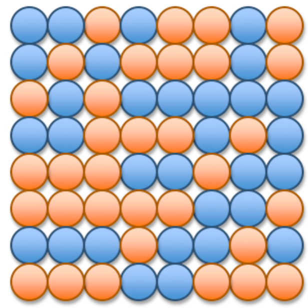
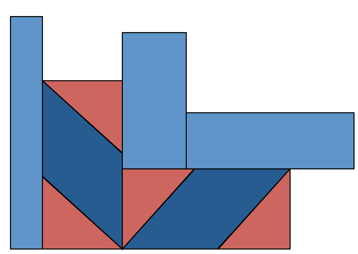


UNIT 14A
The Limits of Computing: Intractability

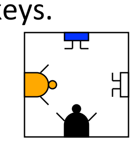
Decision Problems

- A specific set of computations are classified as decision problems.
- An algorithm describes a **decision problem** if its output is simply YES or NO, depending on whether a certain property holds for its input.
- Example:
  Given a set of N shapes, can these shapes be arranged into a rectangle?



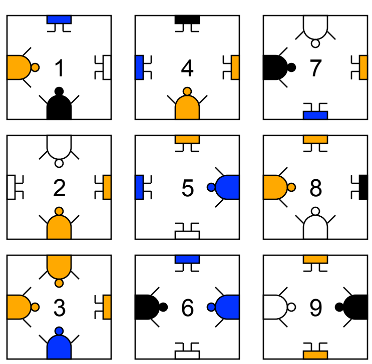
The Monkey Puzzle

- Given:
  - A set of N square cards whose sides are imprinted with the upper and lower halves of colored monkeys.
  - N is a square number, such that $N = M^2$.
  - Cards cannot be rotated.



- Problem:
  - Determine if an arrangement of the N cards in an M X M grid exists such that each adjacent pair of cards display the upper and lower half of a monkey of the same color.

Example

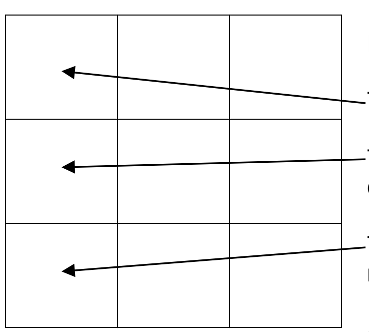


Algorithm

Simple brute-force algorithm:

- Pick one card for each cell of M X M grid.
- Verify if each pair of touching edges make a full monkey of the same color.
- If not, try another arrangement until a solution is found or all possible arrangements are checked.
- Answer "YES" if a solution is found. Otherwise, answer "NO" if all arrangements are analyzed and no solution is found.

Analysis



If there are N = 9 cards (M = 3):

To fill the first cell, we have 9 card choices.

To fill the second cell, we have 8 card choices left.

To fill the third cell, we have 7 card choices remaining.

etc.

The total number of unique arrangements for N = 9 cards is:

____

Analysis (cont' d)

For N cards, the number of arrangements to examine is N! (N factorial)

If we can analyze one arrangement in a microsecond:

N	Time to analyze all arrangements
9	362,880 μs
16	20,922,789,888,000 μs
25	15,511,210,043,330,985,984,000,000 μs

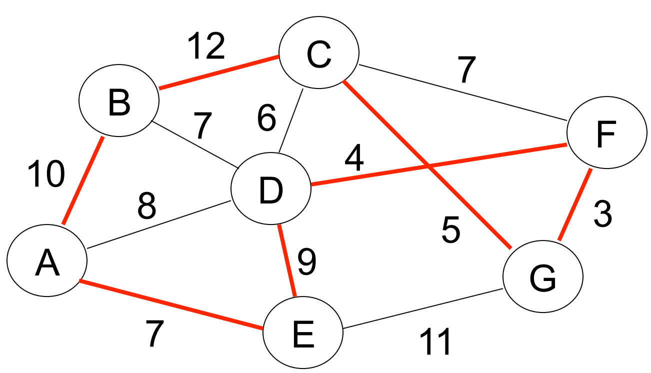
Classifications

- Algorithms that are $O(N^k)$ for some fixed k are **polynomial-time** algorithms.
  - O(1), O(log N), O(N), O(N log N), $O(N^2)$
  - reasonable, **tractable**
- All other algorithms are **super-polynomial-time** algorithms.
  - $O(2^N)$, $O(N^N)$, O(N!)
  - unreasonable, **intractable**

Traveling Salesperson

- Given: a weighted graph of nodes representing cities and edges representing flight paths (weights represent cost)
- Is there a route that takes the salesperson through every city and back to the starting city with cost no more than K?
  - The salesperson can visit a city only once (except for the start and end of the trip).

Traveling Salesperson



Is there a route with cost at most 52? YES (Route above costs 50.)
Is there a route with cost at most 48? YES? NO?

Analysis

- If there are N cities, what is the maximum number of routes that we might need to compute?
- Worst-case: There is a flight available between every pair of cities.
- Compute cost of every possible route.
  - Pick a starting city
  - Pick the next city (N-1 choices remaining)
  - Pick the next city (N-2 choices remaining)
  - ...
- Maximum number of routes: ___________

Map Coloring

- Given a map of N territories, can the map be colored using K colors such that no two adjacent territories are colored with the same color?
- K=4: Answer is always yes.
- K=2: Only if the map contains no point that is the junction of an odd number of territories.

Map Coloring

- Given a map of N territories, can the map be colored using **3** colors such that no two adjacent territories are colored with the same color?

Analysis

- Given a map of N territories, can the map be colored using **3** colors such that no two adjacent territories are colored with the same color?
  - Pick a color for territory 1 (3 choices)
  - Pick a color for territory 2 (3 choices)
  - ...
- There are __________ possible colorings.

Satisfiability

- Given a Boolean formula with N variables using the operators AND, OR and NOT:
  - Is there an assignment of boolean values for the variables so that the formula is true (satisfied)?
    Example: (A AND B) OR (NOT C AND A)
  - Truth assignment: A = True, B = True, C = False.
- How many assignments do we need to check for N variables?
  - Each symbol has 2 possibilities .... ___ assignments

The Big Picture

- Intractable problems are solvable if the amount of data (N) that we're processing is small.
- But if N is not small, then the amount of computation grows exponentially and the solutions quickly become intractable (i.e. out of our reach).
- Computers can solve these problems if N is not small, but it will take far too long for the result to be generated.
  - We would be long dead before the result is computed.

What's Next

- For a specific decision problem, is there single tractable (polynomial-time) algorithm to solve any instance of this problem?
- If one existed, can we use it to solve other decision problems?
- What is one of the big computational questions to be answered in the 21st century?