Lecture Notes on Unbounded Arrays

15-122: Principles of Imperative Computation
Frank Pfenning

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1 Introduction

Most lectures so far had topics related to all three major categories of learning goals for the course: computational thinking, algorithms, and programming. The same is true for this lecture. With respect to algorithms, we introduce *unbounded arrays* and operations on them. Analyzing them requires *amortized analysis*, a particular way to reason about sequences of operations on data structures. We also briefly talk about again about *data structure invariants* and *interfaces*, which are crucial computational thinking concepts.

2 Unbounded Arrays

In the second homework assignment, you were asked to read in some files such as the *Collected Works of Shakespeare* or the *Scrabble Players Dictionary*. What kind of data structure do we want to use when we read the file? In later parts of the assignment we want to look up words, perhaps sort them, so it is natural to want to use an array of strings, each string constituting a word. A problem is that before we start reading we don't know how many words there will be in the file so we cannot allocate an array of the right size! One solution uses either a queue or a stack. We discussed this in Lecture 9 on Queues. Another solution is to use an *unbounded array*. While arrays are a language primitive, unbounded arrays are a *data structure* that we need to implement.

Thinking about it abstractly, an unbounded array should be like an array in the sense that we can get and set the value of an arbitrary element via its index i. We should also be able to add a new element to the end of the array, and delete an element from the end of the array.

We use the unbounded array by creating an empty one before reading a file. Then we read words from the file, one by one, and add them to the end of the unbounded array. Meanwhile we can count the number of elements to know how many words we have read. We trust the data structure not to run out of space unless we hit some hard memory limit, which is unlikely for the kind of task we have in mind, given modern operating systems. When we have read the whole file the words will be in the unbounded array, in order, the first word at index 0, the second at index 1, etc.

The general implementation strategy is as follows. We maintain an array of a fixed length *limit* and an internal index *size* which tracks how many elements are actually in the array. When we add a new element we increment *size*, when we remove an element we decrement *size*. The tricky issue is how to proceed when we are already at the limit and want to add another element. At that point, we allocate a new array with a larger limit and copy the elements we already have to the new array. For reasons we explain later in this lecture, every time we need to enlarge the array we *double* its size. Removing and element from the end is simple: we just decrement *size*. There are some issues to consider if we want to shrink the array, but this is optional.

3 Implementing Unbounded Arrays

According to our implementation sketch, an unbounded array needs to track three forms of data: an integer *limit*, an integer *size* and an array of strings. We can put these together in a `struct` with fields *limit*, *size* and A as the *fields* of the struct. It is declared with

```
typedef string elem;

struct ubarray {
  int limit;                    /* limit > 0 */
  int size;                     /* 0 <= size && size <= limit */
  elem[] A;                     /* \length(A) == limit */
};
```

The `typedef` says that, in this case, the elements are strings, but indicates that the implementation should really be independent of the kind of element stored in the array.

There are some *data structure invariants* that we maintain, although they may be temporarily violated as the elements of the structure are manipulated at a low level. Generally, when we pass a pointer to the data structure or assign it to a variable we expect these invariants to hold. C0, however, has no intrinsic support for ensuring these invariants. Instead, our method is to define a function to test them and then verify adherence to the invariants in contracts as well as loop invariants and assertions. Here, the function `is_uba` serves that purpose.

As a general idiom, when we use structs, we define a new type to stand for a pointer to the struct. This is because most of the time we work with pointers to structs rather than structs themselves. Here we call this new type `uba`.

```
typedef struct ubarray* uba;

bool is_uba (uba L)
//@requires L->limit == \length(L->A);
{
  return L->limit > 0 && 0 <= L->size && L->size <= L->limit;
}
```

Note that consistency between the `L->limit` and the length of `L->A` can only be tested in a precondition (or explicit `@assert`) since `\length` can only appear in annotations.

To create a new unbounded array, we allocate a `struct ubarray` and an array of a supplied initial limit.

```
uba uba_new (int initial_limit)
//@requires initial_limit > 0;
//@ensures is_uba(\result);
{
  assert(initial_limit > 0);
  uba L = alloc(struct ubarray);
  L->limit = initial_limit;
  L->size = 0;
  L->A = alloc_array(elem, L->limit);
  return L;
}
```

We ascertain in the postcondition that the result adheres to the data structure invariants. Since we are implementing a general-purpose data structure, we do not trust that clients will necessarily adhere to the contracts. We therefore check explicitly that the initial limit is strictly positive and signal an error if it is not. This is a common use of explicit `assert` statements. They will *always* be checked at runtime, while `@assert` annotations will not: they are only verified when dynamic checking of contracts is enabled.

