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Surreal Numbers

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1 Numbers from Nothing

We will define a wild collection of numbers that turns out to be much, much larger than any other, conventional number system, it will include natural counterparts to the reals, ordinals, cardinals, infinitesimals, and so on. This construction is due to the late J. H. Conway, see [2] and [4], one of the most unconventional minds in 20th century mathematics. The main idea is to generalize Dedekind's construction [3] of the reals: Dedekind describes a real number by a collection of rational numbers, intuitively the ones strictly less than the target real. These sets are called Dedekind cuts and each cut uniquely determines a real number. The beauty of this construction is that it requires only rationals, simple sets of rationals, and some basic order properties. By comparison, constructing the reals from Cauchy sequences is much more cumbersome, see Numbers.

In Conway's construction, we do not start with the rationals, instead we start with absolutely nothing. On the other hand, we keep repeating the operation over and over again, producing more and more numbers along the way, which can the be used to form more sets of numbers, which produce more numbers, and so on and so forth, through transfinitely many steps. To make this work, the upper/lower bound idea has to be slightly modified: the numbers in the lower/upper bound sets may not be arbitrarily close to each other as in Dedekind's method, so we are looking for the "simplest" number that lies between the two sets, whatever that may mean exactly. As in Dedekind's approach, we need a comparison operation between these new numbers: everything in the lower bound set needs to be smaller than everything in the upper bound set.

In section 2, we will only consider the first infinitely many levels of the construction. The arguments there require only ordinary induction on the naturals. As it turns out, not much happens in that first phase, all we get is the dyadic rationals. Still, it is much easier to develop some intuitive grasp of what is going. In section 3 we will tackle transfinite levels and start to see some new and bizarre objects. . The induction arguments here are much more intricate and require a bit of sophisication. The real power of induction becomes visible on these transfinite levels.

2 Finite Levels

Here are the original definitions, more or less verbatim.

Definition 2.1 (Surreals) *A surreal number consists of two sets of surreal numbers, the left and right set. No element in the right set can be less-than-or-equal-to any element in the left set.*

Definition 2.2 (Comparison) *A surreal number x is less-than-or-equal-to a surreal number y if there is no element u of the left set of x so that y is less-than-or-equal-to u; symmetrically, there is no element v of the right set of y so that v is less-than-or-equal-to x.*

We need a bit of notation to make these definitions less opaque. Let's write $\mathbb{S}$ for the collection of all surreal numbers, whatever that may turn out to be. To display the decomposition of a surreal x according to the first definition we write

$$x = (\, L \mid R \,) \qquad \text{or} \qquad x = (\, X_L \mid X_R \,)$$

The order relation will be denoted by $x \preccurlyeq y$. Dire warning: so far we know next to nothing about this relation, in particular we do not yet know that it is reflexive and transitive as the notation suggests. The order condition on the left/right sets now reads

$$\forall\, u \in X_L, v \in X_R \,(v \not\preccurlyeq u)$$

It is worth spelling out the second definition in greater detail:

$$x \preccurlyeq y \Leftrightarrow \forall\, u \in X_L, v \in Y_R \,(y \not\preccurlyeq u \wedge v \not\preccurlyeq x)$$
$$x \not\preccurlyeq y \Leftrightarrow \exists\, u \in X_L, v \in Y_R \,(y \preccurlyeq u \vee v \preccurlyeq x)$$

This makes intuitive sense: x is supposed to be strictly larger than all the elements in X_L, so y should not be bounded by any of these. Ditto for the second condition. Another useful convention is to write $X \preccurlyeq Y$ for sets X and Y to mean that $\forall\, x \in X, y \in Y \,(x \preccurlyeq y)$.

At first glance, the definition of a surreal may seem circular and thus utterly useless: we already need surreals to get surreals. But note that a surreal is not comprised of two other surreals, but of two *sets* of surreals. So we can get off the ground by choosing these sets to be empty. Here is our first surreal, the first element of $\mathbb{S}$:

$$(\, \emptyset \mid \emptyset \,)$$

It is customary to leave the spaces around the slash empty rather then spelling out the empty set. Also, with a view towards maintaining some measure of reader sanity, we introduce a handy alias for this surreal:

$$0 = (\;\mid\;)$$

One might object that the symbol 0 is already heavily overloaded in math, but we will see that no harm comes from this identification.

