Performance of Gittins policy in G/G/1 and G/G/k, with and without setup times

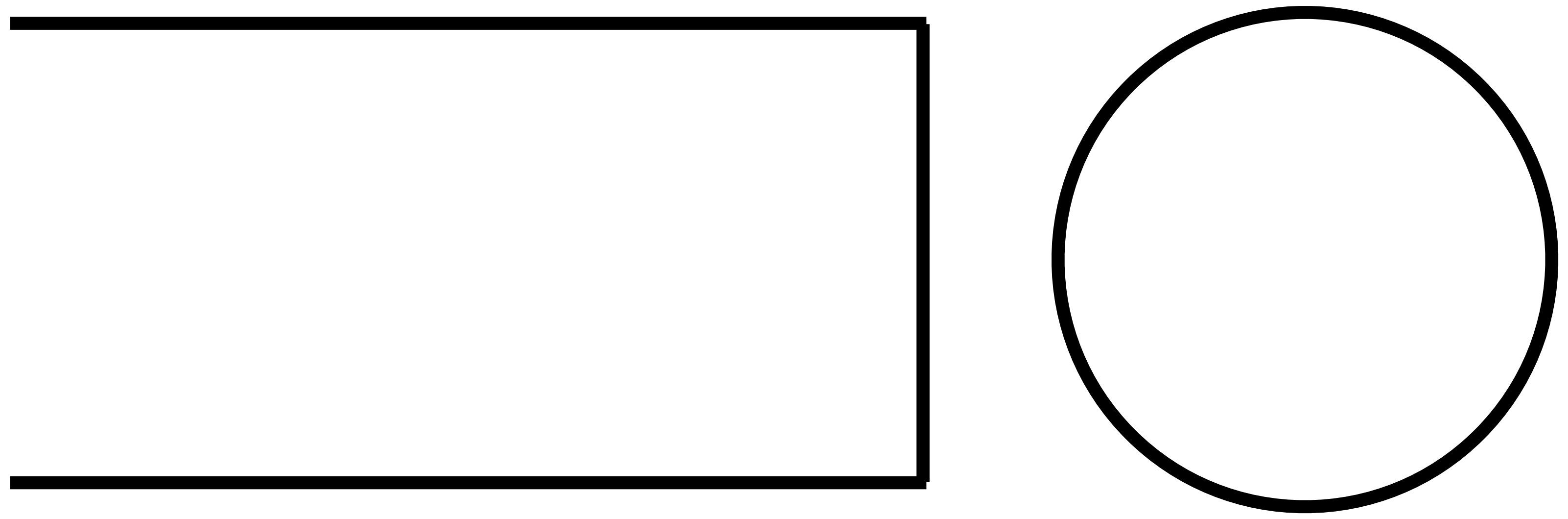
IFIP PERFORMANCE 2023, Nov 14th

Presenter: Yige Hong (Carnegie Mellon University)
Joint work with Ziv Scully (Cornell University)

Motivating question

How should we schedule jobs with unknown sizes in complicated systems?

Scheduling problem



Scheduling problem

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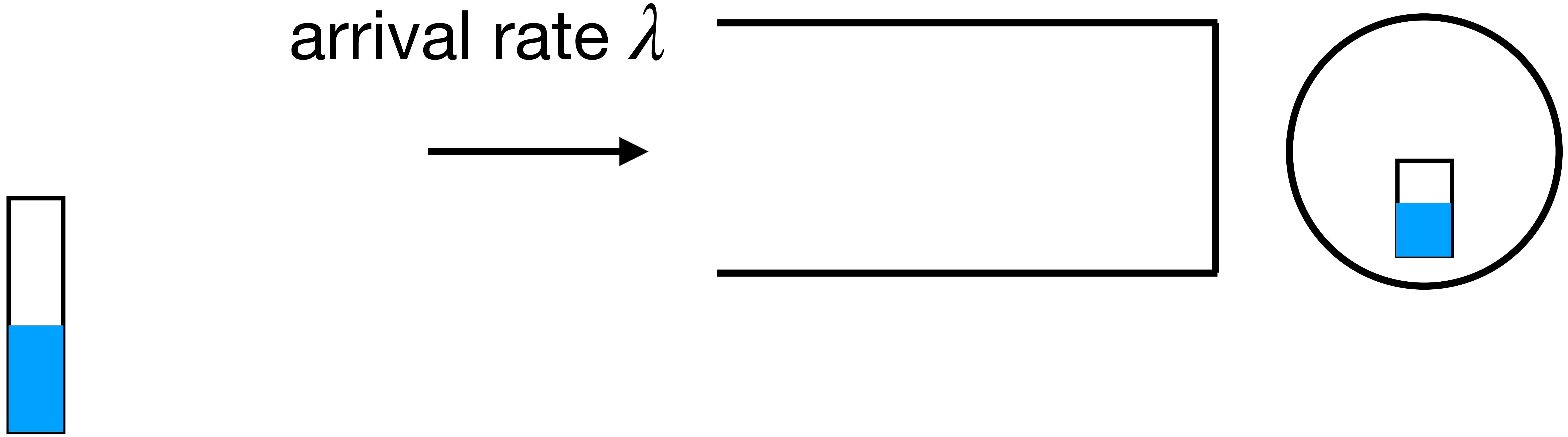
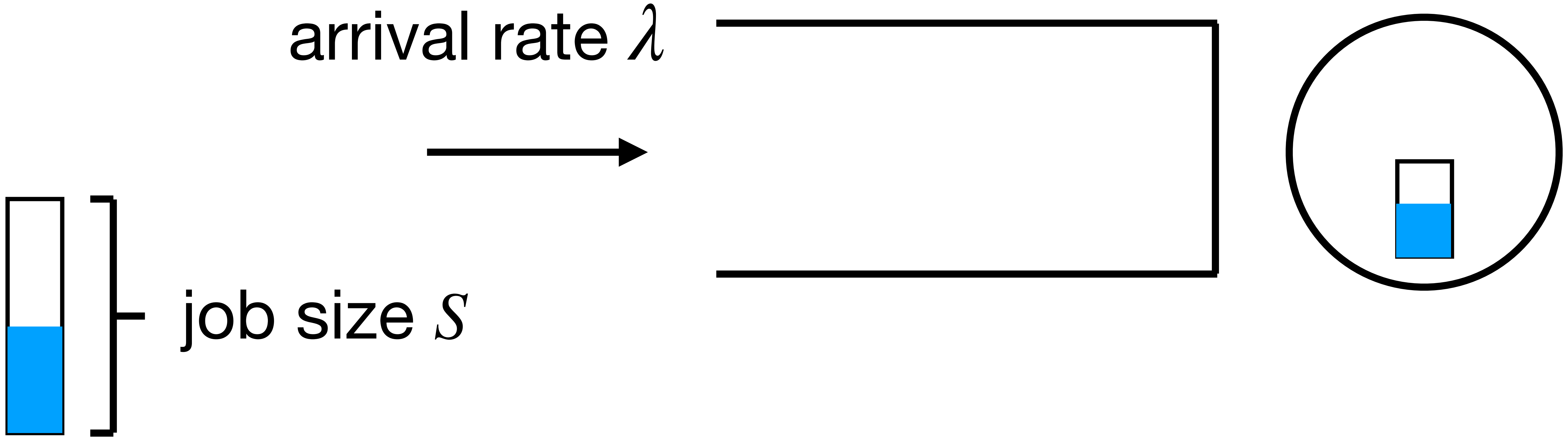


arrival rate λ

Scheduling problem



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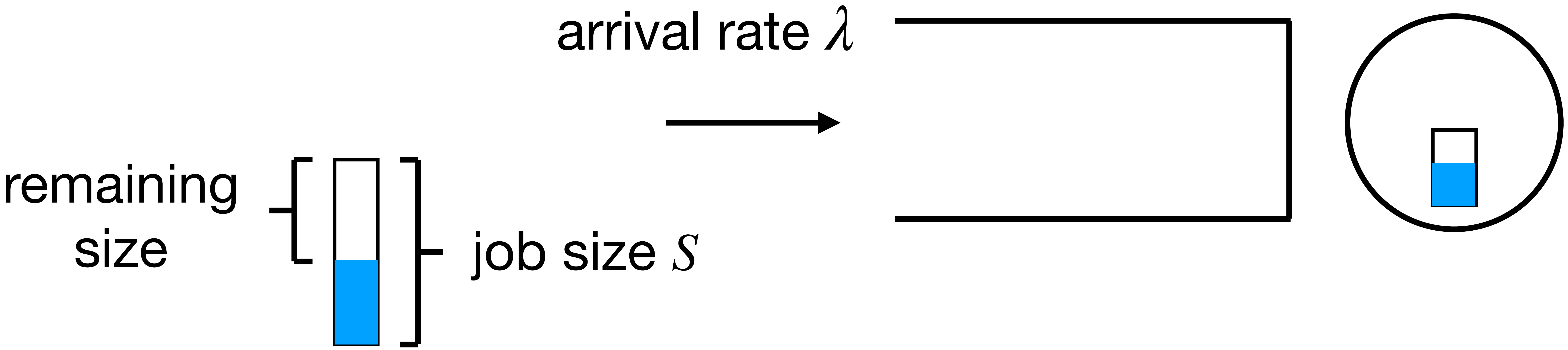
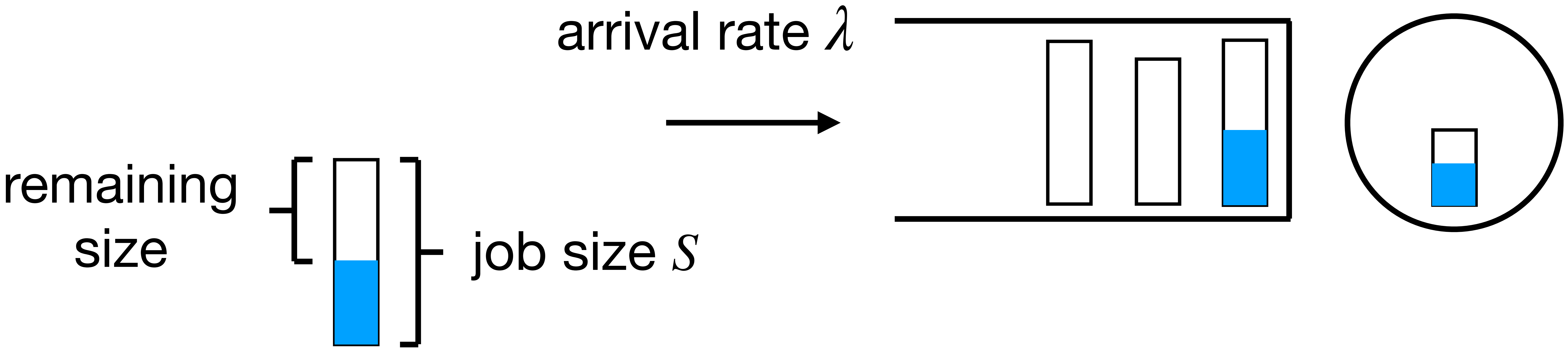


arrival rate λ

remaining size

job size S

Scheduling problem



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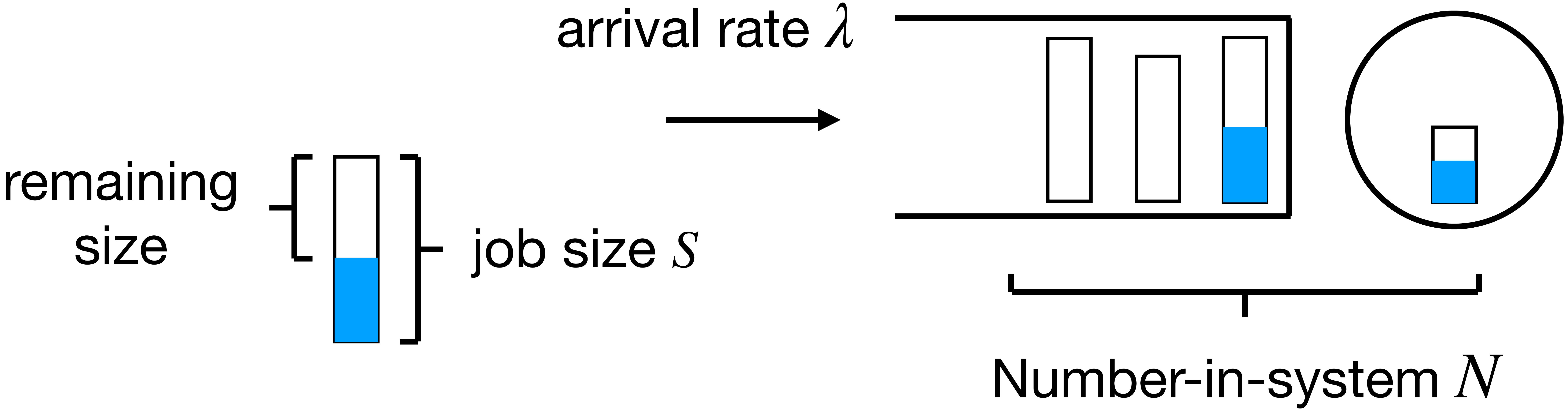

