

Errors: What they are, and how to deal with them

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Synopsis

- 1) Introduction
- 2) Rules for quoting errors
- 3) Combining errors
- 4) Statistical analysis of random errors
- 5) Use of graphs in experimental physics

Text book: this lecture series is self-contained, but enthusiasts might like to look at "An Introduction to Error Analysis" by John R Taylor (Oxford University Press, 1982) ISBN 0-935702-10-5, a copy of which is in the laboratory.

1) Introduction

What is an error?

All measurements in science have an uncertainty (or “error”) associated with them:

- Errors come from the limitations of measuring equipment, or
- in extreme cases, intrinsic uncertainties built in to quantum mechanics
- Errors can be minimised, by choosing an appropriate method of measurement, but they cannot be eliminated.

Notice that errors are not “mistakes” or “blunders”.

Two distinct types:

Random:

These are errors which cause a measurement to be as often larger than the true value, as it is smaller. Taking the average of several measurements of the same quantity reduces this type of error (as we shall see later).

Example: the reading error in measuring a length with a metre rule.

Systematic:

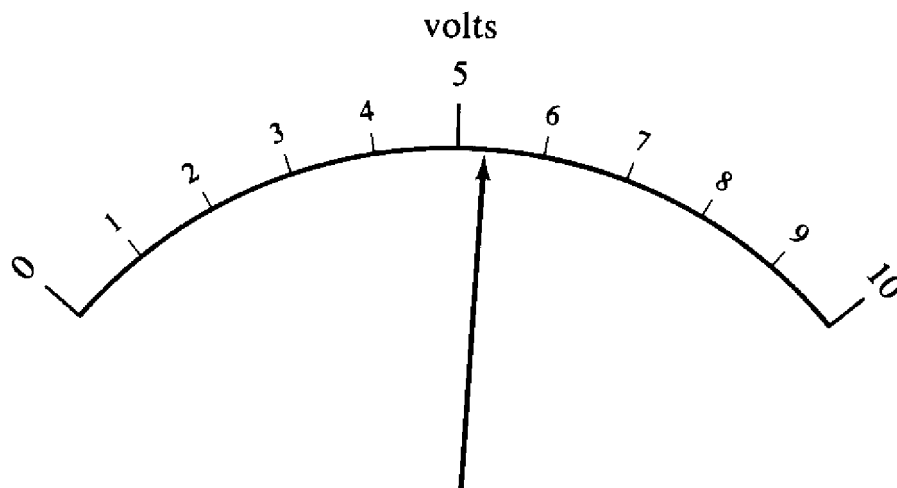
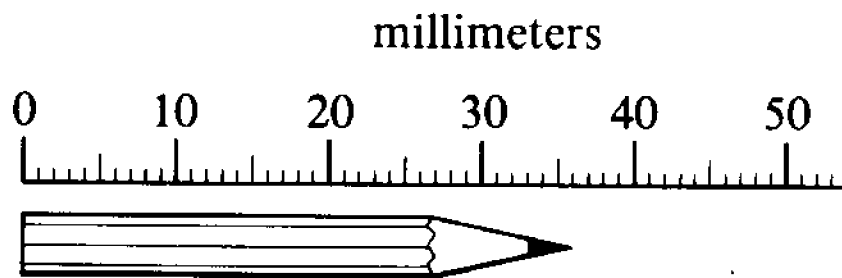
Systematic errors cause the measurement always to differ from the truth in the same way (i.e. the measurement is always larger, or always smaller, than the truth).

Example: the error caused by using a poorly calibrated instrument.

It is random errors that we will discuss most.

Estimating errors

e.g. measurements involving marked scales:



The process of estimating positions between markings is called “interpolation”

Repeatable measurements

There is a sure-fire way of estimating errors when a measurement can be repeated several times. e.g. measuring the period of a pendulum, with a stopwatch.

Example:

Four measurements are taken of the period of a pendulum. They are:

2.3 sec, 2.4 sec, 2.5 sec, 2.4 sec

In this case:

best estimate (average) = 2.4 sec

probable range 2.3 to 2.5 sec

We will come back to the statistics of repeated measurements later.