At any rate, now we have one bonified surreal, $(\ \mid\ )$, which has appeared out of nothing at level 0 of the construction, a fact we will express by writing $(\ \mid\ ) \in \mathbb{S}_0$. Now the induction starts: we use the numbers in $\mathbb{S}_n$ to define new numbers in $\mathbb{S}_{n+1}$. In our case, $n = 0$ and we can only build slightly more interesting left and right sets:

$$(\ \mid (\ \mid\ ))\qquad ((\ \mid\ ) \mid\ )$$

This looks awful, aliases can help:

$$-1 = (\ \mid 0\,)\qquad 1 = (\,0 \mid\ )$$

Note that these left/right sets trivially obey the order conditions, so these are legitimate surreals, but $(\,0 \mid 0\,)$ is not. Also note that we do not use curly braces for the sets, our notation for surreals is bad enough as is.

Now we have three surreals in $\mathbb{S}_1$ and we can concoct marginally more complicated left/right sets to get yet more surreals at level 2. Here is a table listing the new surreals, with aliases.

$$\begin{aligned} 2 &= (\,1 \mid\ ) = (((|)|)|) \\ 1/2 &= (\,0 \mid 1\,) = ((|)|((|)|)) \\ -1/2 &= (\,-1 \mid 0\,) = ((|(|))|(|)) \\ -2 &= (\ \mid -1\,) = (|(|(|))) \end{aligned}$$

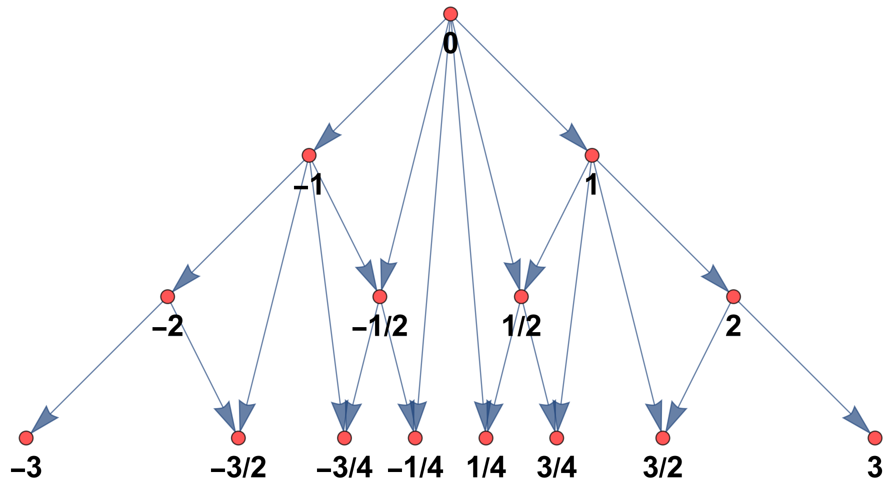
Where do these aliases come from? As mentioned, the surreals encompass several traditional number systems, and in particular the reals and ordinals. So whenever possible we would like to associate a surreal with a traditional number. As it turns out, during the first infinitely many levels we only get surreals that correspond nicely to certain rationals. Things get more interesting at transfinite levels—which is a good thing, otherwise we would just be reinventing the wheel, and a very awkward one at that.

So how do we concoct values for the surreals at finite levels? Going back to the idea of generalized Dedekind cuts, assuming we already have values for the elements of the left/right sets, we want to choose a number strictly between these values. In other words,

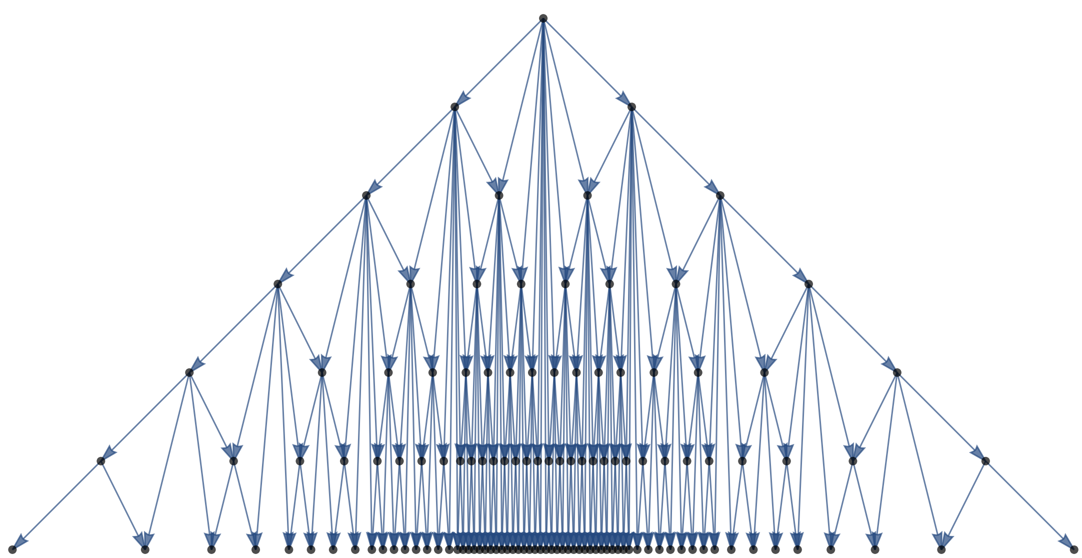
$$\mathsf{val}(X_L) < \mathsf{val}(x) < \mathsf{val}(X_R)$$