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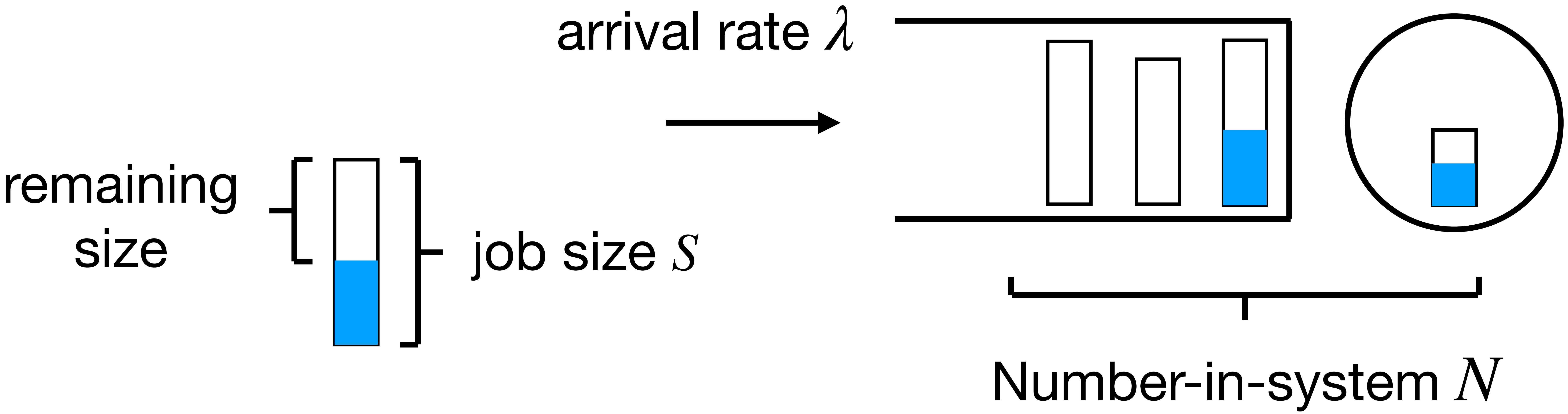
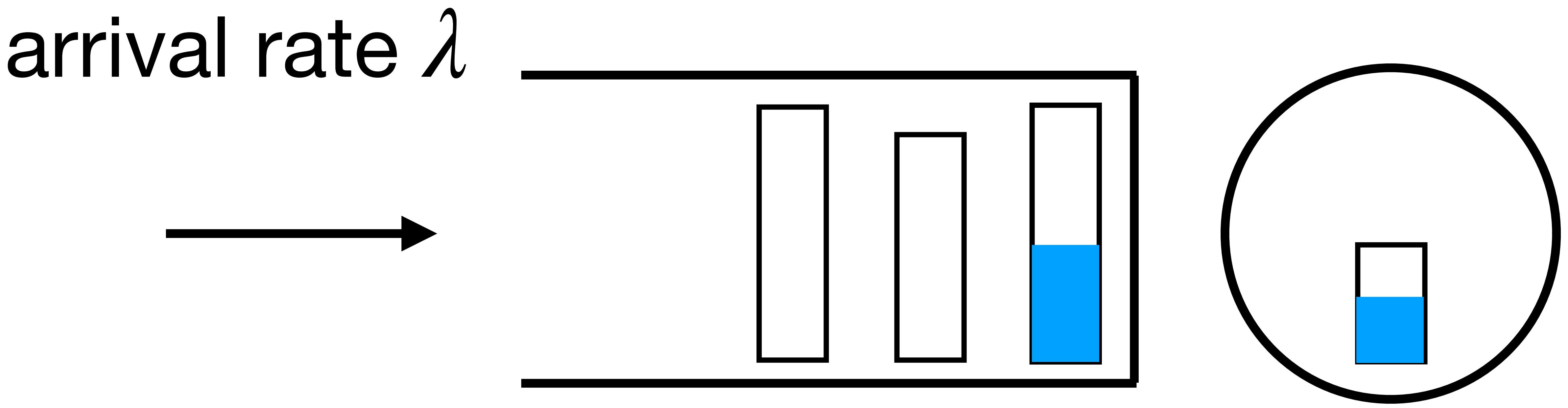


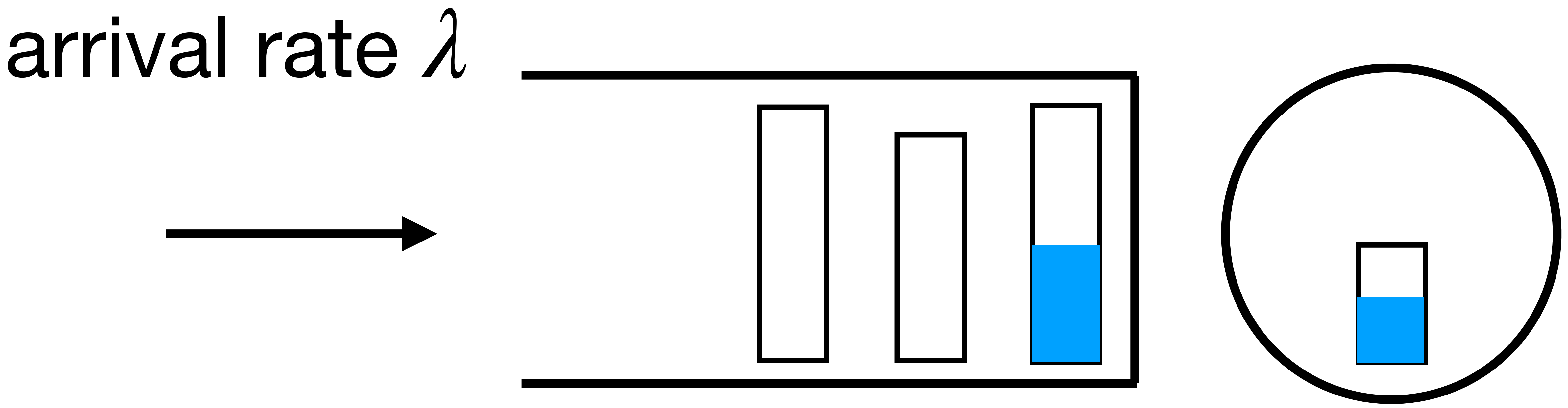
Goal:
Find the scheduling policy (order of serving jobs)
that minimizes steady-state $\mathbb{E}[N]$

Simpler case: known job sizes



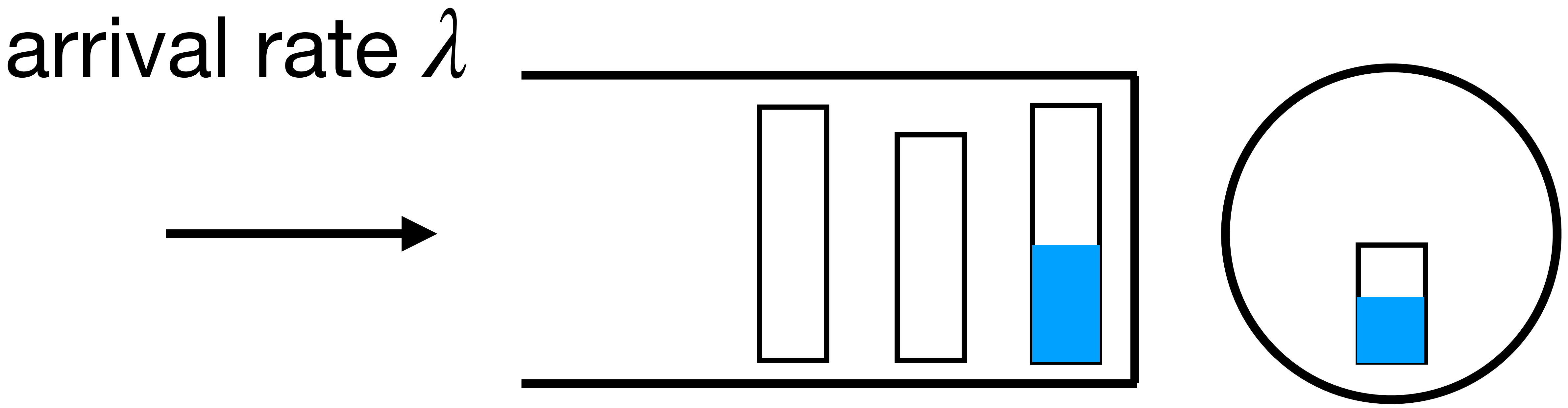
- **Shortest Remaining Processing Time (SRPT)**

Simpler case: known job sizes



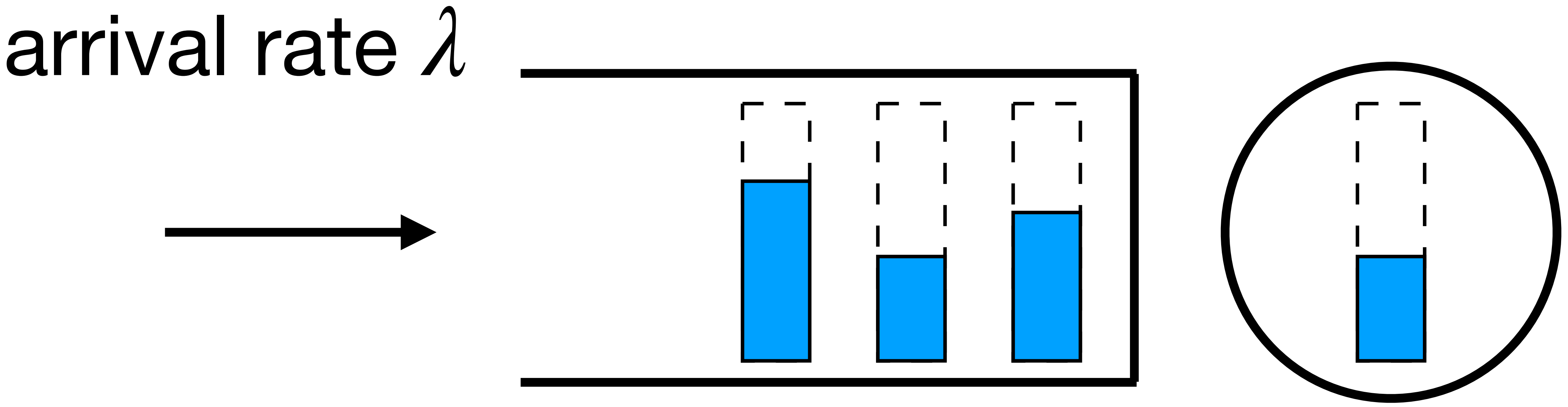
- **Shortest Remaining Processing Time (SRPT)**
  - Minimizes $\mathbb{E}[N]$

Simpler case: known job sizes



- **Shortest Remaining Processing Time (SRPT)**
  - Minimizes $\mathbb{E}[N]$
  - Why: decreases the number as fast as possible