2) Rules for quoting errors

The result of a measurement of the quantity x should be quoted as:

(measured value of x) \pm $x_{\text{best}} \pm \Delta x$

where x_{best} is the best estimate of x , and the probable range is from $x_{\text{best}} - \Delta x$ to $x_{\text{best}} + \Delta x$.

We will be more specific about what is meant by “probable range” later.

Significant figures

Here are two examples of BAD PRACTICE:

(measured g) = 9.82 ~~10.02385~~ ms^{-2}

measured speed = 6051.78 ~~130~~ ms^{-1}

The rules

- Experimental uncertainties should normally be rounded to one significant figure.
- The last significant figure of the answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

For example the answer 92.81 with error of 0.3 should be stated:

92.8 ~~10.3~~

If the error were 3, then the same answer should be stated as:

93 ~~10~~

If the error were 30 then it should be stated as:

One exception: if the leading digit of the error is small (i.e. 1 or 2) then it is acceptable to retain one extra figure in the best estimate.

e.g. “measured length = 7.64 cm” is as acceptable as “28.4 m”

Quoting measured values involving exponents

It is conventional (and sensible) to do as in the following example:

Suppose the best estimate of the measured mass of an electron is 9.11×10^{-31} g with an error of $\pm 5 \times 10^{-33}$ g. Then the result should be written:

This makes it easy to see how large the error is compared with the best estimate.

Convention:

In science, the statement “ $x = 1.27$ ” (without any statement of error) is taken to mean $1.265 \leq x \leq 1.275$.

To avoid any confusion, best practice is always to state an error, with every measurement.

Absolute and fractional errors

The errors we have quoted so far have been absolute ones (they have had the same units as the quantity itself). It is often helpful to consider errors as a fraction (or percentage) of the best estimate. In other words, in the form:

$$\text{fractional error in } x = \frac{|x|}{|x_{\text{best}}|}$$

(the modulus simply ensures that the error is positive - this is conventional)

The fractional error is (sort of) a measure of the quality of a result.

One could then write the result of a measurement as:

$$\text{measured value of } x = x_{\text{best}} \pm \frac{|x|}{|x_{\text{best}}|}$$

NOTE: it is not conventional to quote errors like this.

Comparison of measured and accepted values

It is often the objective of an experiment to measure some quantity and then compare it with the accepted value (e.g. a measurement of g , or of the speed of light).

Example:

In an experiment to measure the speed of sound in air, a student comes up with the result:

$$\text{Measured speed} = 329 \pm 3 \text{ m s}^{-1}$$

The accepted value is 331 m s^{-1} . Consistent or inconsistent?

If the student had measured the speed to be $345 \pm 3 \text{ m s}^{-1}$ then the conclusion is...?

Note: accepted values can also have errors.

3) Combining errors

In this section we discuss the way errors “propagate” when one combines individual measurements in various ways in order to obtain the desired result. We will start off by looking at a few special cases (very common ones). We will then refine the rules, and finally look at the general rule for combining errors.

Error in a sum or difference

A quantity q is related to two measured quantities x and y by

$$q = x \pm y$$

The measured value of x is $x_{\text{best}} \pm \Delta x$, and that of y is $y_{\text{best}} \pm \Delta y$.

What is the best estimate, and the error in q .

The highest probable value of q is:

$$x_{\text{best}} + \Delta x + y_{\text{best}} + \Delta y$$

and the lowest probable value is:

$$x_{\text{best}} - \Delta x + y_{\text{best}} - \Delta y$$

A similar argument (check for yourselves) shows that the error in the difference $x - y$ is also $\frac{\Delta x}{x} + \frac{\Delta y}{y}$.