Unfortunately, we have no requirement that says that the values in the left and right sets get arbitrarily close, so the new rational may not be determined uniquely. For example, in $(\,0 \mid 1\,)$ we ostensibly could choose any rational between 0 and 1. To get around this problem, we invoke our principle of simplicity: we want the simplest number that obeys the bounds, so we take $(0 + 1)/2 = 1/2$. At the extreme ends we similarly pick a simple number that obeys the bounds. So $(\,0 \mid\ )$ needs to be larger that 0 but has no upper bound—we resolve this problem by choosing the next smallest integer after 0.

There are other legitimate constructions such as $(\,-1, 0 \mid 1\,)$, but this produces the same value as $(\,0 \mid 1\,)$ according to our averaging rule: a lower bound of 0 is stricter than a lower bound of -1. The reason we only need left/right sets of size at most 1 is that we can replace, say, L by the maximum element of L if such a thing exists. We will see in section 3 that this is indeed the case for finite sets of surreals. Here is a picture of $\mathbb{S}_3$, the next level, that shows how the new surreals arise from the previous levels. For example, $(((|)|((|)|))|((|)|))$ with value 3/4 arises from $((|)|((|)|))$ and $((|)|)$, with values 1/2 and 1, respectively.



We are using aliases everywhere, try to figure out what the actual surreals look like. Here is the analogous tree for the first 6 levels of the construction, without any labels to avoid visual clutter. We obtain a sort of light cone, that contains more an more numbers.



Picking out just one of the numbers at level 5, one can appreciate the importance of aliases; the actual numbers are basically impossible to parse:

$$5/16 = (\,1/4 \mid 3/8\,) = (((|)|((|)|((|)|)))|(((|)|((|)|((|)|)))|((|)|((|)|))))$$

At this point it looks like we can obtain all rationals of the form a/b where b is a power of 2, but nothing else. These numbers are commonly known as the dyadic rationals. This is a bit underwhelming, wait for the real action to start in the next section.

Exercise 2.1 Show that one can replace X_L by $\max X_L$ and X_R by $\min X_R$ assuming that $\preccurlyeq$ has the usual order properties.

Exercise 2.2 Determine the number of new surreals at level n.

Exercise 2.3 Explain exactly what these new surreals look like.

3 Transfinite Levels

So far, we have handled only finite levels and produced an infinite set of surreals $\mathbb{S}_\omega = \bigcup_{n \geq 0} \mathbb{S}_n$. Here we use the standard notation ω for the first infinite ordinal. Unfortunately, $\mathbb{S}_\omega$ is fairly lame, even a simple fraction like 1/3 has not yet appeared. But what happens if we take one more step, using left/right sets that are subsets of $\mathbb{S}_\omega$ to produce $\mathbb{S}_{\omega+1}$? For example, we can choose

$$L = \{\, x \in \mathbb{S}_\omega \mid x < 1/3 \,\} \qquad R = \{\, x \in \mathbb{S}_\omega \mid x > 1/3 \,\}$$

Since the values of these surreals get arbitrarily close to 1/3, it makes perfect sense to say that the value of $(\, L \mid R \,)$ is 1/3. In fact, all rationals and even all reals will appear in this manner. So we may think of $\mathbb{R} \subseteq \mathbb{S}_\omega$. But there is more: we also get $(\, \mathbb{N} \mid \,) = (\, 0, 1, 2, \ldots \mid \,)$, the first infinite surreal number. In the opposite direction, $(\, 0 \mid 1, 1/2, 1/4, \ldots \,)$ represents a positive but infinitesimally small number, an object that is rather useful when one tries to do calculus without limits.

