by an ordered pair (r, s) of integers. A vertex (r, s) represents the state in which bidders 1 through $r - 1$ have together

acquired s units. In addition, vertex $(1, 0)$ is the source and vertex $(N + 1, K)$ is the sink. Therefore the set of vertices is

$$\{(1, 0)\} \cup \left(\bigcup_{r=2}^N \bigcup_{s=0}^K \{(r, s)\} \right) \cup \{(N + 1, K)\}.$$

Edge $((r, s), (r + 1, s + t))$ belongs to bidder r when $t > 0$, and belongs to the auctioneer for $t = 0$.

For each vertex (r, s) , there is an edge directed to each vertex $(r + 1, s + t)$ such that $0 \leq t \leq K - s$. Such an edge corresponds to bidder r obtaining t units. The length of that edge is V_r^t (where $V_r^0 \equiv 0$).

Each source-sink path in G corresponds to a feasible assignment of K objects among the bidders. Since G is acyclic, the longest path problem is well defined on it. Also each path from the source to the sink uses at most one edge belonging to any given agent. Since agents are substitutes in this environment, it follows from the shortest path section of this paper that the dual variables that correspond to the constraint that a bidder can own only one edge on a path is his marginal product (in a suitably chosen dual solution).

The problem of finding the longest path can be formulated as

$$\begin{aligned} \max \quad & \sum_{r=1}^{N-1} \sum_{t=0}^K \sum_{s=0}^{K-t} V_r^t x_{(r,s),(r+1,s+t)} + \sum_{s=0}^K V_N^{K-s} x_{(N,s),(N+1,K)} \\ \text{s.t.} \quad & \sum_{s=0}^K x_{(N,s),(N+1,K)} = 1 \\ & \sum_{t=0}^s x_{(r-1,s-t),(r,s)} - \sum_{t=0}^{K-s} x_{(r,s),(r+1,s+t)} = 0 \quad \forall s, \forall r > 2 \\ & x_{(1,0),(2,s)} - \sum_{t=0}^{K-s} x_{(2,s),(3,s+t)} = 0 \quad \forall s \\ & \sum_{s=0}^K \sum_{t=1}^{K-s} x_{(r,s),(r+1,s+t)} \leq 1 \quad \forall 1 < r \leq N \\ & \sum_{t=1}^K x_{(1,0),(2,t)} \leq 1 \\ & x_e \geq 0 \quad \forall e \end{aligned}$$

The first constraint requires that exactly one edge is chosen to enter the sink $(N + 1, K)$. The second class of constraints requires that in each intermediate node, the number of entering arcs equals the number of leaving arcs; the third constraint requires the same for vertices whose incident arcs come from the source. The fourth class limits—to at most one—the number of edges giving bidder $r > 1$ a positive number of units, and the fifth is the same constraint for bidder 1. These last constraints are redundant.

To write down the dual, we associate with each constraint of type

- $\sum_{s=0}^K x_{(N,s),(N+1,K)} = 1$, an unrestricted variable $\nu_{N+1,K}$,
- $\sum_{t=0}^s x_{(r-1,s-t),(r,s)} - \sum_{t=0}^{K-s} x_{(r,s),(r+1,s+t)} = 0$, $\forall s, \forall r > 2$, an unrestricted variable $\nu_{r,s}$,
- $x_{(1,0),(2,s)} - \sum_{t=0}^{K-s} x_{(2,s),(3,s+t)} = 0$, $\forall s$, an unrestricted variable $\nu_{2,s}$,
- $\sum_{s=0}^K \sum_{t=1}^{K-s} x_{(r,s),(r+1,s+t)} \leq 1$, $\forall 1 < r \leq N$, a variable $\lambda_r \geq 0$,
- $\sum_{t=1}^K x_{(1,0),(2,t)} \leq 1$, a variable $\lambda_1 \geq 0$.

