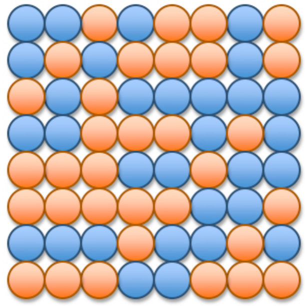
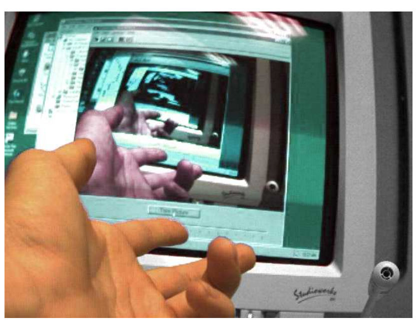


UNIT 5A
Recursion: Basics

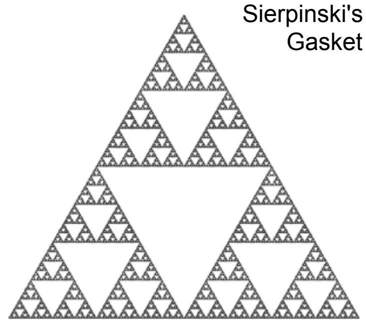
Recursion

- A recursive operation is an operation that is defined in terms of itself.



http://fusionanomaly.net/recursion.jpg

Sierpinski's Gasket



Recursive Definitions

- Every recursive definition includes two parts:
  - Base case (non-recursive)
    A simple case that can be done without solving the same problem again.
  - Recursive case(s)
    One or more cases that are "simpler" versions of the original problem.
    - By "simpler", we sometimes mean "smaller" or "shorter" or "closer to the base case".

GCD

```
def gcd2(x, y):
  if y == 0:
     return x
  else:
     return gcd2(y, x % y)
```

base case: `if y == 0: return x`

recursive case (a "simpler" version of the same problem): `else: return gcd2(y, x % y)`

Factorial

- Definition: n! = n(n-1)(n-2)…(2)(1)
- Since (n-1)(n-2)…(2)(1) = (n-1)!
  - n! = n(n-1)!, for n > 0
  - n! = 1 for n = 0 (base case)
- Example:

4! = 4(3!) = 4(6) = 24
3! = 3(2!) = 3(2) = 6
2! = 2(1!) = 2(1) = 2
1! = 1(0!) = 1(1) = 1

Factorial in Python

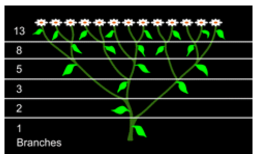
```
def factorial(n):
    if n == 0:
        return 1
    else:
        return n * factorial(n-1)
```