More importantly, we can keep on going, repeating our construction over and over again to produce more and more surreals. To keep track of these transfinite levels of the construction one really should use ordinals, generalizations of the naturals to the infinite domain. For example, we have the first infinite ordinal ω, followed by $\omega + 1$, $\omega + 2$ and so on. In the realm of surreals, one can think of ω as being represented by $(\, \mathbb{N} \mid \,) = (\, 0, 1, 2, \ldots \mid \,)$. This identification is a bit dangerous, though. Ordinals carry a natural arithmetic, and though we can represent them by surreals, the arithmetic there is different, see section 4. For example, addition commutes in the surreals, but not for ordinals. And we can form $(\, \mathbb{N} \mid \,) - 1$ and even $(\, \mathbb{N} \mid \,)/2$, which makes no sense for ordinals.

At any rate, we have just seen that $\mathbb{S}_{\omega+1}$ contains all reals (or, more precisely, their surreal counterparts). We will avoid all technicalities and simply assume that ordinals share the essential property of the natural: one can argue by induction. More precisely, suppose that whenever all surreals at levels $\sigma < \tau$ have some property, this property is inherited at level τ. In this case we may conclude that the property holds at all levels. To see why, suppose our property fails at some level τ. By the basic structure of ordinals, there must be a least such level. But then the property holds at all smaller levels, and thus is inherited at level τ, a contradiction.

We can define the rank of apparition of a surreal x to be the least σ such that $x \in \mathbb{S}_\sigma$, in symbols $\mathsf{rap}(x)$. Ranks provide an elegant way to make our intuition about a number being simpler than another precise: lower rank means less complicated.

So far, we have mostly been hand-waving. This is the right approach to develop intuition, but at some point we need to prove actual theorems. Our first project is to explore the properties of the order $\preccurlyeq$.

Lemma 3.1 (Pre-Order)

The relation $x \preccurlyeq y$ is a pre-order: it is reflexive and transitive.

Proof. First, reflexivity. We need to show $x \preccurlyeq x$.

Suppose x has minimal rank such that $x \not\preccurlyeq x$. Then, by the definition of comparison, we have $x \preccurlyeq u$ for some $u \in X_L$, or $v \preccurlyeq x$, $v \in X_R$. But then $u \not\preccurlyeq X_L$ and in particular $u \not\preccurlyeq u$, contradicting minimality. The second case is similar.

For transitivity suppose $x \preccurlyeq y$ and $y \preccurlyeq z$. We need to show that $x \preccurlyeq z$.

Let x, y, z of minimal rank such that $x \preccurlyeq y$ and $y \preccurlyeq z$, but $x \not\preccurlyeq z$. There are two similar cases, the first one is $z \preccurlyeq u$ for some $u \in X_L$. Then transitivity already fails at y, z, u. In the second case we get v, x, y where $v \in Z_R$. In either case, we have a contradiction to minimality.

□

The argument in the last part of the proof is rather terse. We are dealing with three surreals, so we actually need to handle three ranks, and only one of them will decrease. In the finite case there is no problem: we can simply argue in terms of the sum of the ranks, $\mathsf{rap}(x) + \mathsf{rap}(y) + \mathsf{rap}(z)$, see exercise 3.2. Alas, the sum of ranks approach fails when ranks are transfinite ordinals, but as the exercise indicates, we can get around this problem.

Recall that a surreal $x = (\, X_L \mid X_R \,)$ is supposed to be strictly between X_L and X_R. To make this precise, define the strict version of our order as $u \prec v$ if $u \preccurlyeq v \wedge v \not\preccurlyeq u$. This may seem unnecessarily complicated, but recall that $\preccurlyeq$ is not anti-symmetric: $0 \preccurlyeq (\, -1 \mid 1 \,) \preccurlyeq 0$ but $0 \neq (\, -1 \mid 1 \,)$.

Lemma 3.2 (Sandwich) $X_L \prec x \prec X_R$

Proof.

We first show that $X_L \preccurlyeq x \preccurlyeq X_R$.

Let's deal with the first inequality only. Suppose x has minimal rank where this inequality fails, so for some $u \in X_L$ we have $u \not\preccurlyeq x$. Then $x \preccurlyeq v \in U_L$ or $X_R \ni v \preccurlyeq u$. But the second case is impossible by the definition of a surreal. Since u has smaller rank, we have $U_L \preccurlyeq u$, so $v \preccurlyeq u$ and thus $x \preccurlyeq u$. But then for any $v \in X_L$ we have $u \not\preccurlyeq v$, contradicting $u \in X_L$.