The dual is

$$\begin{aligned}
& \min \nu_{N+1,K} + \sum_{r=1}^N \lambda_r \\
& \text{s.t. } \nu_{r+1,s+t} - \nu_{r,s} + \lambda_r \geq V_r^t \quad \forall t \geq 1, \forall (r,s) \\
& \quad \nu_{r+1,s} - \nu_{r,s} \geq 0 \quad \forall (r,s) \\
& \quad \lambda_r \geq 0 \quad \forall r \in N \\
& \quad \nu_{r,s} \geq 0 \quad \forall (r,s) \\
& \quad \nu_{1,0} = 0
\end{aligned}$$

A feature of the construction to be noted is that a bidder $r > 1$ owns many different edges that correspond to consuming t units of the good. Bidder 1, however, has exactly one edge that corresponds to consuming t units. We want to equally price all edges that correspond to bidder $r > 1$ consumes t units. To do this, we later ensure that for $r > 1$, $\nu_{r+1,s+t} - \nu_{r,s} = \nu_{r+1,s+t+1} - \nu_{r,s+1}$.

For an initial feasible solution, set $\hat{\nu} = 0$, corresponding to initial prices on all items of zero. Let $\hat{\lambda}_r = \max_{s,t} [V_r^t - \hat{\nu}_{r+1,s+t} + \hat{\nu}_{r,s}]^+$. Let $I = \{r : \hat{\lambda}_r = 0\}$

be the set of agents without positive surplus, and let bidder r 's "demand set" be denoted

$$J^r = \{((r, s), (r + 1, s + t)) : t \geq 1 \wedge \nu_{r+1, s+t} - \nu_{r, s} + \lambda_r = V_r^t\}.$$

This is the set of edges that maximize the surplus of bidder r . Denote the set of edges that bidder r is "willing" to consume as

$$L^r = \{((r, s), (r + 1, s + t)) : t \geq 1 \wedge \nu_{r+1, s+t} - \nu_{r, s} \leq V_r^t\}.$$

These edges leave their owner with non-negative surplus.

If $L^r = \emptyset$ set $\alpha_r = 0$. Otherwise set $\alpha_r = \max\{t : ((r, s), (r + 1, s + t)) \in J^r\}$. The value α_r is the greatest number of items that bidder r wants to buy at current prices. Similarly, set $\beta_r = 0$ if $L^r = \emptyset$ and otherwise $\beta_r = \min\{t : ((r, s), (r + 1, s + t)) \in J^r\}$. The value β_r is the smallest number of items that bidder r wants to buy at current prices. If $\sum_r \beta_r \leq K \leq \sum_r \alpha_r$ we have the optimal solution and we are done.

Using the current dual values, determine for each bidder r the number $\gamma_r = [K - \sum_{p \neq r} \beta_p]^+$. Notice that γ_r is the number of units that bidder r has to buy so that all items are sold. Ausubel calls this the number of items *clinched* by bidder r . Let $\gamma = K - \sum_{r=1}^n \gamma_r$ be the total number of nonclinched items.

Let E^r denote the set of edges owned by agent r and E^0 be the edges that the auctioneer holds. If the current dual solution is optimal, we know, by complementary slackness, for every primal optimal solution \bar{x} ,

1. $r \in I \Rightarrow \sum_{e \in E^r} \bar{x}_e \leq 1$,
2. $r \notin I \Rightarrow \sum_{e \in E^r} \bar{x}_e = 1$,
3. $e \in J^r \Rightarrow \bar{x}_e \geq 0$, and
4. $e \in E^r \setminus J^r \Rightarrow \bar{x}_e = 0$.

The corresponding restricted primal (**RP**) will be:

$$\begin{aligned}
& \max -(y_{N+1,K}^+ + y_{N+1,K}^-) - \gamma \sum_{r=1}^N \sum_{s=0}^K (y_{rs}^+ + y_{rs}^-) \\
& \text{s.t. } \sum_{s=0}^K x_{(N,s),(N+1,K)} + (y_{N+1,K}^+ - y_{N+1,K}^-) = 1 \\
& \sum_{t=0}^s x_{(r-1,s-t),(r,s)} - \sum_{t=0}^{K-s} x_{(r,s),(r+1,s+t)} + (y_{rs}^+ - y_{rs}^-) = 0 \quad \forall s, \forall r > 2 \\
& x_{(1,0),(2,s)} - \sum_{t=0}^{K-s} x_{2,s,3,s+t} + (y_{1s}^+ - y_{1s}^-) = 0 \quad \forall s \\
& x_e \geq 0 \quad \forall e \in E \\
& \sum_{e \in E^r} x_e \leq 1 \quad \forall r \in I \\
& \sum_{e \in E^r} x_e = 1 \quad \forall r \notin I \\
& x_e \geq 0 \quad \forall e \in J^r \\
& x_e = 0 \quad \forall e \in E^r \setminus J^r \\
& y \geq 0
\end{aligned}$$

Denote by $G(J)$ a graph with the same set of vertices as G but only those edges that belong to J . If an arc $(r, s), (r + 1, s + t)$ is present in $G(J)$ this means that under current prices, it is a surplus-maximizing arc for agent r . That is at current prices, agent r maximizes his surplus by buying t units.