Unknown job sizes



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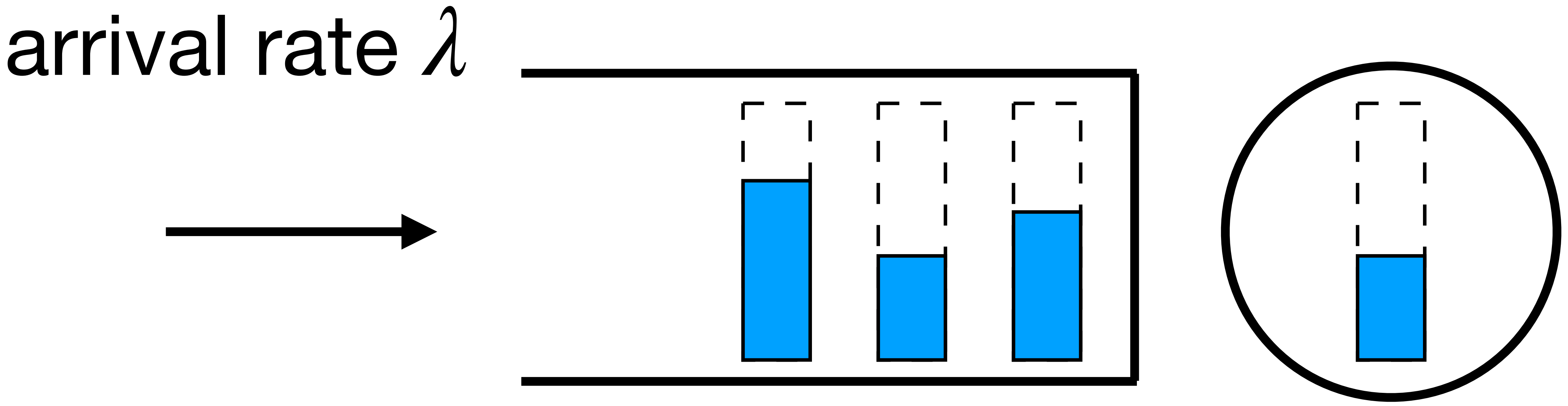
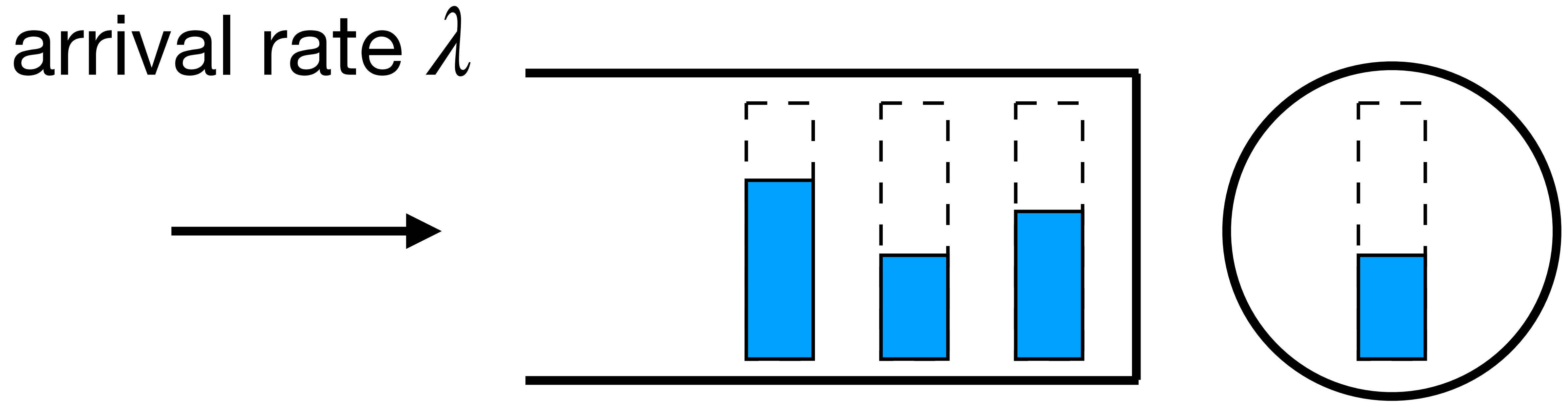


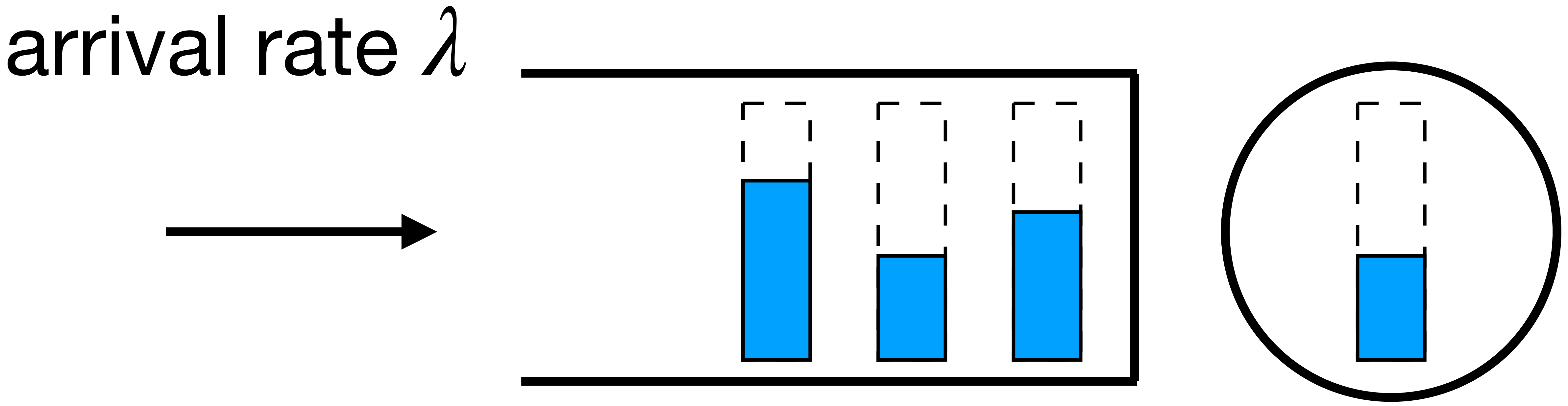
- Sampled from a known distribution

Unknown job sizes



- Sampled from a known distribution
- Prioritize jobs that are *likely* to have small remaining size

Unknown job sizes



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- Prioritize jobs that are *likely* to have small remaining size
- Infer remaining size from *age*

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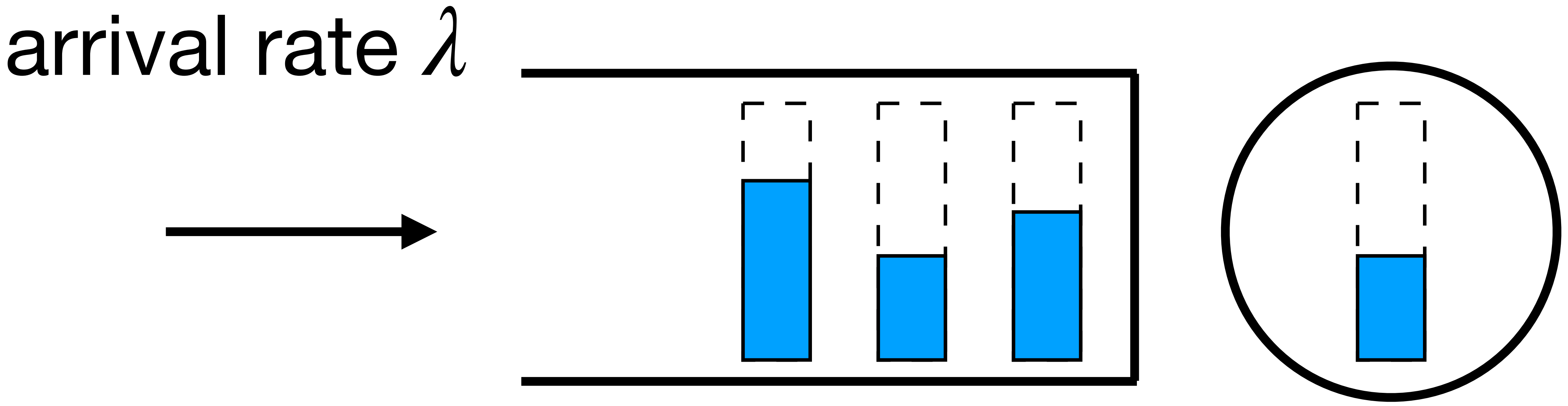

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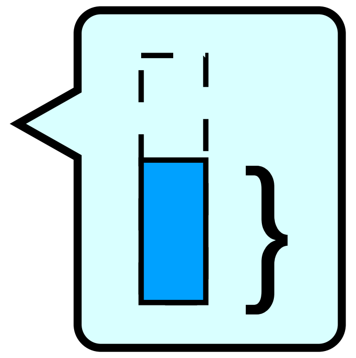

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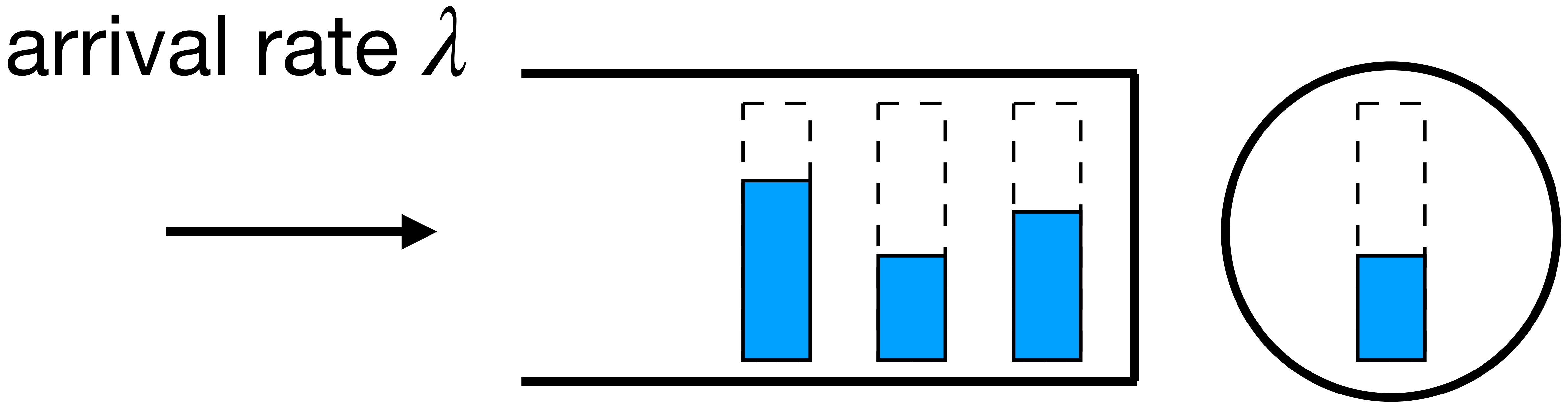

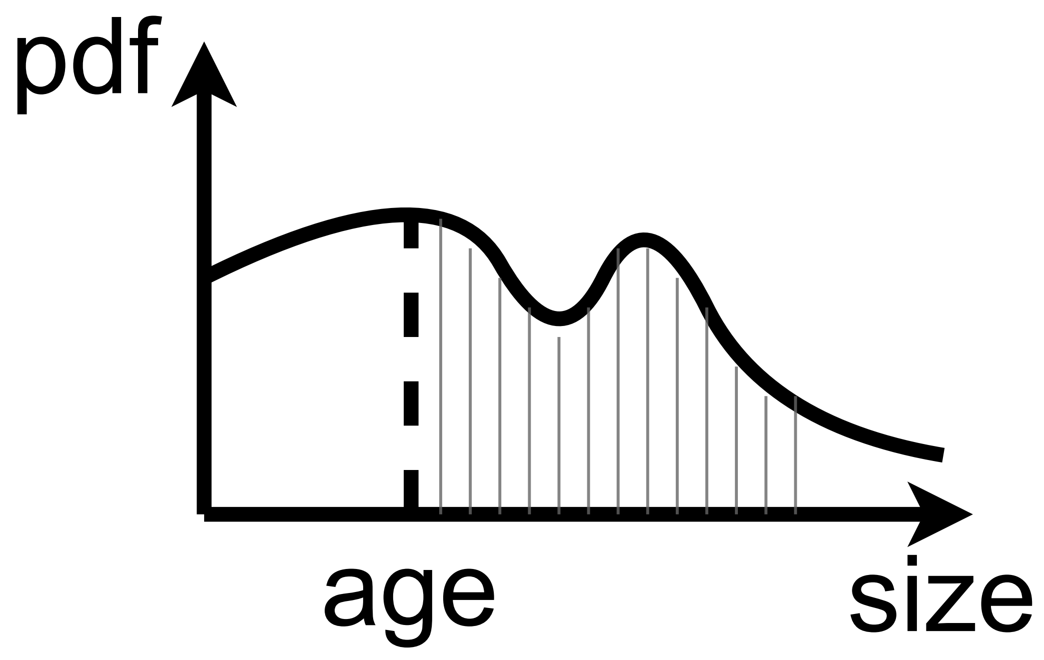

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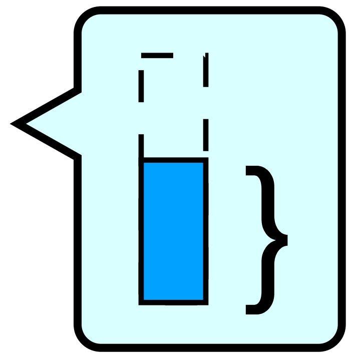