From the definitions, $x \not\preccurlyeq u$ for all $u \in X_L$, and we are done. □

The sandwich result also implies that we have a total order.

Lemma 3.3 (Total Order) *For all surreals x and y, $x \preccurlyeq y$ or $y \preccurlyeq x$.*

Proof. Suppose $x \not\preccurlyeq y$ for some x and y. Then $y \preccurlyeq u \in X_L$ or $Y_R \ni v \preccurlyeq x$. In either case, it follows from the sandwich lemma that $y \preccurlyeq x$, as required.

□

Now that we know we are dealing with a total order, we can restate our definition in a more intuitive way:

$$x \preccurlyeq y \Leftrightarrow X_L \prec y \wedge x \prec Y_R$$

If we wanted to deal with the lack of anti-symmety we could collapse all equivalent elements into a single number:

$$x \equiv y \Leftrightarrow x \preccurlyeq y \preccurlyeq x$$

It will come as no surprise that there is no harm in identifying equivalent surreals in this manner. For example, replacing all elements of L and R by equivalent ones we get two new sets L' and R'. Then $(\, L \mid R\,) \equiv (\, L' \mid R'\,)$.

Recall that for finite levels it suffices to deal with left/right sets of cardinality at most 1. Clearly, this fails for transfinite levels:

$$(\, 0, 1, 2, 3, \ldots \mid \,)$$

cannot be obtained from a finite left set. And there are many equivalent numbers such as $(\, 1, 2, 4, 8, \ldots \mid \,)$, $(\, 2, 3, 5, 7, \ldots \mid \,)$.

Exercise 3.1 Prove the remark about equivalent surreals being interchangeable.

Exercise 3.2 Suppose we have a ternary recursive function f on the naturals, which is explicitly defined for $f(0, b, c)$ and $f(a, b, 0)$. Otherwise, a call to $f(a, b, c)$ results in a call to $f(b, c, d)$, $d < a$, and a call $f(d, a, b)$, $d < c$. Show that any such function is guaranteed to terminate by considering the *weight* $a + b + c$ of the argument.

Exercise 3.3 Find a different proof for the last exercise that does not use weights. You proof should carry over to the situation where the arguments are ordinals rather than naturals.

4 Surreal Arithmetic

A vast set of numbers that includes (or rather, can be interpreted as including) reals, ordinals and infinitesimals is certainly interesting, but to be really useful we need arithmetic operations As it turns out, one can naturally define addition and multiplication on the surreals, but there is a bit of work involved.

How should addition $x + y$ be defined? The basic idea is not too complicated: We would want to shift the left set of x by y and the left set of y by x, and combine the two to obtain the left set for $x + y$. The right set will defined analogously, and both lean heavily on induction. Of course, we have to check that everything works out as intended.

Definition 4.1 (Surreal Addition)

$x + y = (\, X_L + y, x + Y_L \mid X_R + y, x + Y_R \,)$

For example $1 + 1 = (\, 0 \mid \,) + (\, 0 \mid \,) = (\, 0 + 1, 1 + 0 \mid \,) = (\, 1 \mid \,) = 2$. Unfortunately, the expression on the right might violate the order condition for left/right sets, so we will refer to it as a pseudo-surreal. Note that we can still argue about order properties of the sets involved, we just have to make sure that pseudo-surreals in the end turn out to be surreals. At any rate, note that it immediately follows from the definition and the usual induction that addition is commutative. The next result is key to deal with pseudo-surreals; alas, its proof is rather messy.

Lemma 4.1 (Translation Invariance)

For all x, y, z: $x \preccurlyeq y$ if, and only if, $x + z \preccurlyeq y + z$.

Proof.

To keep notation manageable, write $F[x, y, z]$ for the forward implication from left to right, and $B[x, y, z]$ for the opposite direction.

We will only sketch the implication from left to right. So suppose $x \preccurlyeq y$, meaning that

$$X_L \prec y \wedge x \prec Y_R$$

From the definition, we have

$$x + z = (\, X_L + z, x + Z_L \mid X_R + z, x + Z_R \,)$$
$$y + z = (\, Y_L + z, y + Z_L \mid Y_R + z, y + Z_R \,)$$

We need to show that

$$X_L + z, x + Z_L \prec y + z$$

There are two cases, $u \in X_L + z$ or $u \in x + Z_L$. In either case, we need to show that

$$U_L \prec y + z \wedge u \prec (y + z)_R = Y_R + z, y + Z_R$$

This can be handled by induction in a manner similar to the proof of transitivity.