Error in a product or quotient

This time the quantity q is related to two measured quantities x and y by:

$$q = xy$$

Writing x and y in terms of their fractional errors:

$$\text{measured value of } x = x_{\text{best}} \left(1 \pm \frac{\Delta x}{x_{\text{best}}} \right)$$

$$\text{and measured value of } y = y_{\text{best}} \left(1 \pm \frac{\Delta y}{y_{\text{best}}} \right)$$

so the largest probable value of q is:

$$x_{\text{best}} y_{\text{best}} \left(1 + \frac{\Delta x}{x_{\text{best}}} + \frac{\Delta y}{y_{\text{best}}} \right)$$

while the smallest probable value is:

$$x_{\text{best}} y_{\text{best}} \left(1 - \frac{\Delta x}{x_{\text{best}}} - \frac{\Delta y}{y_{\text{best}}} \right)$$

Just consider the two bracketed terms in the first of these:

$$\left(1 + \frac{\Delta x}{x_{\text{best}}} + \frac{\Delta y}{y_{\text{best}}} \right) = 1 + \frac{\Delta x}{x_{\text{best}}} + \frac{\Delta y}{y_{\text{best}}} + \frac{\Delta x}{x_{\text{best}}} \frac{\Delta y}{y_{\text{best}}}$$

The last term contains the product of two small numbers, and can be neglected in comparison with the other three terms. We therefore conclude that the largest probable value of q is:

$$x_{\text{best}} y_{\text{best}} \left(1 + \frac{\Delta x}{x_{\text{best}}} + \frac{\Delta y}{y_{\text{best}}} \right)$$

and by a similar argument, the smallest probable value is:

$$x_{\text{best}} y_{\text{best}} \left(1 - \frac{\Delta x}{x_{\text{best}}} - \frac{\Delta y}{y_{\text{best}}} \right)$$

For the case of $q = \frac{x}{y}$ exactly the same result for the error in q is obtained.

Summary

(These rules work for results that depend on any number of measured quantities, not just two)

The error in a power

Suppose $q = x^n$. Now x^n is just a product of x with itself (n times). So applying the rule for errors in a product, we obtain:

$$\frac{\Delta q}{|q_{\text{best}}|} = n \frac{\Delta x}{|x_{\text{best}}|}$$

Refinement of the rules for independent errors

Actually we have been a bit pessimistic by saying that we should add the errors: If x and y are independent quantities, the error in x is just as likely to (partially) cancel the error in y , as it is to add to it.

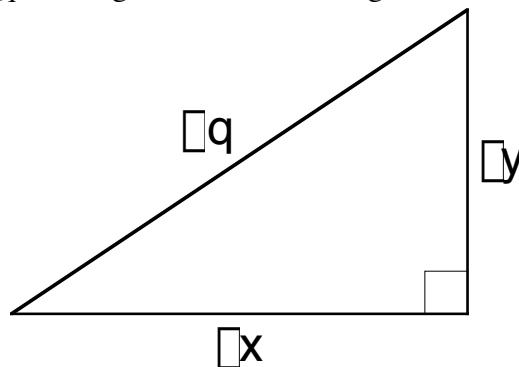
In fact the correct way of combining independent errors is “to add them in quadrature”. In other words, instead of saying (for the case of sums and differences):

$$\Delta q = \Delta x + \Delta y$$

we say:

$$\Delta q = \sqrt{\Delta x^2 + \Delta y^2} \quad (\text{This is what is meant by adding in quadrature})$$

If we think of Δx , Δy and Δq as being the sides of a triangle:



we can see that we have a right-angled triangle, and the rule for adding in quadrature is just Pythagoras' theorem.

Note:

- the error in the result, Δq is greater than the errors in either of the measurements, but
- it is always less than the sum of the errors in the measurements.

So the process of adding in quadrature has reduced the error, as we required.

It is possible to show rigorously that this is the correct procedure for combining errors.

Summary of the refined rules:

For sums and differences, the absolute error in the result is obtained by taking the root of the sum of the squares of the absolute errors in the original quantities.