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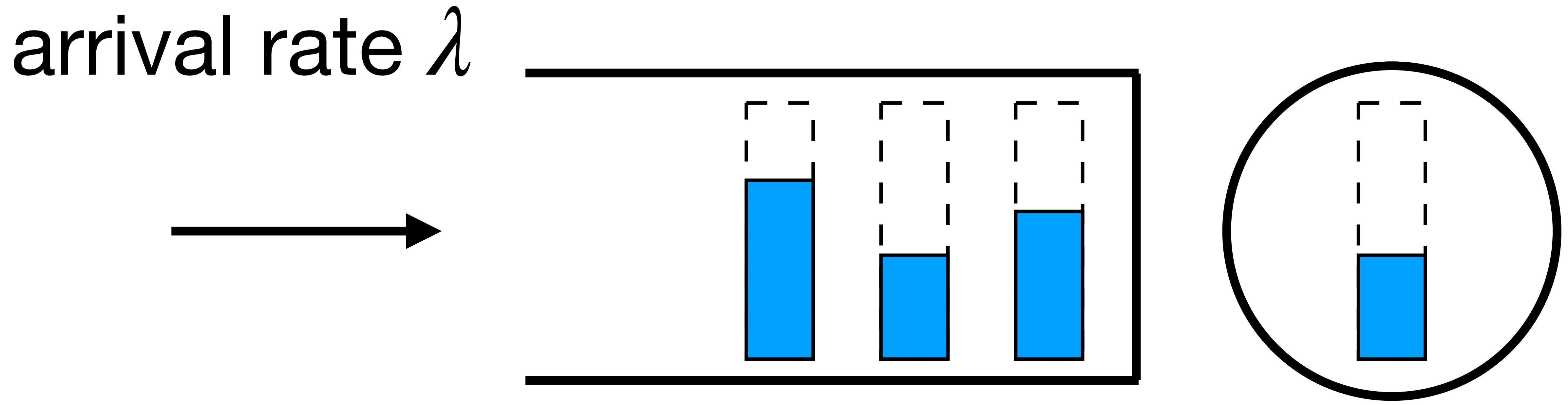
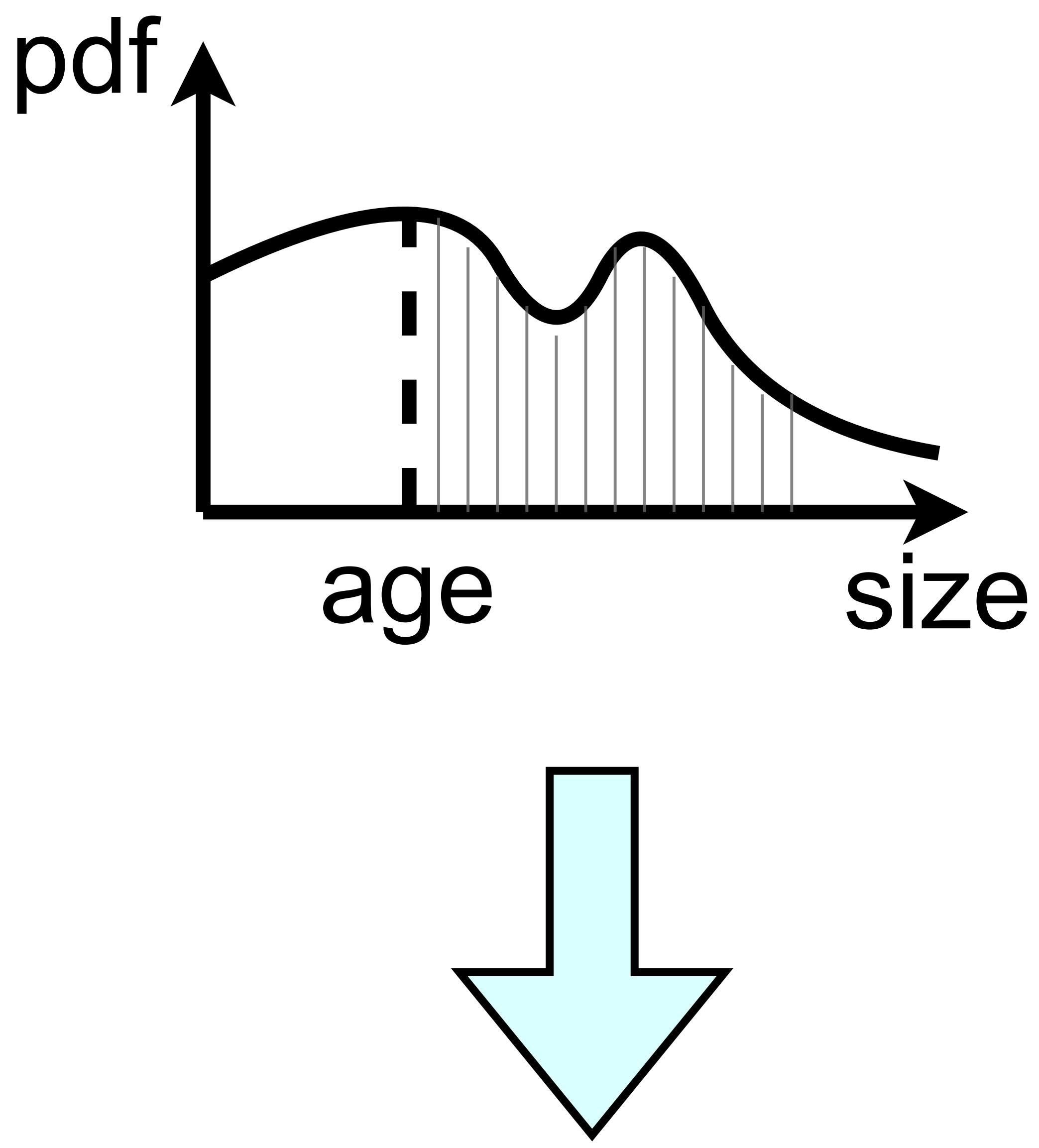


arrival rate λ



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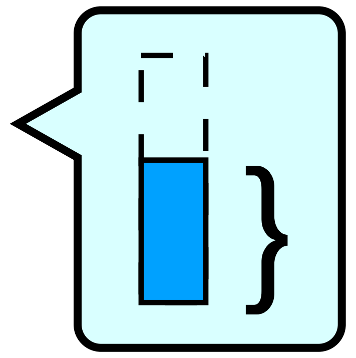

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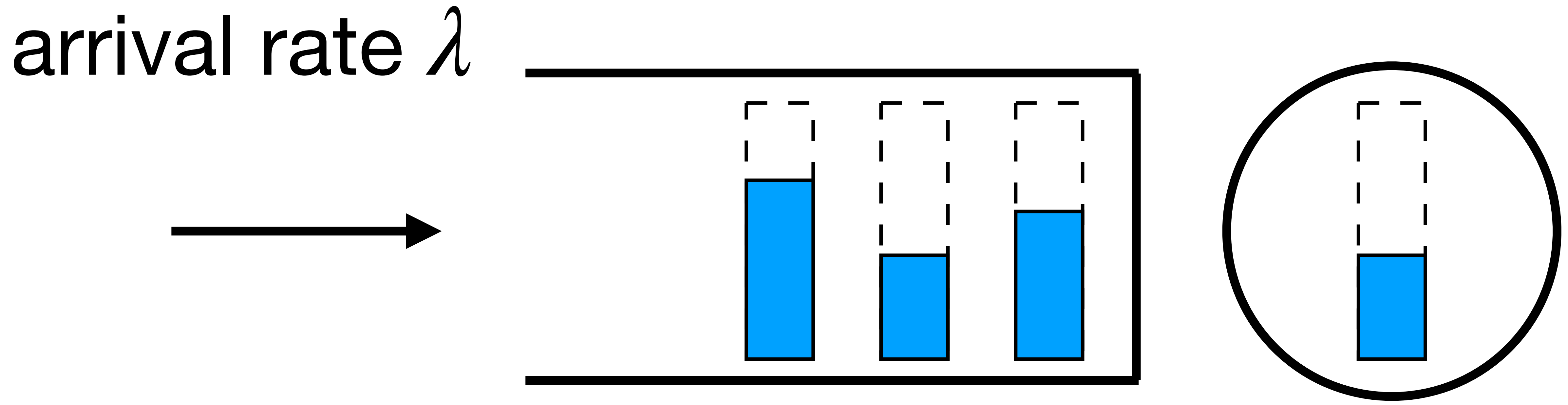

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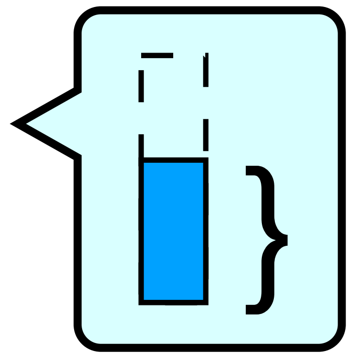

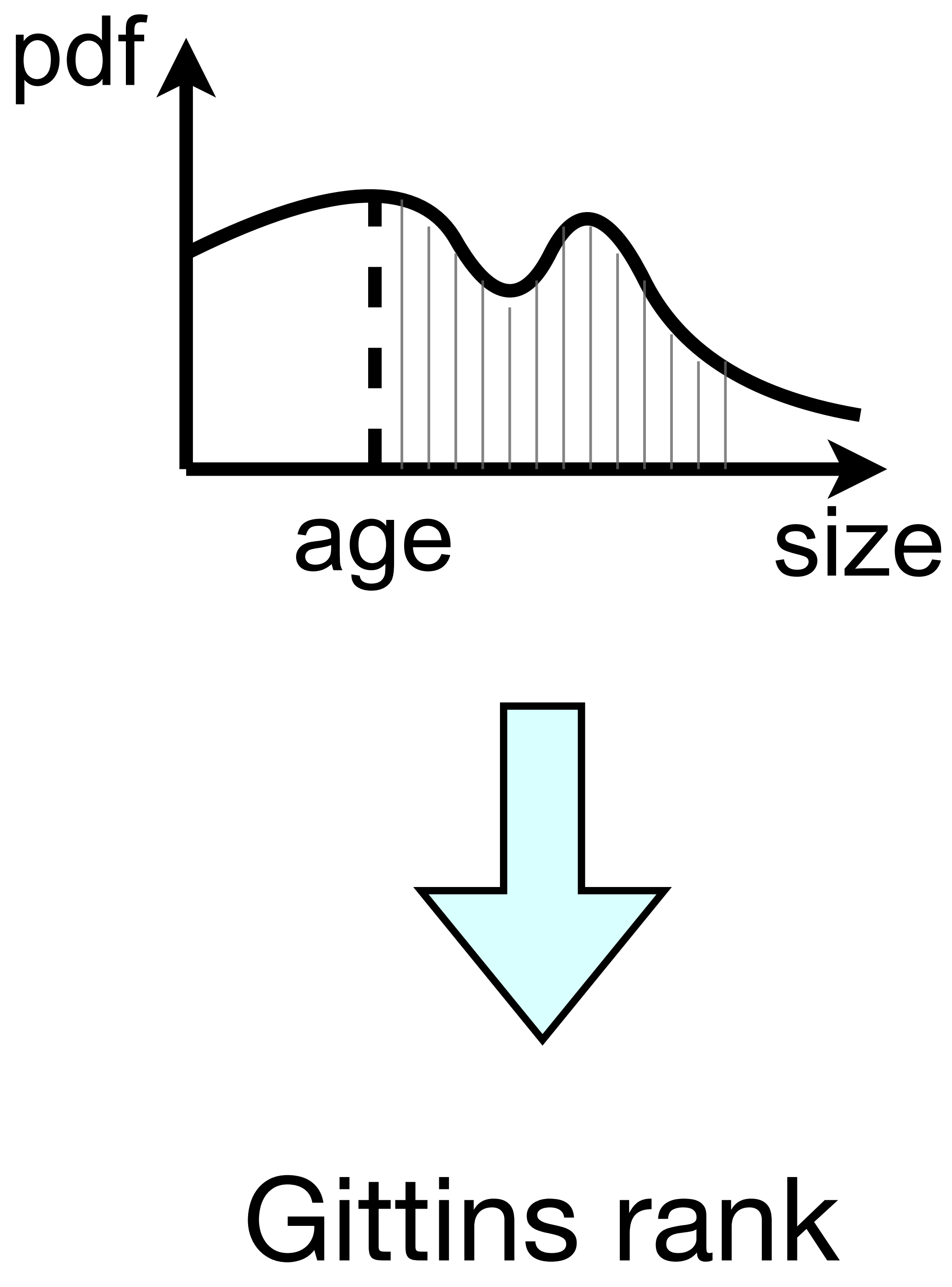

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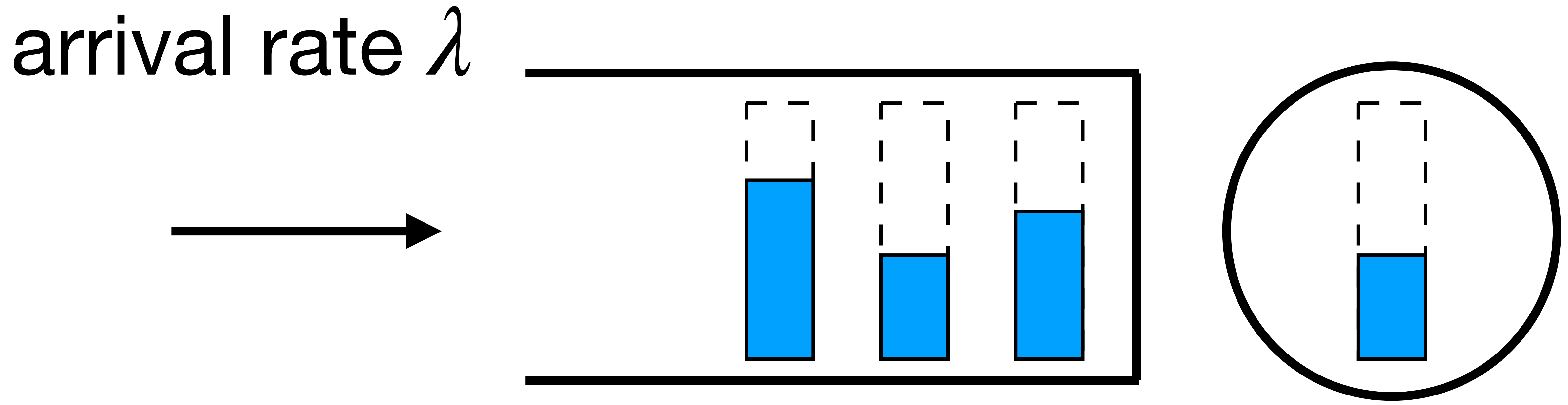