□

One can now verify that addition as defined indeed produces surreals rather than just pseudo-surreals, the necessary inequalities all hold. So we have a commutative operation on $\mathbb{S}$, but in order to get anything of algebraic interest we need at least associativity. This is somewhat tedious to check, but rather straightforward.

Lemma 4.2 (Associativity)

Addition on $\mathbb{S}$ is associative: $x + (y + z) = (x + y) + z$.

Proof. We unfold the definitions, using induction to apply associativity at lower levels:

$$\begin{aligned}
x + (y + z) &= x + (\, Y_L + z, y + Z_L \mid Y_R + z, y + Z_R \,) \\
&= (\, X_L + (y + z), x + (Y_L + z), x + (y + Z_L) \mid X_R + (y + z), x + (Y_R + z), x + (y + Z_R) \,) \\
&= (\, (X_L + y) + z, (x + Y_L) + z, (x + y) + Z_L \mid (X_R + y) + z, (x + Y_R) + z, (x + y) + Z_R \,) \\
&= (\, X_L + y, x + Y_L \mid X_R + y, x + Y_R \,) + z \\
&= (x + y) + z
\end{aligned}$$

□

It is easy to check that 0 is a neutral element, so we actually have a monoid $\langle \mathbb{S}, + \rangle$. Monoids are fine, but groups are better. Is there a notion of $-x$ in the surreals? Keeping in mind our lower bound/upper bound idea, this is actually quite easy.

Definition 4.2 (Negation) *Define* $-x = (\, -X_R \mid -X_L \,)$ *and* $x - y = x + (-y)$.

As usual, we have to make sure that we obtain surreals from this definition, not just pseudo-surreals. It is easy to see by induction that $x \preccurlyeq y \Leftrightarrow -y \preccurlyeq -x$, and that suffices to show that $-x$ is really a surreal (no put intended). It follows immediately that negation is an involution on $\mathbb{S}$: $-(-x) = x$.

We are hoping to produce a group, so we still have to check the following. Note that we are dealing with equivalence here, there is no reason to expect an identity.

Lemma 4.3 $x - x \equiv 0$.

Proof. Using $-x = (\, -X_R \mid -X_L \,)$, we unfold the definition.

$$x - x = (\, X_L - x, x - X_R \mid X_R - x, x - X_L \,)$$

We need to show $0 \preccurlyeq x - x \preccurlyeq 0$. This results in 4 inequalities, one of which is $X_L - x \prec 0$. Assume this is false, so that $0 \preccurlyeq u - x$ for some $u \in X_L$. Unfolding the definition of $\preccurlyeq$, this produces

$$0 \prec (u - x)_R = U_R - x, u - X_L$$

and we have a contradiction since u has lower rank and $u \in X_L$. The other inequalities are similar.

□

At long last, we have our big theorem.

Theorem 4.1 *The surreal numbers under addition from a commutative group.*

Some would quibble that the carrier set of a group is required to be just that, a set, but the surreals do not form a set, they are a proper class. True, but completely irrelevant, just one of the idiosyncracies of using set theory as a foundational system.

And even the group structure is not the end of the story: once turn $\mathbb{S}$ into a field by defining an appropriate notion of multiplication. Here is the definition.

Definition 4.3 *Define* $z = x \cdot y$ *as follows:*

$$X_L = X_L \cdot y + x \cdot Y_L - X_L \cdot Y_L, X_R \cdot y + x \cdot Y_R - X_R \cdot Y_R$$
$$X_R = X_L \cdot y + x \cdot Y_R - X_L \cdot Y_R, X_R \cdot y + x \cdot Y_R - X_L \cdot Y_L$$

This is quite complicated; surprisingly subtraction is involved. One might expect that arguments involving multiplication are more convuluted than for addition. We won't go there.

Exercise 4.1 Fill in all the gaps in the proofs of this section.

Exercise 4.2 Similarly, make sure that the inductions used are actually legitimate.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for your Mathematical Plays*. Academic Press, 1982.

[2] J. H. Conway. *On Numbers and Games*. A.K. Peters, 2001.

[3] R. Dedekind. *Essays on the Theory of Numbers*. Open Court Publishing, Chicago, 1901.

[4] D. E. Knuth. *Surreal Numbers*. Addison-Wesley, 1974.