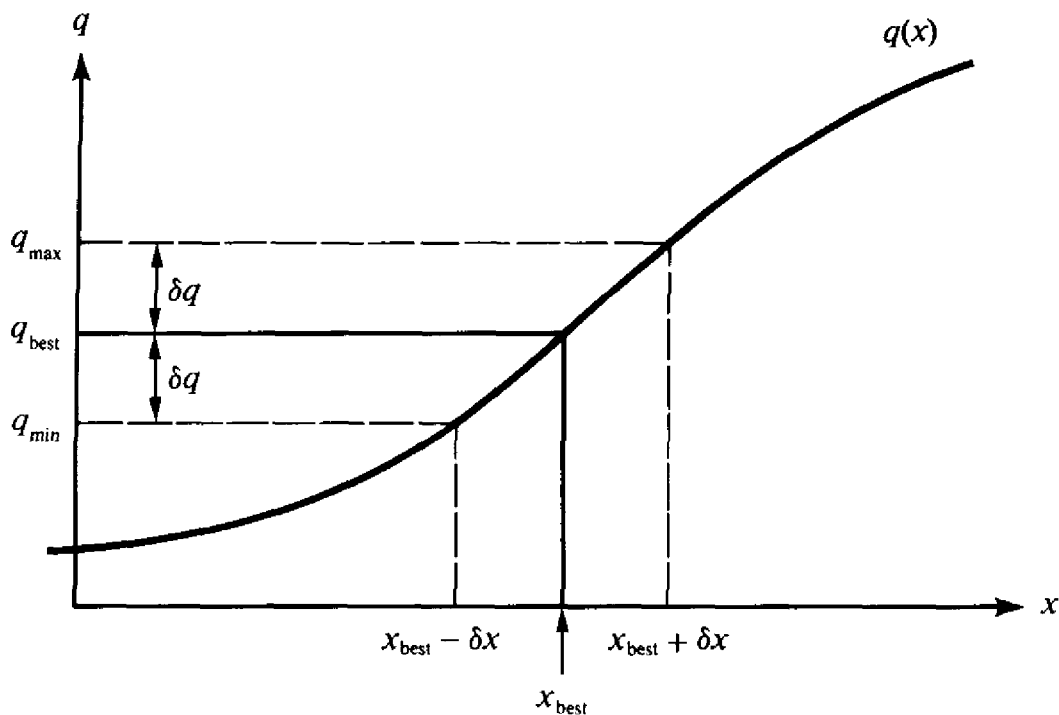
For products and quotients, the fractional error in the result is obtained taking the root of the sum of the squares of the fractional errors in the original quantities.

(These rules work for results that depend on any number of measured quantities, not just two)

NB: the rule for powers remains as above, because each of the 'x's in x^n are the same (and have the same error), so this is not a product of independent quantities.

The uncertainty in any function of one variable

The graph below shows the function $q(x)$ vs. x :



The dashed lines show how the error in the measured quantity x affects the error in q .

Now $\Delta q = q(x_{\text{best}} + \Delta x) - q(x_{\text{best}})$

But from calculus, we know that the definition of $\frac{dq}{dx}$ is:

$$\frac{dq}{dx} = \lim_{\Delta x \rightarrow 0} \frac{q(x + \Delta x) - q(x)}{\Delta x}$$

So for a sufficiently small error in the measured quantity x , we can say that the error in any function of one variable is given by:

$$\Delta q = \left| \frac{dq}{dx} \right| \Delta x$$

(where the modulus sign ensures that Δq is positive - by convention all errors are quoted as positive numbers)

General formula for combining uncertainties

We can extend this to calculating the error for a function of more than one variable, by realising that if $q = q(x, \dots, z)$ (several measured variables) then the effect that an error in one measured quantity (say the variable u) is given by:

$$\Delta q(\text{u only}) = \left| \frac{\partial q}{\partial u} \right| \Delta u$$

Errors in each of the measured variables has its own effect on Δq , and these effects are combined as follows, to give evaluate the total error in the result q :

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x} \Delta x \right)^2 + \dots + \left(\frac{\partial q}{\partial z} \Delta z \right)^2}$$

In other words, we evaluate the error caused by each of the individual measurements and then combine them by taking the root of the sum of the squares.

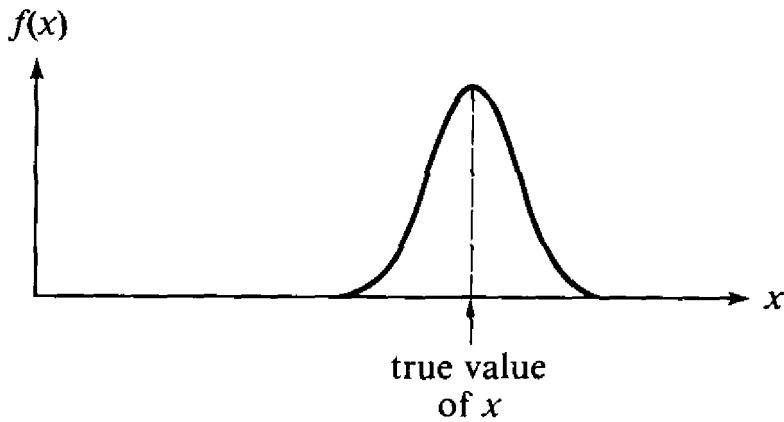
It can be shown that the rules for combining errors in the special cases discussed above come from this general formula.