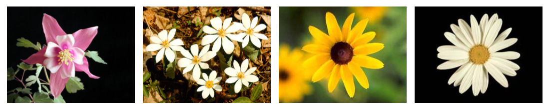
OR

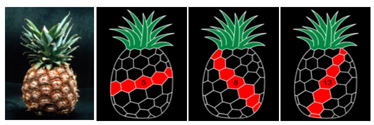
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
```

Fibonacci Numbers

- A sequence of numbers such that each number is the sum of the previous two numbers in the sequence, starting the sequence with 0 and 1.
- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, etc.



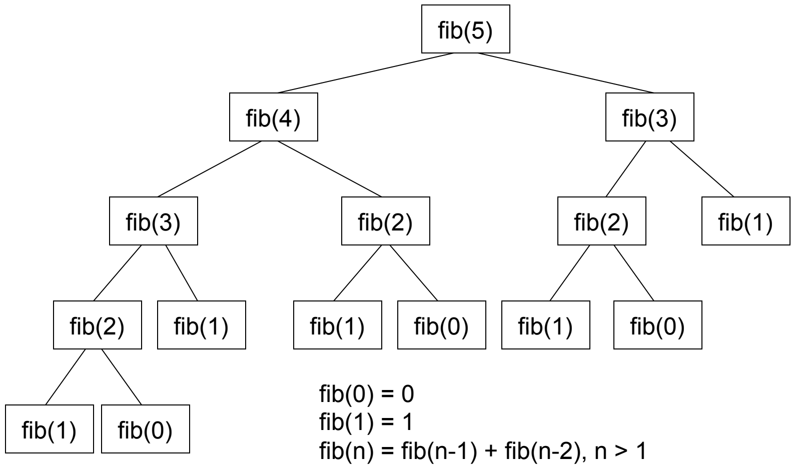




Recursive Definition

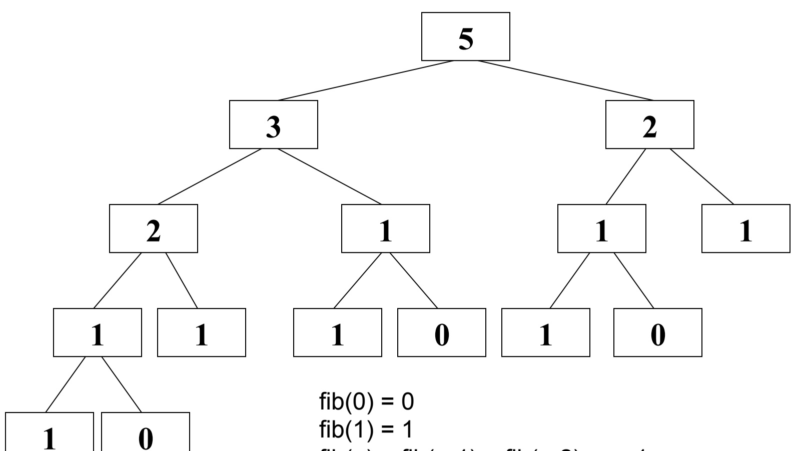
- Let fib(n) = the n^{th} Fibonacci number, $n \geq 0$
  - fib(0) = 0 (base case)
  - fib(1) = 1 (base case)
  - fib(n) = fib(n-1) + fib(n-2), $n > 1$

Recursive Definition



fib(0) = 0
fib(1) = 1
fib(n) = fib(n-1) + fib(n-2), n > 1

Recursive Definition



fib(0) = 0
fib(1) = 1
fib(n) = fib(n-1) + fib(n-2), n > 1

Fibonacci Numbers in Python

```
def fib(n):
    if n == 0 or n == 1:
      return n
    else:
      return fib(n-1) + fib(n-2)
```

In python3, let's print out the first 50 Fibonacci numbers:

```
for i in range(0,50):
    print(fib(i))
```

Why does it take longer to print each subsequent value?

Computing the sum of a list