Getting and setting an element of an unbounded array is straightforward. However, we do have to verify that the array access is *in bounds*. This is stricter than checking that it is within the allocated array (below `limit`), because everything beyond the current size should be considered to be *undefined*. These array elements have not yet been added to the array, so reading or writing them is meaningless. We show only the operation of writing to an unbounded array, `uba_set`.

```
void uba_set(uba L, int index, elem e)
//@requires is_uba(L);
{
  assert(0 <= index && index < L->size);
  L->A[index] = e;
  return;
}
```

More interesting is the operation of adding an element to the end of an unbounded array. For that we need a function to resize an unbounded array. This function takes an unbounded array L and a new limit `new_limit`. It is required that the new limit is strictly greater than the current size, to make sure we have enough room to preserve all current elements and one more for the next one to add. We also stipulate that the size does not change by stating `L->size == \old(L->size)` in the postcondition.

```
void uba_resize(uba L, int new_limit)
//@requires is_uba(L);
//@requires new_limit > L->size;
//@ensures is_uba(L);
//@ensures L->limit == new_limit && L->size == \old(L->size);
//@ensures L->size < L->limit;
{
  elem[] B = alloc_array(elem, new_limit);
  for (int i = 0; i < L->size; i++)
    //@loop_invariant 0 <= i && i <= L->size;
    {
      B[i] = L->A[i];
    }
  L->limit = new_limit;
  /* L->size remains unchanged */
  L->A = B;
  return;
}
```

Finally we are ready to write the function that adds an element to the end of an unbounded array. We first check whether there is room for another element and, if not, double the size of the underlying array of strings. The contract just states that the array is valid before and after the operation.

```
void addend(uba L, elem e)
//@requires is_uba(L);
//@ensures is_uba(L);
{
  if (L->size == L->limit)
    uba_resize(L, 2*L->limit);
  L->A[L->size] = e;
  L->size++;
  return;
}
```

We discuss the function that removes an element from an array later in this lecture.

4 Amortized Analysis

It is easy to see that reading from or writing to an unbounded array at a given index is a constant-time operation. However, adding an element to an array is not. Most of the time it takes constant time $O(1)$, but when we have run out of space it take times $O(\mathsf{size})$ because we have to copy the old elements to the new underlying array. On the other hand, it doesn't seem to happen very often. Can we characterize this situation more precisely? This is the subject of *amortized analysis*.

In order to make the analysis as concrete as possible, we want to count the number of writes to an array, that is, the number of assignments $A[-] = -$ that are performed. Calling the operation to add a new element to an unbounded array an *insert*, we claim:

The cost of n insert operations into an unbounded array is $O(n)$.

How do we prove that? A simple insert (when there is room in the array) requires a single write operation, so we count it as 1. Similarly, we count the act of copying one element from one array to another as 1 operation, because it requires one write operation.. Now performing a sequence of inserts, starting with an empty array of, say, size 4 looks as follows.

call	op's	*size*	*limit*
`addend(L,"a")`	1	1	4
`addend(L,"b")`	1	2	4
`addend(L,"c")`	1	3	4
`addend(L,"d")`	1	4	4
`addend(L,"e")`	5	5	8
`addend(L,"f")`	1	6	8
`addend(L,"g")`	1	7	8
`addend(L,"h")`	1	8	8
`addend(L,"i")`	9	9	16

We have taken 4 extra operations when inserting `"e"` in order to copy `"a"` through `"d"`. Overall, we have performed 21 operations for inserting 9 elements. Would that be $O(n)$ by the time we had inserted n elements?

We approach this by giving us an overall budget of $c * n$ operations ("tokens") before we start to insert n elements. Every time we perform a write operation we spend a token. If we perform all n insert without running out of tokens, we have achieved the desired amortized complexity.

One difficult is to guess the right constant c. We already know that $c = 1$ or $c = 2$ will not be enough, because in the sequence above we must spend 21 tokens to insert 9 elements. Let's try $c = 3$, so we start with 27 tokens.

call	op's	tokens left	*size*	*limit*
`addend(L,"a")`	1	26	1	4
`addend(L,"b")`	1	25	2	4
`addend(L,"c")`	1	24	3	4
`addend(L,"d")`	1	23	4	4
`addend(L,"e")`	5	18	5	8
`addend(L,"f")`	1	17	6	8
`addend(L,"g")`	1	16	7	8
`addend(L,"h")`	1	15	8	8
`addend(L,"i")`	9	6	9	16

We see that we spend 4 tokens when adding `"e"` to copy `"a"` through `"d"`, and we add a new one for the insertion of `"e"` itself.