Exercises on combining errors

- 1) $Z = \frac{AB}{C}$ with $A = 5.0 \pm 0.5$
 $B = 12,120 \pm 10$ [Z = 6100 ± 700]
 $C = 10.0 \pm 0.5$
- 2) $Z = AB^2C^{1/2}$ with $A = 10 \pm 1$
 $B = 20 \pm 2$ [Z = 22000 ± 5000]
 $C = 30 \pm 3$
- 3) $Z = A + B \pi C$ with $A = 11 \pm 1$
 $B = 20 \pm 2$ [Z = 1 ± 4]
 $C = 30 \pm 3$
- 4) $Z = A + 2B \pi 3C$ with $A = 1.50(\pm 0.01) \pi 10^{\pi 6}$
 $B = 1.50(\pm 0.01) \pi 10^{\pi 6}$ [Z = 0(±4) π 10^{π 8}]
 $C = 1.50(\pm 0.01) \pi 10^{\pi 6}$
- 5) $Z = A \left[\frac{2B \pi C}{D} \right]$ with $A = 10 \pm 1$
 $B = 20 \pm 2$ [Z = 390 ± 60]
 $C = 30 \pm 3$
 $D = 40 \pm 4$
- 6) $Z = A \sin(B) \text{Ln}(C)$ with $A = 1.5 \pm 0.1$
 $B = 30 \pm 1$ degrees [Z = 1.7 ± 0.1]
 $C = 10.0 \pm 0.5$

4) Statistical analysis of random errors

In this section, we discuss a number of useful quantities in dealing with random errors. It is helpful first to consider the distribution of values obtained when a measurement is made:



Typically this will be 'bell-shaped' with a peak near the average value.

For measurements subject to many small sources of random error (and no systematic error) is called the NORMAL or GAUSSIAN DISTRIBUTION. It is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - X)^2}{2\sigma^2}\right]$$

The distribution is centred about $x = X$, is bell-shaped, with a width proportional to σ .

Further comments about the Normal Distribution

- The $\frac{1}{\sigma\sqrt{2\pi}}$ in front of the exponential is called the normalisation constant. It makes areas under the curve equal to the probability (rather than just proportional to it).
- The shape of the Gaussian Distribution indicates that a measurement is more likely to yield a result close to the mean than far from it.
- The rule for adding independent errors in quadrature is a property of the Normal Distribution.
- The concepts of the mean and standard deviation have useful interpretations in terms of this distribution.

The mean

We have already agreed that a reasonable best estimate of a measured quantity is given by the average, or mean, of several measurements:

$$\begin{aligned}x_{\text{best}} &= \bar{x} = \frac{x_1 + x_2 + \dots + x_N}{N} \\ &= \frac{\sum x_i}{N}\end{aligned}$$

where the “sigma” simply denotes the sum of all the individual measurements, from x_1 to x_N .

The standard deviation

It is also helpful to have a measure of the average uncertainty of the measurements, and this is given by the standard deviation:

The deviation of the measurement x_i from the mean is $d_i = x_i - \bar{x}$.

If we simply average the deviations, we will get zero - the deviations are as often positive as negative. A more useful indicator of the average error is to square all the deviations before averaging them, but then to take the square root of the average. This is termed the “ROOT MEAN SQUARE DEVIATION”, or the “STANDARD DEVIATION”:

$$\begin{aligned}\sigma_N &= \sqrt{\frac{\sum (d_i)^2}{N}} \\ &= \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}\end{aligned}$$

Minor Complication

In fact there is another type of standard deviation, defined as

$$\sigma_{N-1} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}}$$

For technical reasons, this turns out to be a better representation of the error, especially in cases where N is small. However, for our purposes either definition is acceptable, and indeed for large N they become virtually identical.