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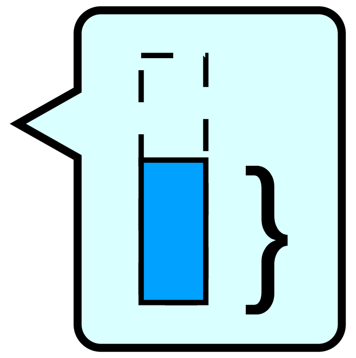
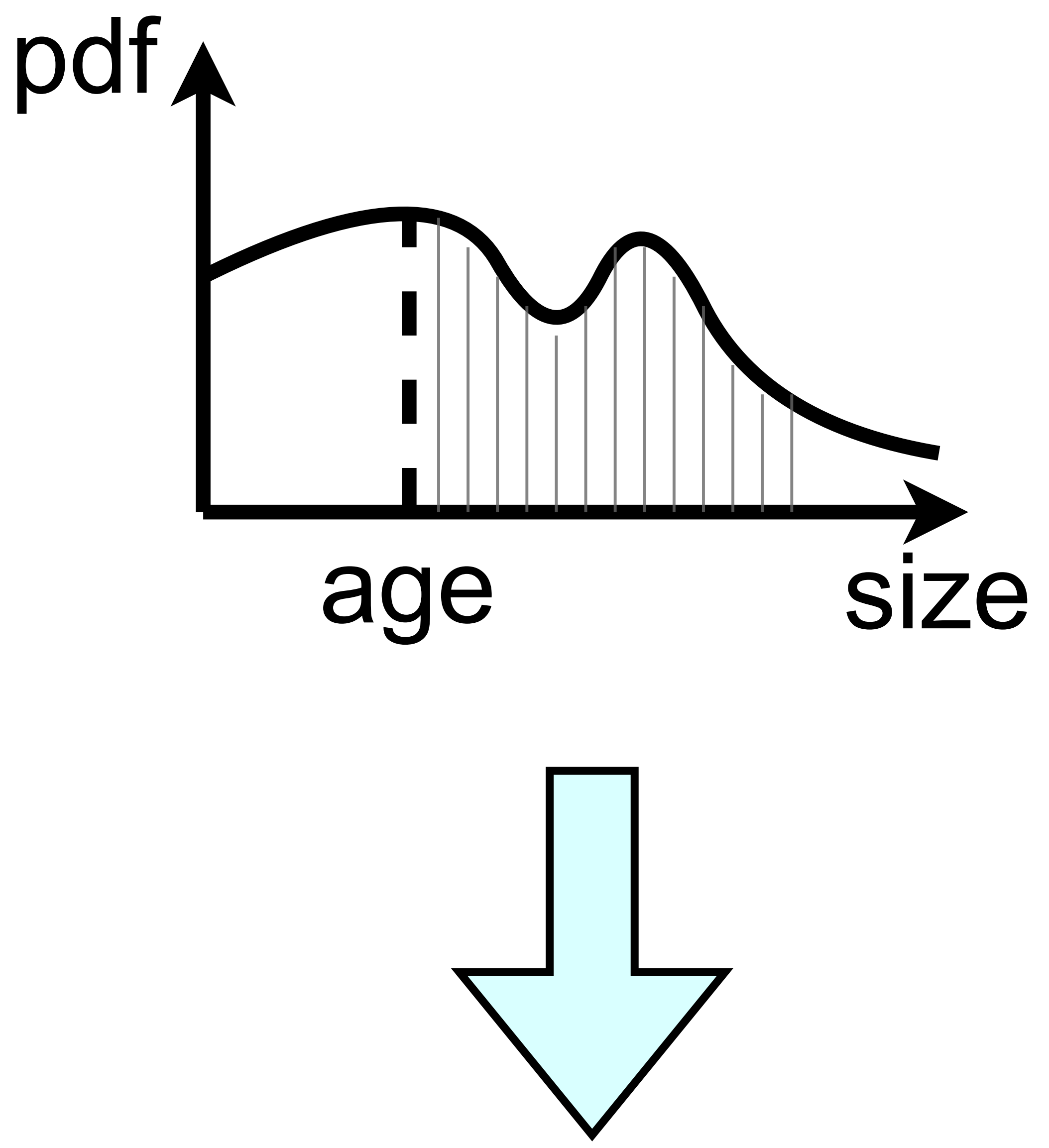
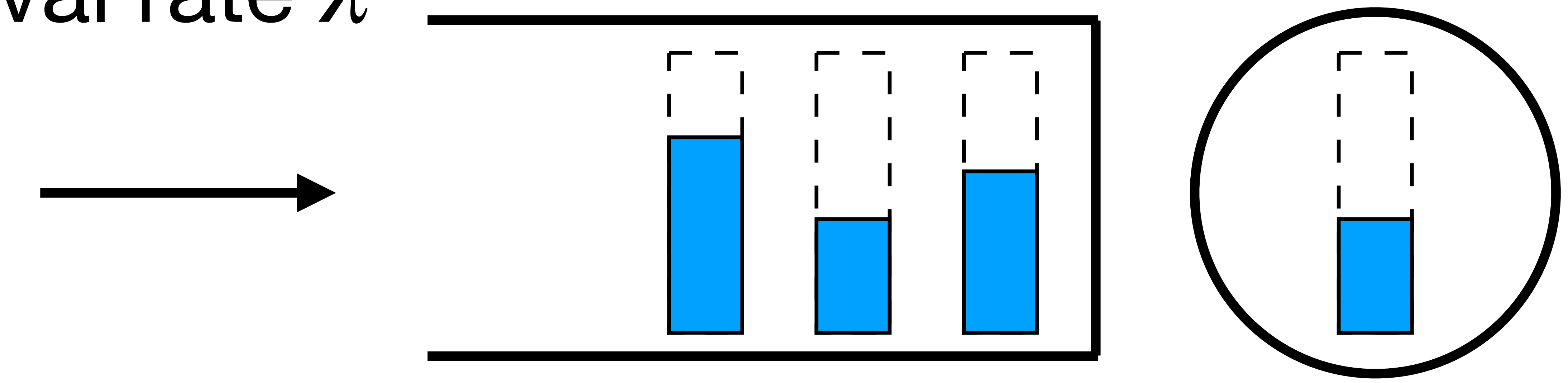