```
def sum(numlist):
    n = len(numlist)
    if n == 0:
      return 0
    else:
      return numlist[0] + sum(numlist[1:n])
```

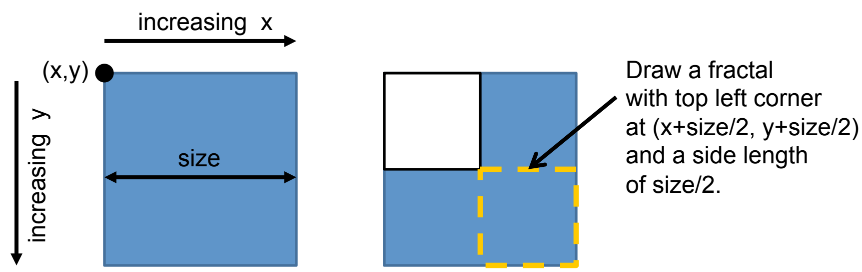
Base case:
The sum of an empty list is 0.

Recursive case:
The sum of a list is the first element + the sum of the rest of the list.

Simple Fractal

To draw a fractal with top-left corner (x,y) and a side length of size:

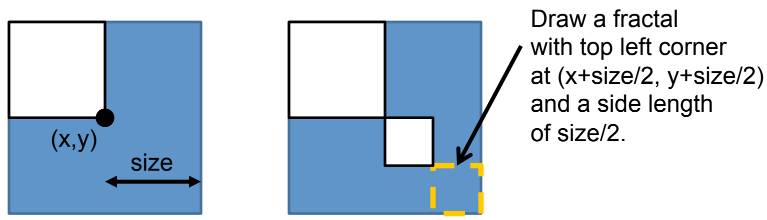
- Draw a white square with top-left corner (x,y) and a side length of size/2.
- Draw another fractal with top-left corner (x+size/2, y+size/2) and a side length of size/2. [recursive step]



Simple Fractal

To draw a fractal with top-left corner (x,y) and a side length of size:

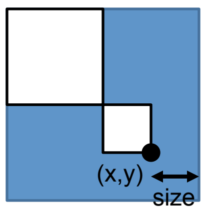
- Draw a white square with top-left corner (x,y) and a side length of size/2.
- Draw another fractal with top-left corner (x+size/2, y+size/2) and a side length of size/2. [recursive step]

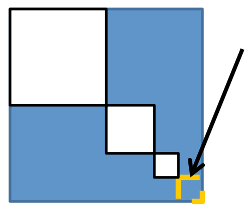


Simple Fractal

To draw a fractal with top-left corner (x,y) and a side length of size:

- Draw a white square with top-left corner (x,y) and a side length of size/2.
- Draw another fractal with top-left corner (x+size/2, y+size/2) and a side length of size/2. [recursive step]



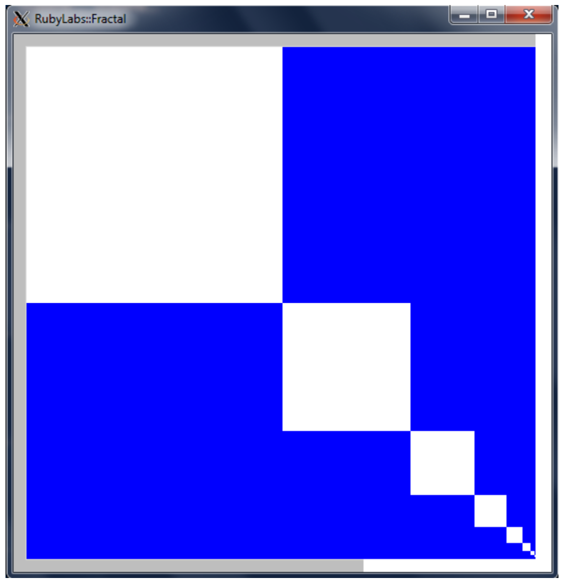


Simple Fractal in Python

(not all code shown)

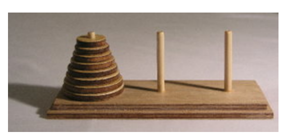
```
def fractal(x, y, size):
    if size < 2:          # base case
        return
    draw_square(x, y, size/2)   # not shown
    fractal(x+size/2, y+size/2, size/2)

def draw_fractal():
    # initial top-left (x,y) and size
    fractal(0, 0, 512)
```



Towers of Hanoi

- A puzzle invented by French mathematician Edouard Lucas in 1883.



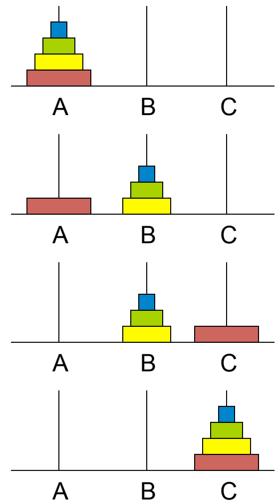
Towers of Hanoi with 8 discs.

- At a temple far away, priests were led to a courtyard with three pegs and 64 discs stacked on one peg in size order.
  - Priests are only allowed to move one disc at a time from one peg to another.
  - Priests may not put a larger disc on top of a smaller disc at any time.
- The goal of the priests was to move all 64 discs from the leftmost peg to the rightmost peg.
- According to the story, the world would end when the priests finished their work.

Towers of Hanoi

Problem: Move n discs
from peg A to peg C using peg B.

1. Move n-1 discs from peg A to peg B using peg C. (recursive step)
2. Move 1 disc from peg A to peg C.
3. Move n-1 discs from peg B to C using peg A. (recursive step)



Towers of Hanoi in Python

```
def towers(n, from_peg, to_peg, using_peg):
  if n >= 1:
    towers(n-1, from_peg, using_peg, to_peg)
    print("Move disc from " + from_peg
      + " to " + to_peg)
    towers(n-1, using_peg, to_peg, from_peg)
```

In python3: `towers(4, "A", "C", "B")`

How many moves do the priests need to move 64 discs?