Jargon:

σ_N is termed “the population standard deviation”. σ_{N-1} is termed “the sample standard deviation”. From now on I will refer to both types as “the standard deviation”, σ .

Δ as the uncertainty in a single measurement

If the measurement follows a Normal Distribution, the standard deviation is just the Δ in

$$f(x) = \frac{1}{\Delta\sqrt{2\pi}} \exp\left[-\frac{(x - X)^2}{2\Delta^2}\right]$$

Standard deviation of the mean

We know that taking the mean of several measurements improves the “reliability” of the measurements. This should be reflected in the error we quote for the mean. It turns out that the standard deviation of the mean of N measurements is a factor \sqrt{N} smaller than the standard deviation in a single measurement. This is often written:

$$\Delta_{\bar{x}} = \frac{\Delta_x}{\sqrt{N}}$$

5) Use of graphs in experimental physics

There are two distinct reasons for plotting graphs in Physics. The first is to test the validity of a theoretical relationship between two quantities.

Example

Simple theory says that if a stone is dropped from a height h and takes a time t to fall to the ground, then:

$$h \propto t^2$$

If we took several different pairs of measurements of h and t , and plotted h vs. t^2 confirmation of the theory would be obtained if the result was a straight line.

Jargon

Determining whether or not a set of data points lie on a straight line is called “Linear Correlation”. We won’t discuss it in this course.

The second reason for plotting graphs is to determine a physical quantity from a set of measurements of two variables having a range of values.

Example

In the example of the time of flight of a falling stone, the theoretical relationship is, in fact:

$$h = \frac{1}{2}gt^2$$

If we assume this to be true we can measure t for various values of h , and then a plot of h vs. t^2 will have a gradient of $\frac{g}{2}$.

Jargon

The procedure of determining the best gradient and intercept of a straight-line graph is called “Linear Regression”, or “Least-Squares Fitting”.

Choosing what to plot

In general the aim is to make the quantity which you want to know the gradient of a straight-line graph. In other words obtain an equation of the form $y = mx + c$ with m being the quantity of interest.

In the above example, a plot of h vs. t^2 will have a gradient of $\frac{g}{2}$.

Some other examples

What should one plot in order to determine the constant λ from the measured quantities x and y , related by the expression $y = A \exp(\lambda x)$?

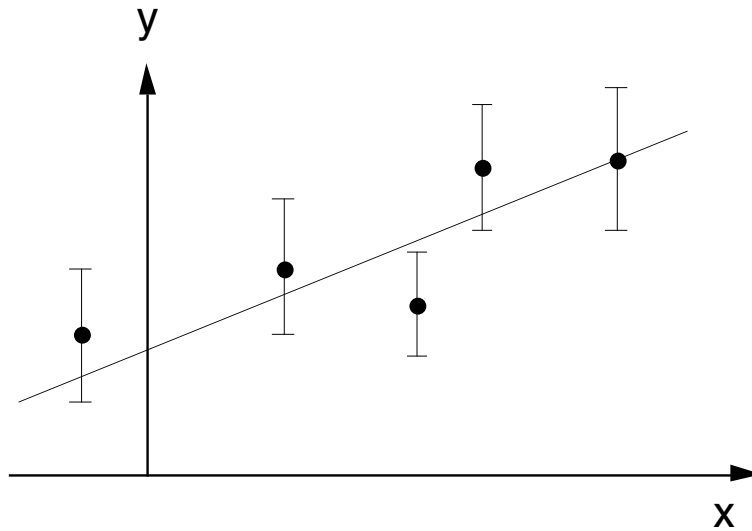
What if one wanted the quantity A rather than λ ?

What if one wanted to know the exponent n in the expression $y = x^n$?