- **Gittins policy**

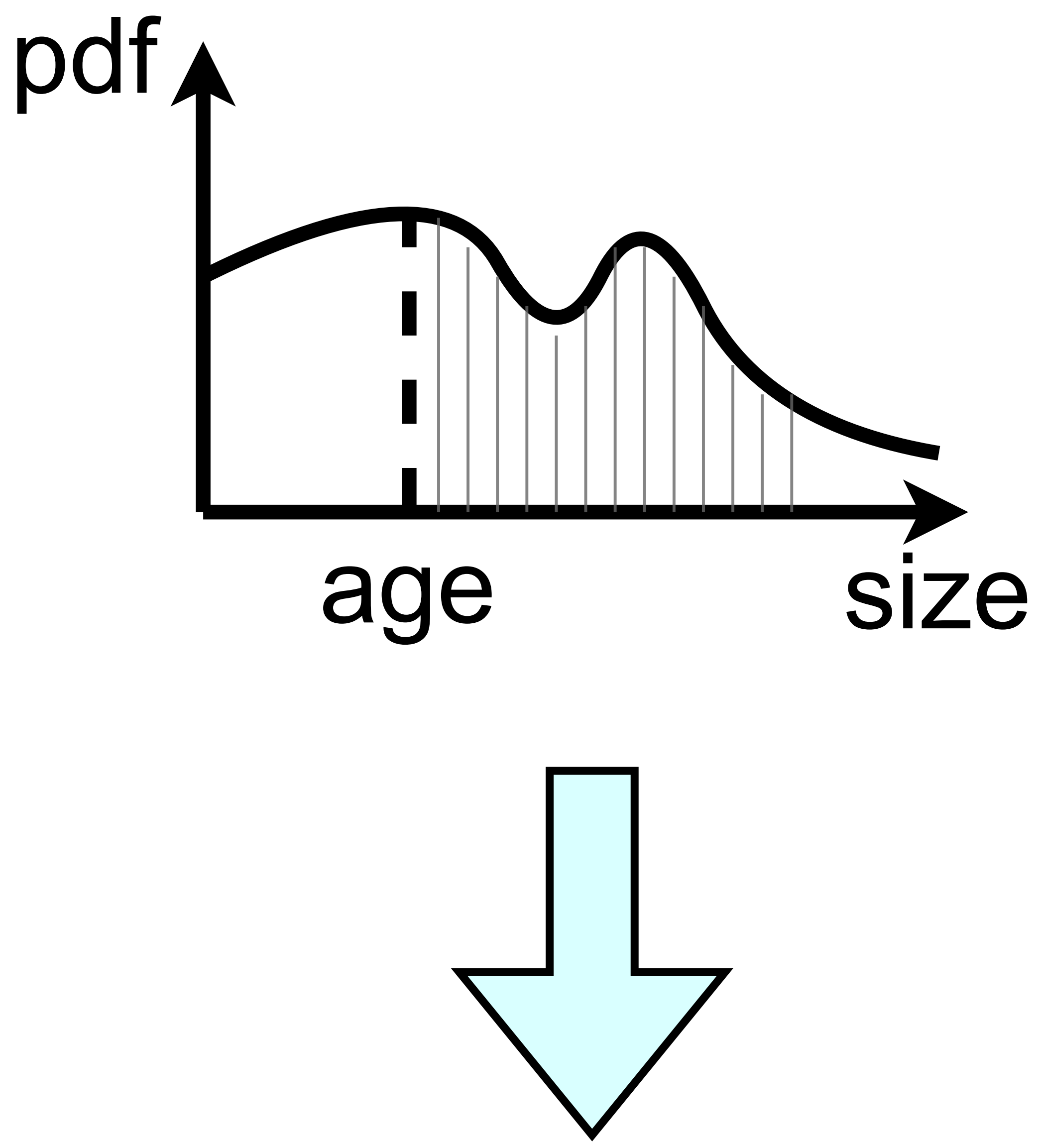


Gittins rank

Unknown job sizes



arrival rate λ



Gittins rank

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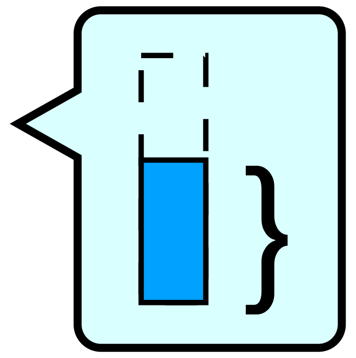
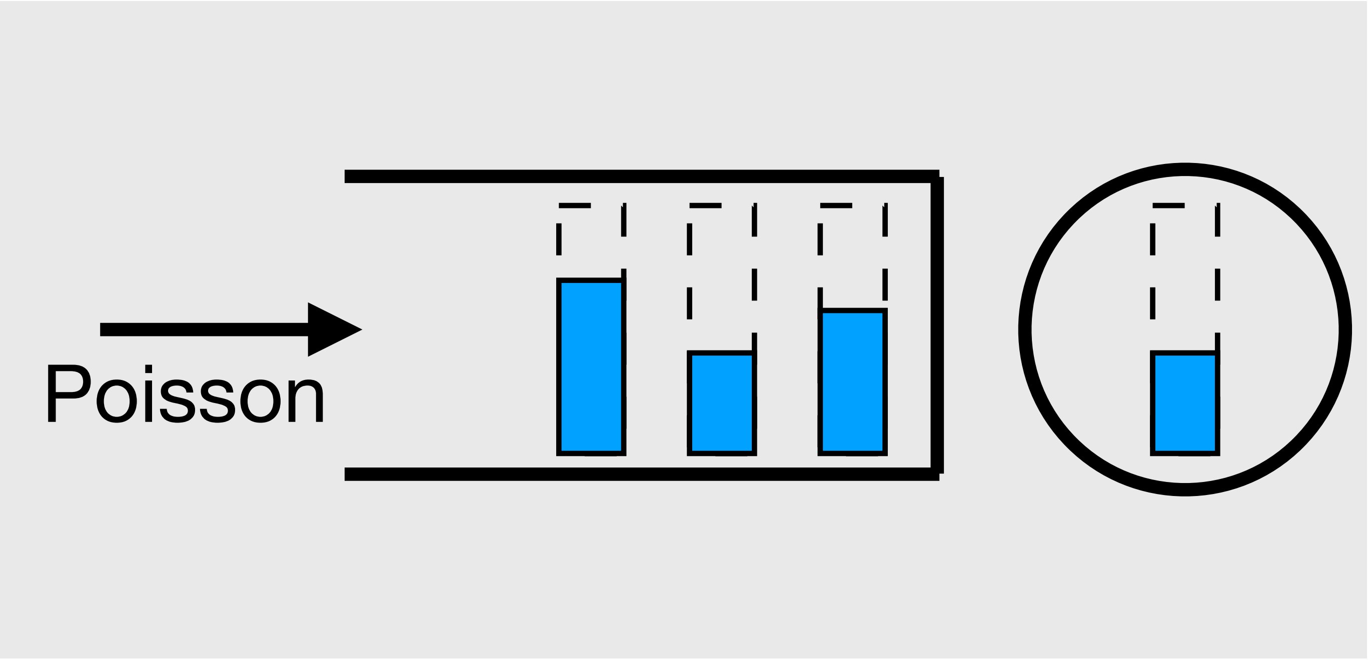
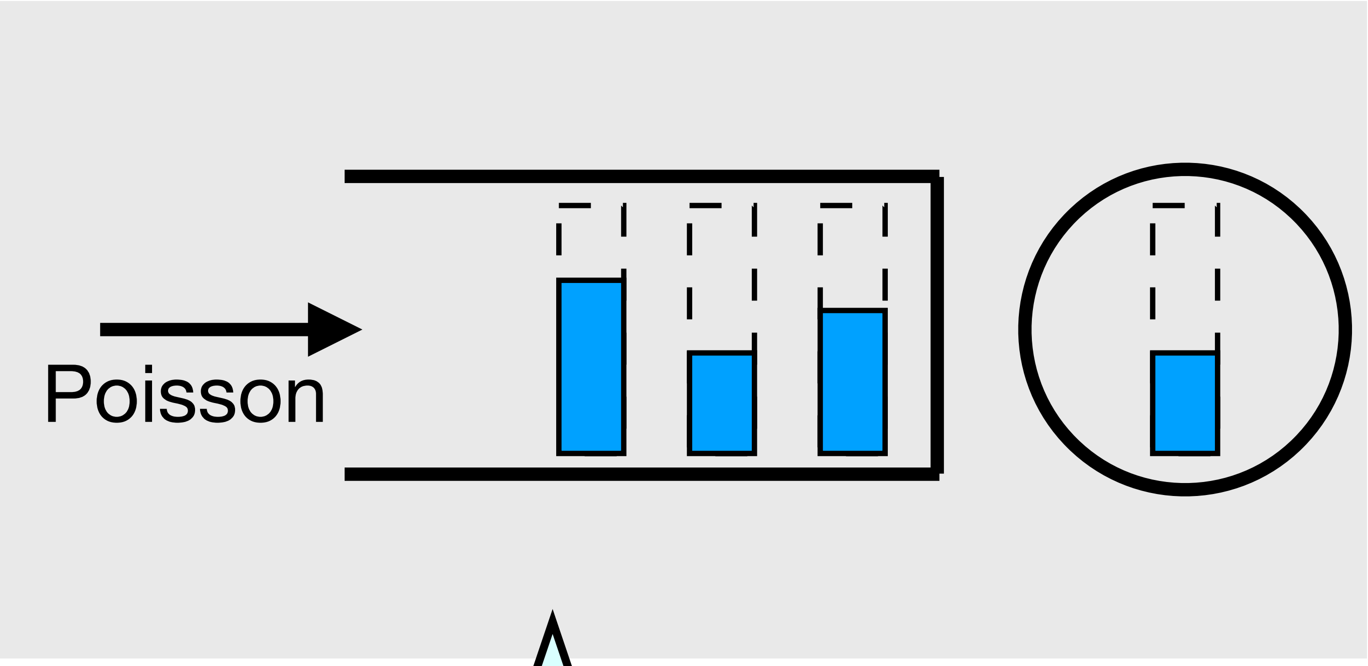


- **Gittins policy**
- Optimal if arrivals are Poisson (M/G/1 system)

M/G/1



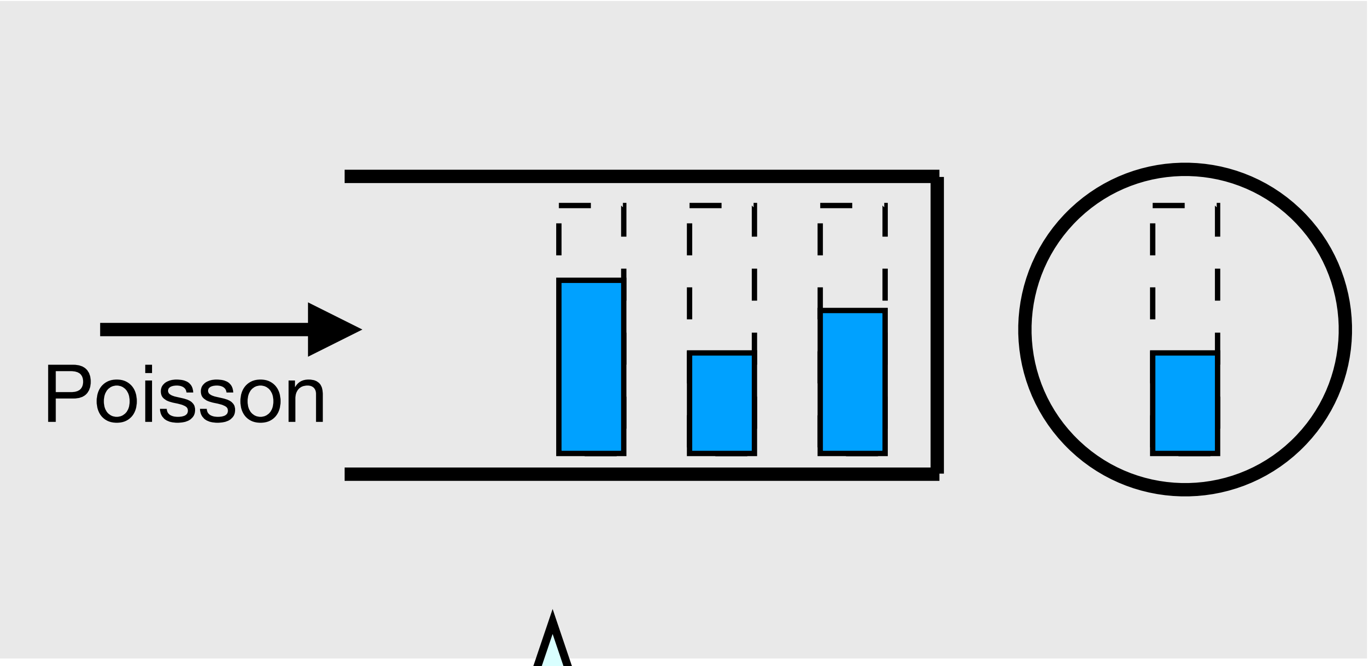
M/G/1



[Gittins79]:
Gittins policy optimal

Complicated systems

M/G/1



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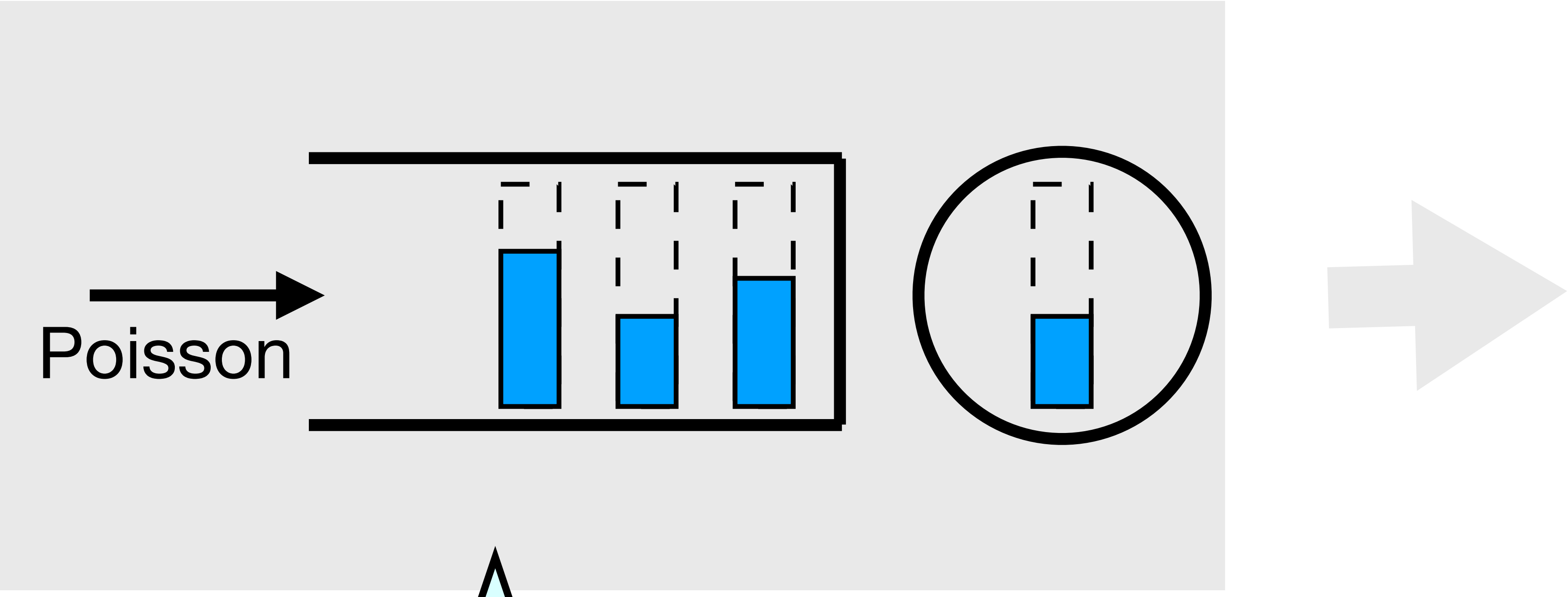

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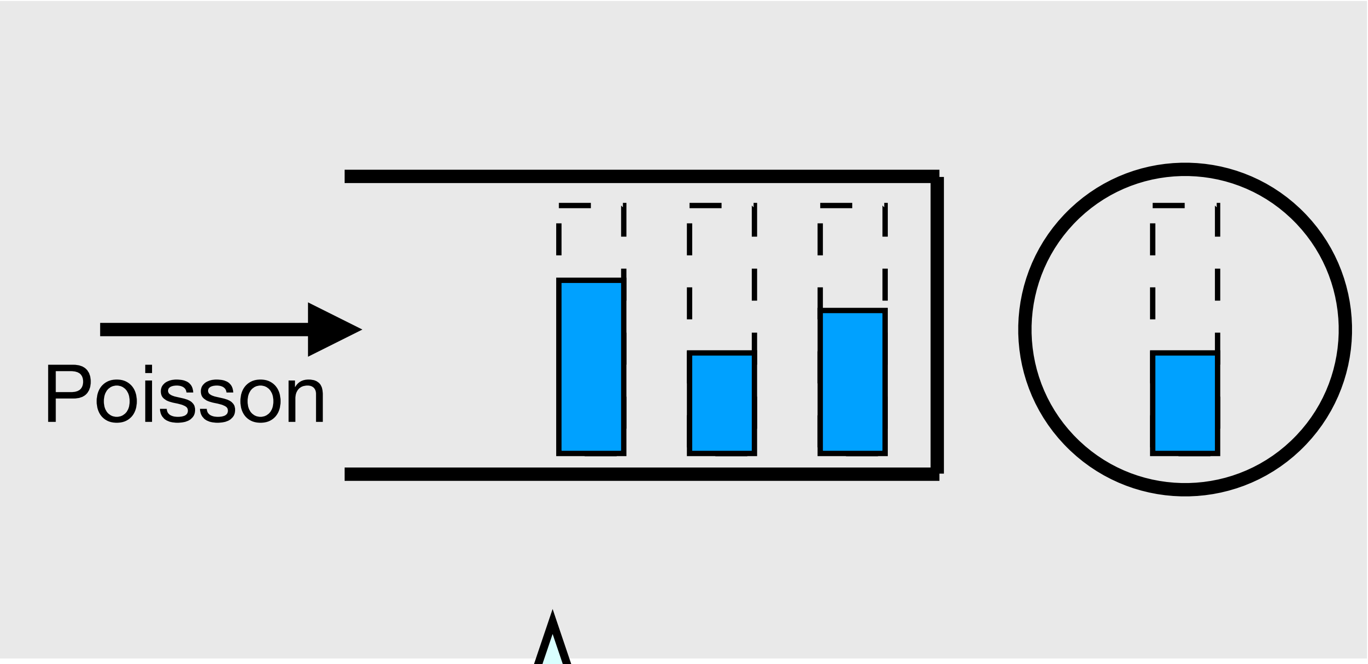