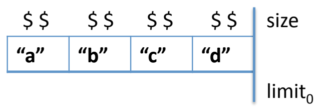
One of the insights of amortized analysis is that we don't need to know the number n of inserts ahead of time. In order to achieve the bound of $c*n$ operations, we simply allow each call to take c operations. If it performs fewer, these remain in the budget and may be spent later! Let's go through the same sequence of calls again.

call	op's	allocated tokens	spent tokens	saved tokens	total saved tokens	*size*	*limit*
`addend(L,"a")`	1	3	1	2	2	1	4
`addend(L,"b")`	1	3	1	2	4	2	4
`addend(L,"c")`	1	3	1	2	6	3	4
`addend(L,"d")`	1	3	1	2	8	4	4
`addend(L,"e")`	5	3	5	-2	6	5	8
`addend(L,"f")`	1	3	1	2	8	6	8
`addend(L,"g")`	1	3	1	2	10	7	8
`addend(L,"h")`	1	3	1	2	12	8	8
`addend(L,"i")`	9	3	9	-6	6	9	16

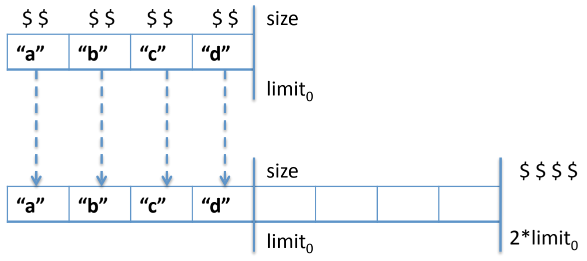
The crucial property we need is that there are $k \geq 0$ tokens left just after have we have doubled the size of the array. We think of this as an *invariant* of the computation: it should always be true, no matter how many strings we insert. In this example we reach 6 tokens after 5 inserts and again after 9 inserts.

To prove this invariant, we must show that it holds the first time we have to double the size of the array, and that it is preserved by the operations.

We when create the array, we give it some initial limit $limit_0$. We run out of space, once we have inserted $limit_0$ tokens, arriving at the following situation.

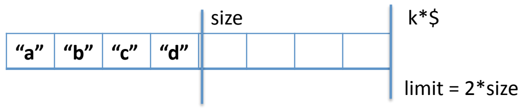


We have accrued $2 * limit_0$ tokens. We have to spend $limit_0$ of them to copy the elements so far, keeping $limit_0 > 0$ in the bank.

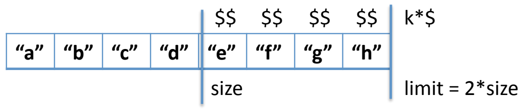


So the invariant holds the first time we double the size.

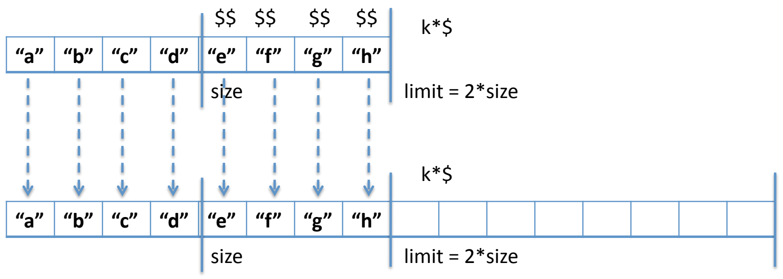
Now assume we have just doubled the size of the array and the invariant holds, that is, we have $k \geq 0$ tokens, and $2 * size = limit$.



After *size* more inserts we are at *limit* and added another $2 * size = limit$ tokens.



On the next insert we double the size of the array and copy *limit* array elements, spending *limit* tokens.



Our bank account is reduced back to k tokens, but we know $k \geq 0$, preserving our invariant.

Since we only save a constant number of tokens on each operation, in addition to the constant time the operation itself takes, we never perform more operations than a constant times the number of operations. So our claim above is true: any sequence of n operations performs at most $O(n)$ steps. We also say that the insert operation has *constant amortized time*.

This completes the argument.

In the example, the number of tokens will now never fall below 6. If we add another 8 elements, we will also put $2 * 8 = 16$ tokens into the bank. We will need to spend these to copy the 16 elements already in the array and we are back down to 6.

Tokens are a conceptual tool in our analysis, but they don't need to be implemented. The fact that there are always 0 or more tokens during any sequence of operations is an invariant of the data structure, although not quite in the same way as discussed before because it tracks sequences of operations rather than the internal state of the structure. In fact, it would be possible to add a new field to the representation of the array that would count tokens and raise an exception if it becomes negative. That would alert us to some kind of mistake, either in our amortized analysis or in our program. This would, however, incur a runtime overhead even when assertions are not checked, so tokens are rarely, if ever, explicitly implemented.