Plotting graphs by hand

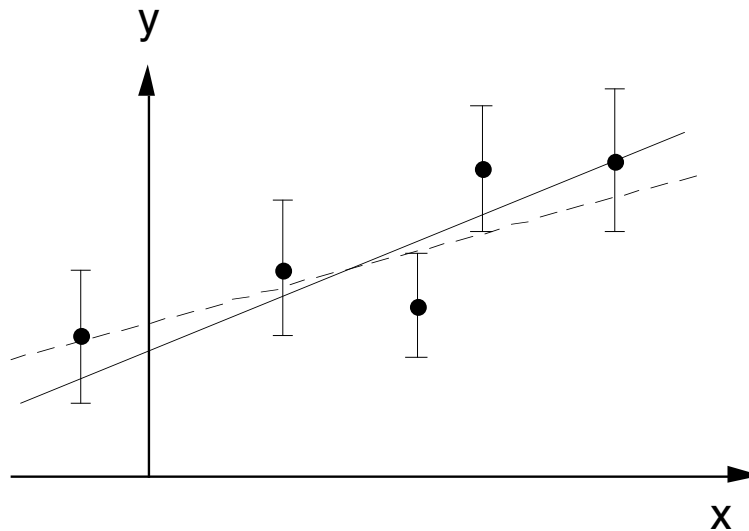
Before we talk about least-squares fitting, it is important to realise that it is perfectly straightforward to determine the best fit gradient and intercept, and the error in these quantities, by eye.

- Plot the points with their error bars.
- to draw the best line, simply position your ruler so that there is a random scatter of points above and below the line:



– about 68% of the points should be within error-bars of the line.

- To find the errors, draw the “worst-fit” line. Swivel your ruler about the “centre of gravity” of your best-fit line until the deviation of points (including their error bars) becomes unacceptable:



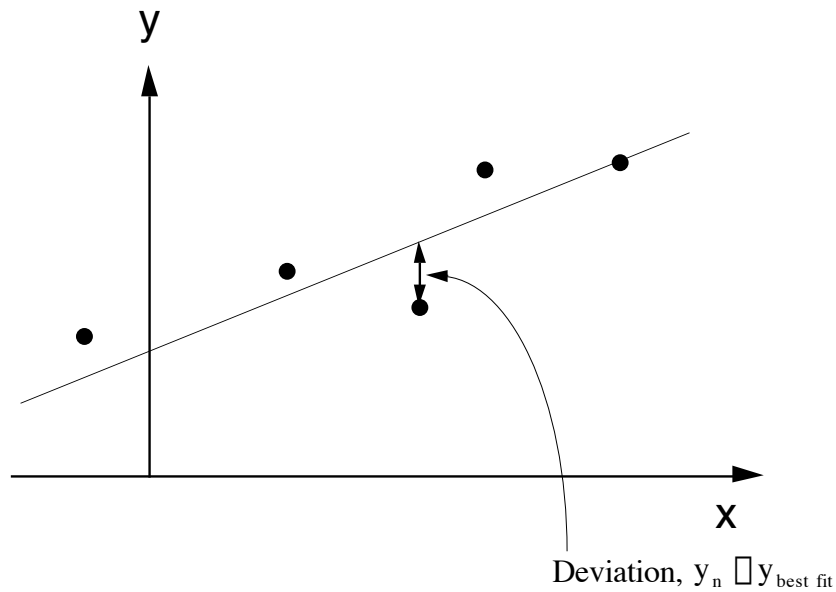
Remember that if your error bars represent standard deviations then you expect 68% of the points to lie within their error bars of the line.

- Now the error in the gradient is given by $\Delta m = |m_{\text{best fit}} - m_{\text{worst fit}}|$, and a similar expression holds for the intercept.

Least-squares fitting

This is the statistical procedure for finding m and c for the equation $y = mx + c$ of the straight line which most closely fits a given set of data $[x_n, y_n]$.

Specifically, the procedure minimises the sum of the squares of the deviations of each value of y_n from the line:



$$= y_n - (mx + c)$$

The fitting procedure obtains values of m and c , and also standard deviations in m and c , σ_m and σ_c .

Points to note:

- The procedure assumes that the x measurements are accurate, and that all the deviation is in y . Bear this in mind when deciding which quantity to plot along which axis. If the two measured quantities are subject to similar errors then it is advisable to make two least-squares fits, swapping the x and y axes around. Then

$$\sigma_m(\text{total}) = \sqrt{\sigma_m(x \text{ vs. } y)^2 + \sigma_m(y \text{ vs. } x)^2}$$
- The least-squares fit assumes that the errors in all the individual values of y_n are the same, and is reflected in the scatter of the points about the line. It does not allow you to put different error bars on each point. You can plot graphs with error bars on the computer, but the fitting procedure ignores them.
- FOR THE ENTHUSIAST: the fitting procedure comes up with analytical expressions for m , c , σ_m and σ_c , (i.e. it isn't a sort of trial and error procedure which only the computer can do). In principle one could calculate these by hand.