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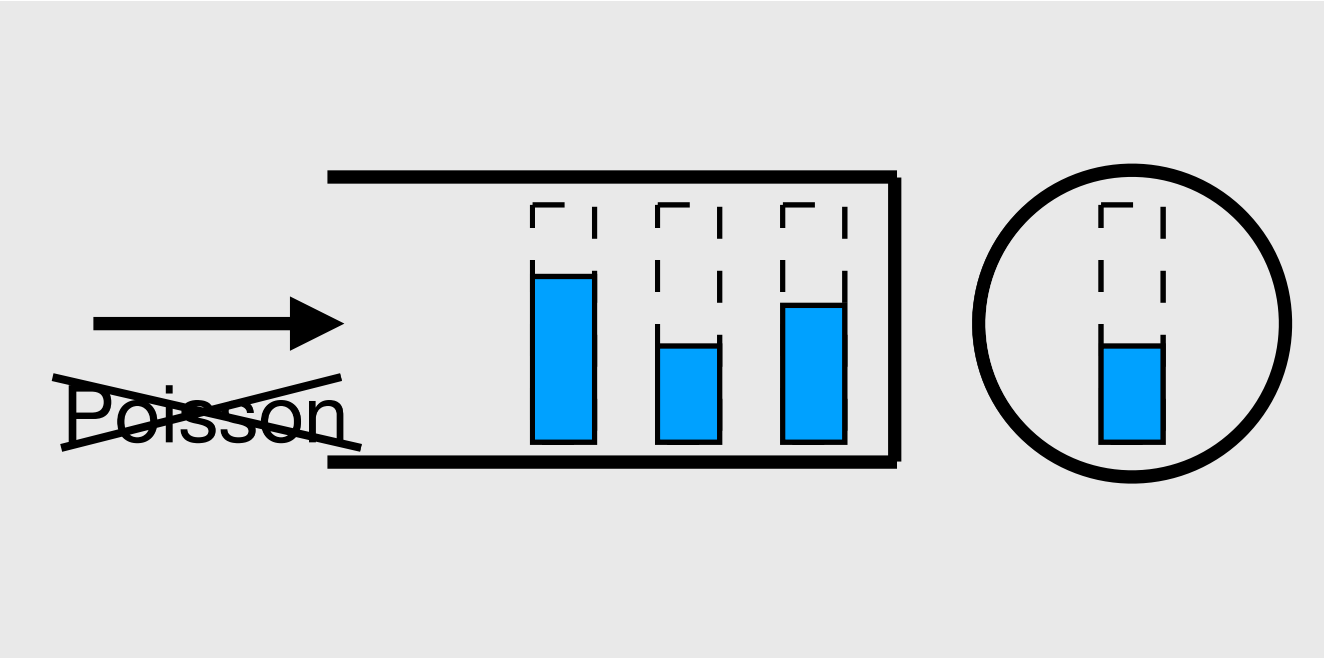
G/G/1



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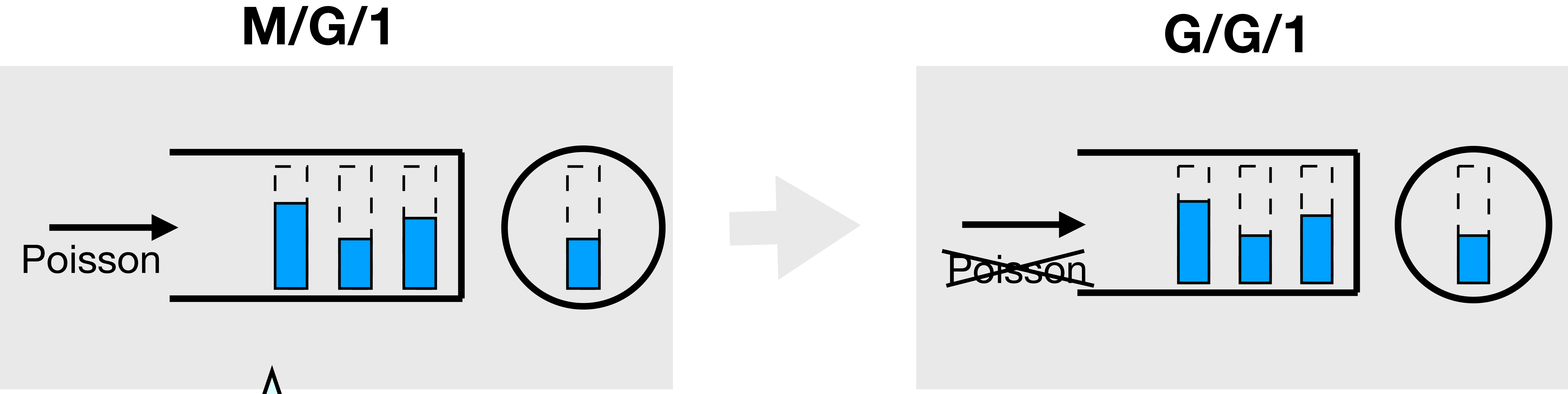

[Gittins79]:
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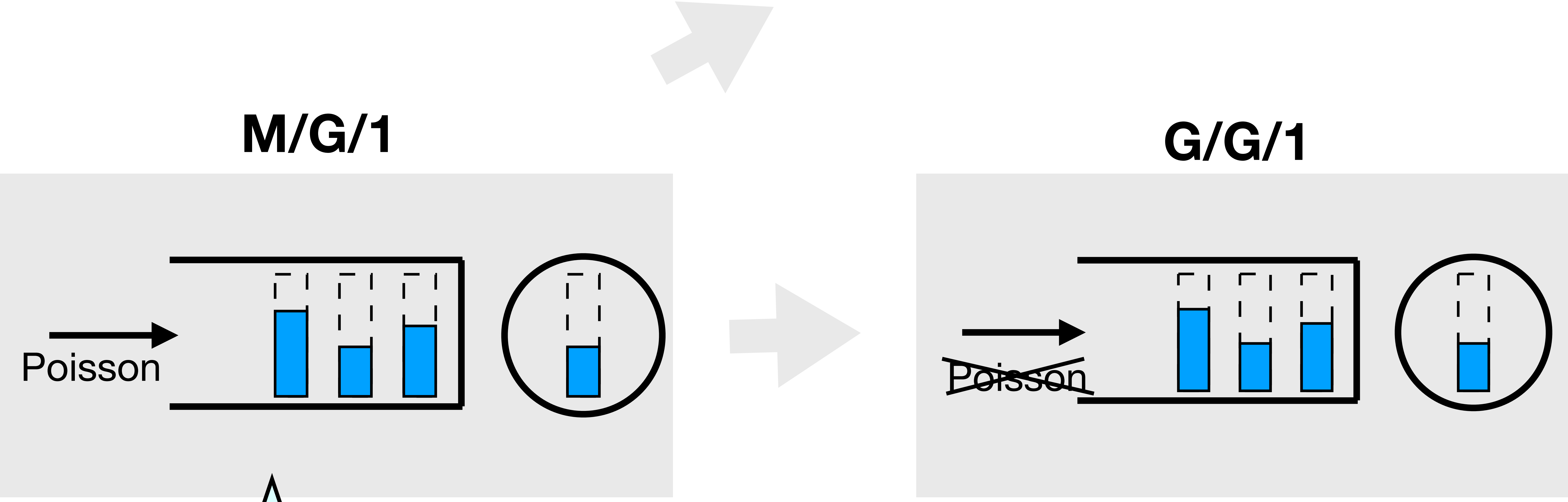
Optimal unknown

Complicated systems



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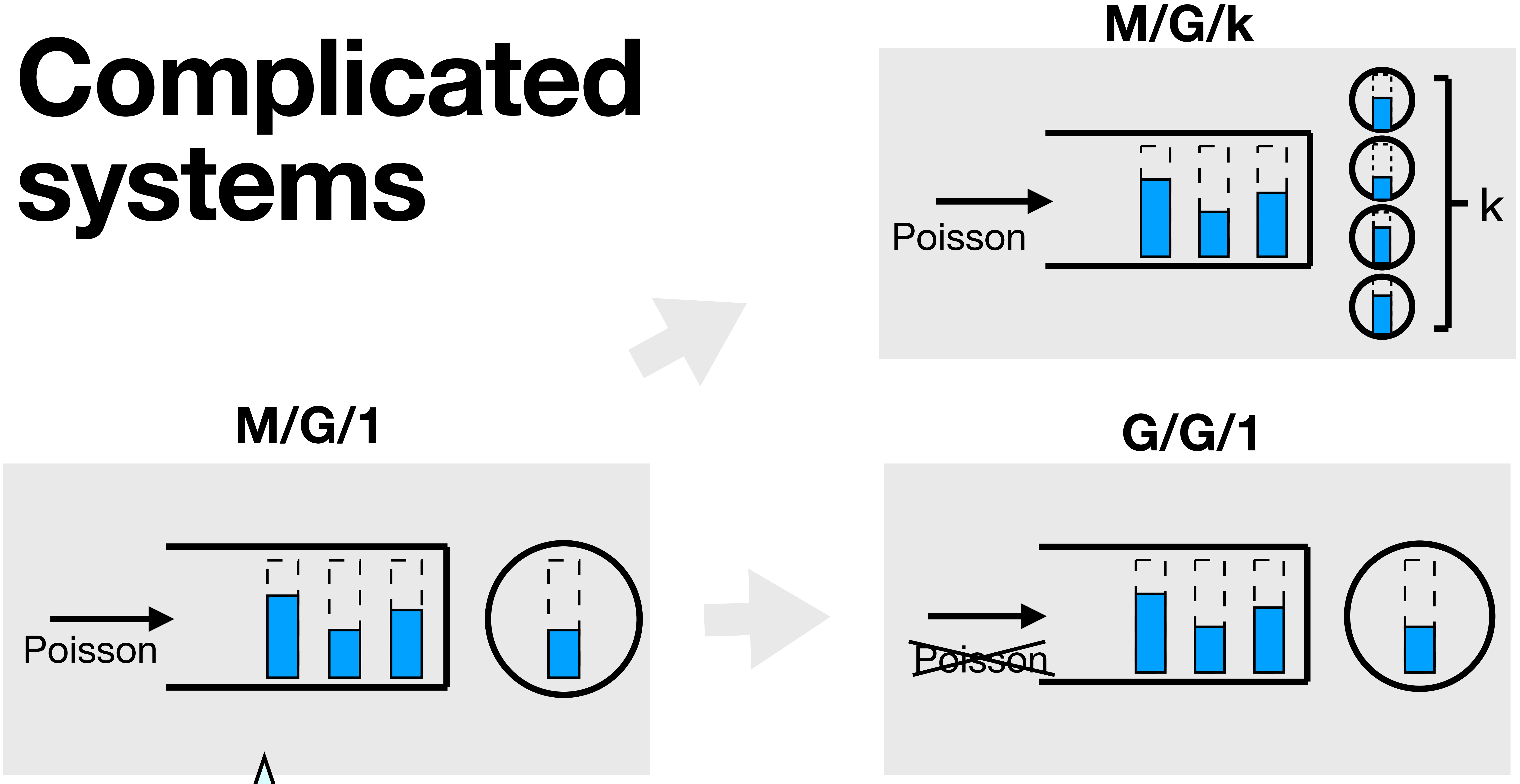

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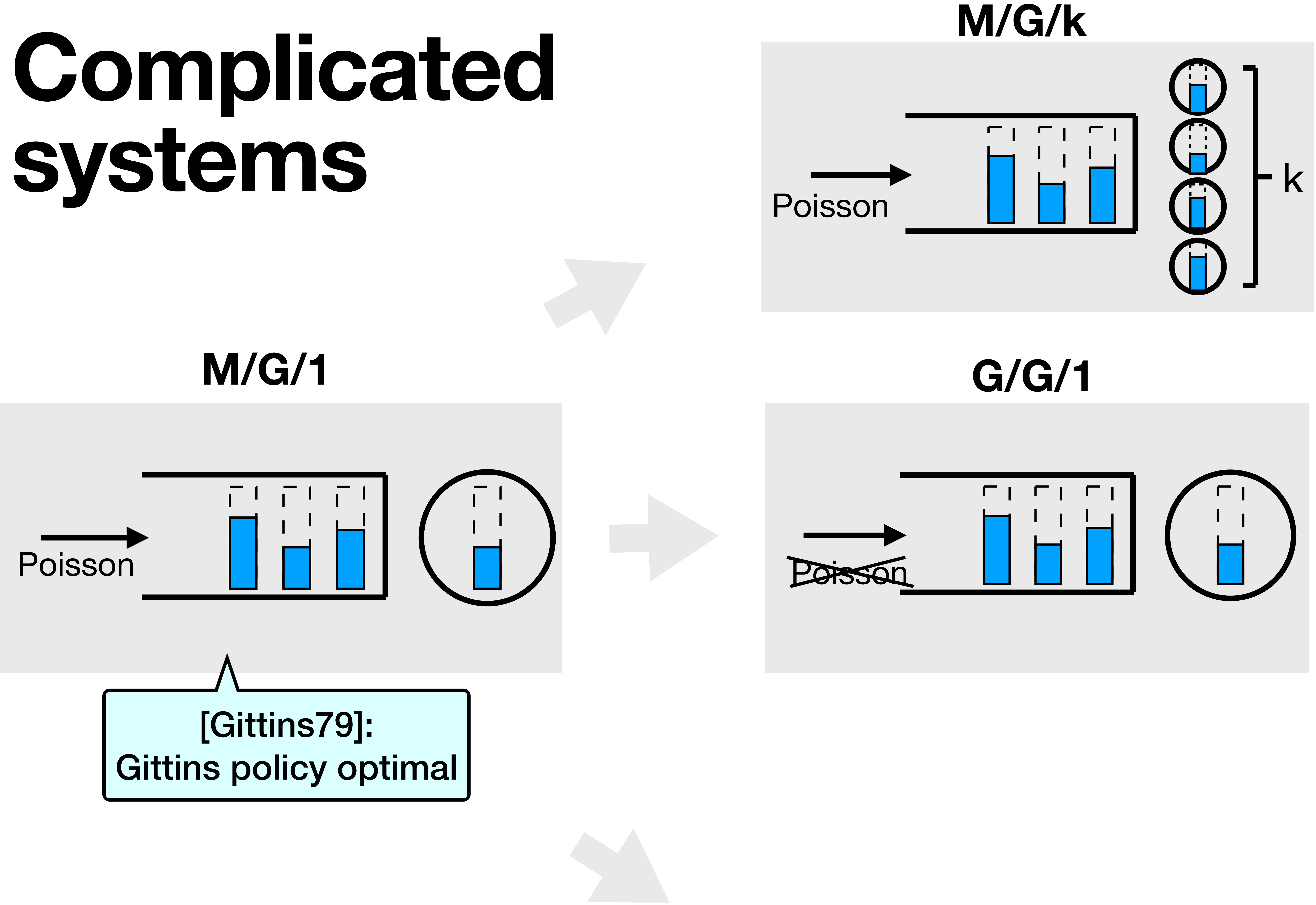