This kind of analysis is important to avoid serious programming mistakes. For example, let's say we decide to increase the size of the array only by 1 whenever we run out of space. The token scheme above does not work, because we cannot set aside enough tokens before we need to copy the array again. And, indeed, after we hit $limit$ the first time, the next sequence of n inserts takes $O(n^2)$ operations, because we copy the array on each step until we reach $2 * limit$.

5 Deleting Elements

Deleting elements from the end of the array is simple, and does not change our amortized analysis, unless we want to shrink the size of the array.

A first idea might be to simply cut the array in half whenever $size$ reaches half the size of the array. However, this cannot work in constant amortized time. The example demonstrating that is an alternating sequence of n inserts and n deletes precisely when we are at the limit of the array. In that case the total cost of the $2 * n$ operations will be $O(n^2)$.

To avoid this problem we cut the size of the array in half only when the number of elements in it reaches $limit/4$. The amortized analysis requires two tokens for any deletion: one to delete the element, and one for any future copy. Then if $size = limit/2$ just after we doubled the size of the array and have no tokens, putting aside one token on every delete means that we have $size/2 = limit/4$ tokens when we arrive at a size of $limit/4$. Again, we have just enough tokens to copy the $limit/4$ elements to the new, smaller array of size $limit/2$.

The code for `remend` ("remove from end"):

```
elem remend(uba L)
//@requires is_uba(L);
//@requires L->size > 0;  /* always check dynamically */
//@ensures is_uba(L);
{
  assert(L->size > 0);
  if (L->size <= L->limit/4)
    uba_resize(L, L->limit/2);
  /* leaving L->A[L->size] in place constitutes a memory leak */
  L->size--;
  elem e = L->A[L->size];
  return e;
}
```

One side remark: before we decrement $size$, we should delete the element from the array by writing `L->strings[L->size] = ""`. In C0, we do not have any explicit memory management. Storage will be reclaimed and used for future allocation when the garbage collector can see that data are no longer accessible from the program. If we remove an element from an unbounded array, but keep the element in the array, the garbage collector can not determine that we will not access it again, because the reason is rather subtle and lies in the bounds check for `uba_get`. In order to allow the garbage collector to free the space occupied by the strings stored in the array, we therefore should overwrite the array element with the empty string `""`, which is the default element for strings. This, however, makes the code specific to strings, which we try to avoid.

6 Interfaces

So far we have shown how to implement and analyze unbounded arrays. From a programming perspective, clients of this data structure should not have to know the exact details of the implementation. It is an important computational thinking principle to hide implementation details from the clients and just present an *interface* with the operations on the data structure. We also note down the complexity of the operations, since they are an important guide to potential clients.

```
typedef string elem;
struct ubarray;
typedef struct ubarray* uba;

uba uba_new(int initial_limit);          /* O(1) */
int uba_size(uba L);                     /* O(1) */
elem uba_get(uba L, int index);          /* O(1) */
void uba_set(uba L, int index, elem e);  /* O(1) */
void addend(uba L, string s);            /* O(1), amortized */
string remend(uba L);                    /* O(1), amortized */
```

Unfortunately, C (and, by association, C0) does not provide a way to enforce that clients do not incorrectly exploit details of the implementation of a data structure. Higher-level languages such as Java or ML have interfaces and data abstraction as one of their explicit design goals. In this course, the use of interfaces is a matter of programming discipline. As we discuss further data structures we generally focus on the interface first, before writing any code. This is because the interface often guides the selection of an implementation technique and the individual functions.

7 Contracts and Interfaces

The interface in the previous section does not provide contracts for the function at the interface. A client should be able to understand the contract by only looking at the interface. This leads to the following:

```
struct ubarray;
typedef struct ubarray* uba;
uba uba_new(int initial_limit)           /* create new unbounded array L */
//@requires initial_limit > 0;
```

```
  ;

int uba_size(uba L)                      /* current size of L */
//@ensures \result >= 0;
  ;

elem uba_get(uba L, int index)           /* "L[index]" */
//@requires 0 <= index && index < uba_size(L);
  ;

void uba_set(uba L, int index, elem e)   /* "L[index] = e" */
//@requires 0 <= index && index < uba_size(L);
  ;

void addend(uba L, elem e);              /* add e at the end of L */

elem remend(uba L)                       /* remove last element in L */
//@requires uba_size(L) > 0;
  ;
```

Contracts on interfaces are cumulative with respect to the contracts on the implementations: both are checked when a function is called through its interface. Note that we do not mention `is_uba`, since this function is not exposed to the client. Client code should only ever be able to obtain valid unbounded arrays if it uses the interface, so preservation of the data structure invariants should be considered an *internal invariant* of the data structure implementation.