Summary

Types of error: Random and Systematic.

Reporting errors:

- Experimental uncertainties should normally be rounded to one significant figure.
- The last significant figure of the answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

e.g. measured electron mass $9.11(±0.05) × 10^{-31}$ kg

Combining Errors:

For sums and differences, the absolute error in the result is obtained by taking the root of the sum of the squares of the absolute errors in the original quantities:

$$\Delta q = \sqrt{\Delta x^2 + \Delta y^2}$$

For products and quotients, the fractional error in the result is obtained taking the root of the sum of the squares of the fractional errors in the original quantities:

$$\frac{\Delta q}{q} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

For powers (i.e. $q = x^n$): $\frac{\Delta q}{|q|} = n \frac{\Delta x}{|x|}$

And for the general case, $q = q(x, \dots, z)$:

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x} \Delta x\right)^2 + \dots + \left(\frac{\partial q}{\partial z} \Delta z\right)^2}$$

Statistical definitions:

The mean:

$$\bar{x} = \frac{\sum x_i}{N}$$

The standard deviation of a single measurement:

$$\Delta x = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N - 1}}$$

and the standard deviation of the mean:

$$\Delta \bar{x} = \frac{\Delta x}{\sqrt{N}}$$

Graphs and least-squares fitting

Follow-up to Errors Lectures

Which Rule?

At the beginning of the lectures I gave you some simplistic rules, then refined them.

IN THE LAB, USE THE REFINED VERSIONS

These are:

For pure sums, pure differences or combinations of sums and differences:
the absolute error in the result is obtained by taking the root of the sum of the squares of the absolute errors in the original quantities:

e.g. if $q = a \pm b + c \pm d$ then $\Delta q = \sqrt{\Delta a^2 + \Delta b^2 + \Delta c^2 + \Delta d^2}$

For pure products, pure quotients or combinations of products and quotients:
the fractional error in the result is obtained taking the root of the sum of the squares of the fractional errors in the original quantities:

e.g. if $q = \frac{a \pm b}{c \pm d}$ then $\frac{\Delta q}{q} = \sqrt{\frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2} + \frac{\Delta c^2}{c^2} + \frac{\Delta d^2}{d^2}}$

For products and quotients involving powers of the measured quantities:

Use the modified rule for products and quotients shown in the following example:

if $q = \frac{a^n \pm b}{c \pm d^m}$ then $\frac{\Delta q}{q} = \sqrt{\frac{n^2 \Delta a^2}{a^2} + \frac{\Delta b^2}{b^2} + \frac{\Delta c^2}{c^2} + \frac{m^2 \Delta d^2}{d^2}}$

For any other formulae:

Use the general expression:

if $q = q(a, \dots, d)$ then $\Delta q = \sqrt{\left(\frac{\partial q}{\partial a} \Delta a\right)^2 + \dots + \left(\frac{\partial q}{\partial d} \Delta d\right)^2}$

The following expressions are examples where you should use this rule:

$q = \frac{a(b \pm c)}{d}$ $q = a^n + b \pm c + d^m$ $q = b \sin c$ $q = a \log b$

Dealing with constants in expressions:

In products and quotients, the constants can be ignored provided you are sure they do not contain errors: fundamental constants like e , or π come into this category.

N.B. numbers given to you in an experiment such as component values are NOT free of errors.

In sums and differences, constants DO play a role:

e.g. $q = a + 2b$

The “2” is a constant and may be assumed exact.

But the absolute error in the quantity $2b$ is twice that in b .

So the error is given by:

$$\Delta q = \sqrt{\Delta a^2 + (2\Delta b)^2}$$

Reporting errors:

- Experimental uncertainties should normally be rounded to one significant figure.
- The last significant figure of the answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

e.g. measured electron mass = $9.11(\pm 0.05) \times 10^{-31} \text{ kg}$