Note that (RP) has value zero if and only if $G(J)$ contains a source-sink path such that each agent from $N \setminus I$ owns one edge on this path. If there is no such path from source to sink in $G(J)$, then we have two possible causes. First, it could be that not even $G(L) \supseteq G(J)$ itself contains a source-sink path. Fortunately, we can avoid this case: In the first iteration, prices are 0 and $G(L)$ is connected. It suffices to keep $G(L)$ connected in subsequent iterations, and we do that below.

Second, it could be that $G(L)$ contains a path, but no such path contains, for each bidder $r \in N \setminus I$, an edge in J^r . In this case, agents demand too much at current prices. To deal with this, we raise item prices, to make large quantities (that is, long arcs) less attractive to agents so that they demand

smaller quantities, which corresponds to demanding arcs $(r, s), (r + 1, s + t)$ with smaller t 's.

Consider now **(DRP)**, the dual of the restricted primal.

$$\begin{aligned}
& \min \nu_{N+1,K} + \sum_{r=1}^N \lambda_r \\
& \text{s.t. } \nu_{r+1,s+t} - \nu_{r,s} + \lambda_r \geq 0 \quad \forall ((r, s), (r + 1, s + t)) \in J^r \\
& \quad \nu_{r+1,s} - \nu_{r,s} \geq 0 \quad \forall (r, s) \\
& \quad \lambda_r \geq 0 \quad \forall r \in I \\
& \quad \lambda_r \geq 0 \quad \forall r \notin I \\
& \quad \nu_{1,0} = 0 \\
& \quad \nu_{r,s} \geq -\gamma \quad \forall (r, s) \\
& \quad \nu_{r,s} \leq \gamma \quad \forall (r, s)
\end{aligned}$$

We need to find a feasible solution $(\bar{\nu}, \bar{\lambda})$ that results in a negative objective function value. As long as no bidder has clinched anything, we uniformly raise item prices by $\theta > 0$. As soon as bidder r clinches some items, the price on these items is fixed for bidder r . Choosing the direction in which the dual variables are to be adjusted takes care of this: $\bar{\nu}_{r,s} = [s - \sum_{q=1}^{r-1} \gamma_q]^+$. By computing $\bar{\lambda}$ accordingly, one can show that we have a negative objective value.

Consider (in the case of no truncation) the difference $\bar{\nu}_{r+1,s+t} - \bar{\nu}_{r,s} = t - \gamma_r$. We raise the price for bundles that bidder r obtains only by the number of non-clinched items in it. This is analogous to Ausubel's ascending auction.

Choosing this dual direction permits us to avoid the uniform prices and steer towards the efficient outcome. From the substitutes property, we know that the final values of the dual variables will give to each agent a payoff that equals his marginal product.

8 Summary

This paper surveys the connections between sealed bid Vickrey auctions and duality in linear programming. By example, we have shown how this relation can be exploited to produce iterative auctions that implement the Vickrey outcome in various scenarios. The approach can be summarized by the following steps.

1. Verify that the *agents are substitutes* condition holds.
2. Formulate a linear program where appropriate dual variables correspond to the marginal products of the agents. Under the substitutes condition, an optimal dual solution exists in which, *simultaneously*, each appropriate variable takes on the value of the corresponding agent's marginal product.
3. Construct a primal-dual algorithm for the linear program. The algorithm must choose an improving direction in the dual that will cause the algorithm to terminate in the dual solution described above. This is the one that, among all optimal dual solutions, is the one that maximizes the combined surplus of the bidders.

If the substitutes condition does not hold, we believe it is unlikely that an iterative auction (in which bidding sincerely is an equilibrium) yielding the Vickrey outcome exists. This is not a formal statement, since the term “iterative auction” requires a precise definition. So, an interesting line of inquiry would be to formalize this last intuition in a precise way.

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