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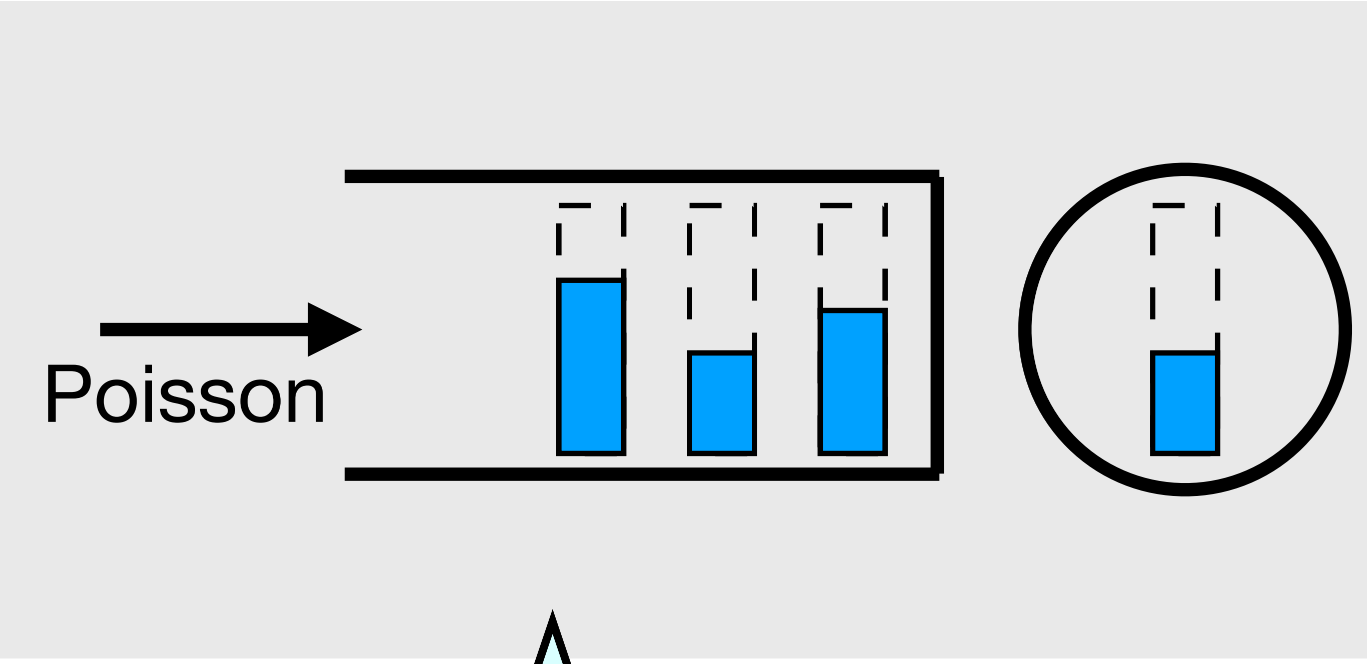

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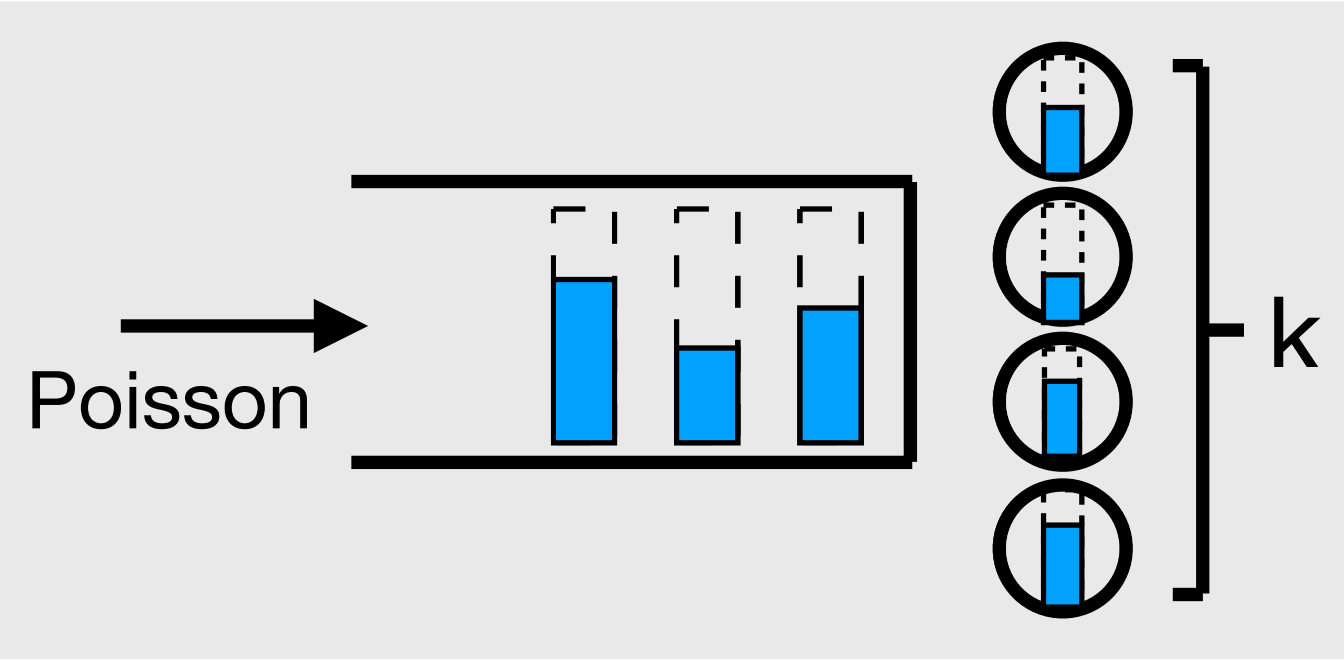
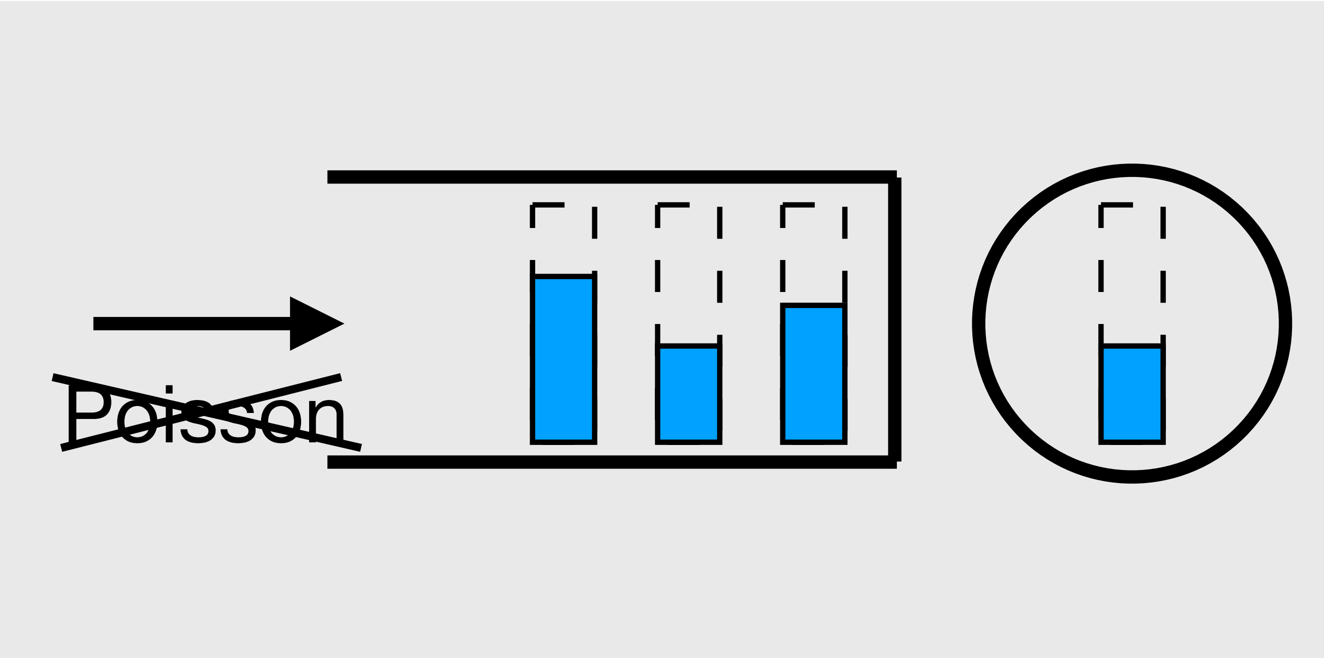
M/G/k

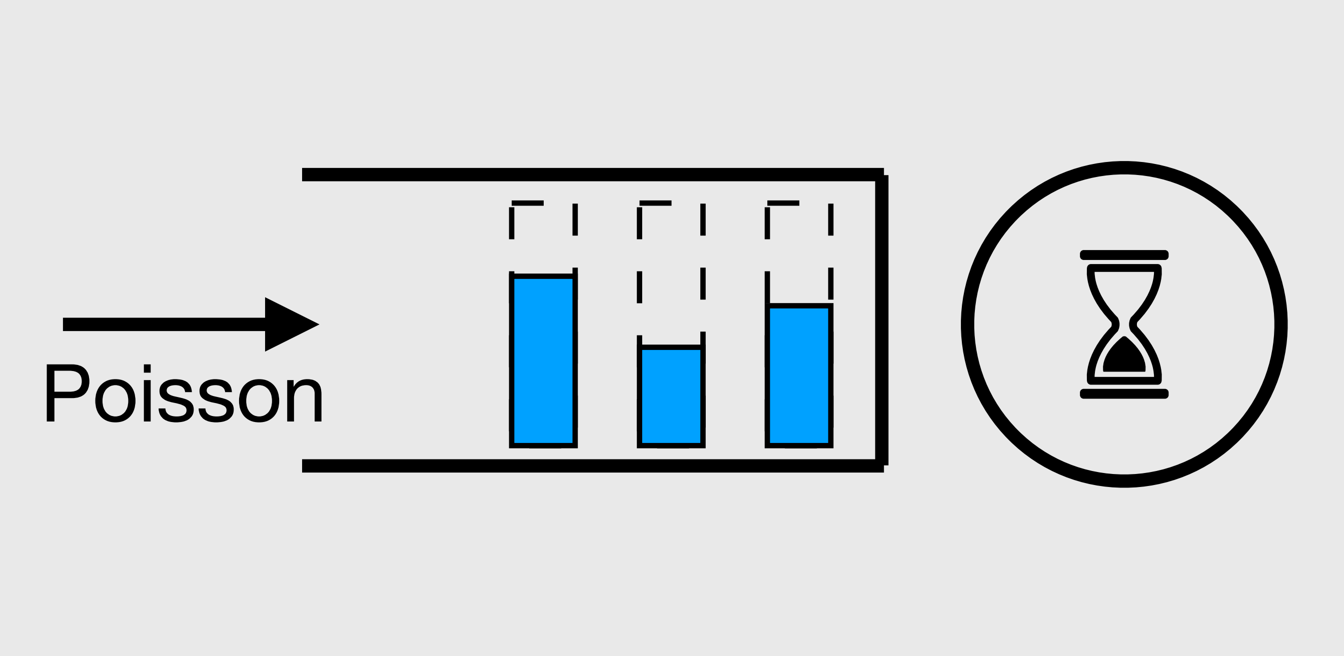


G/G/1

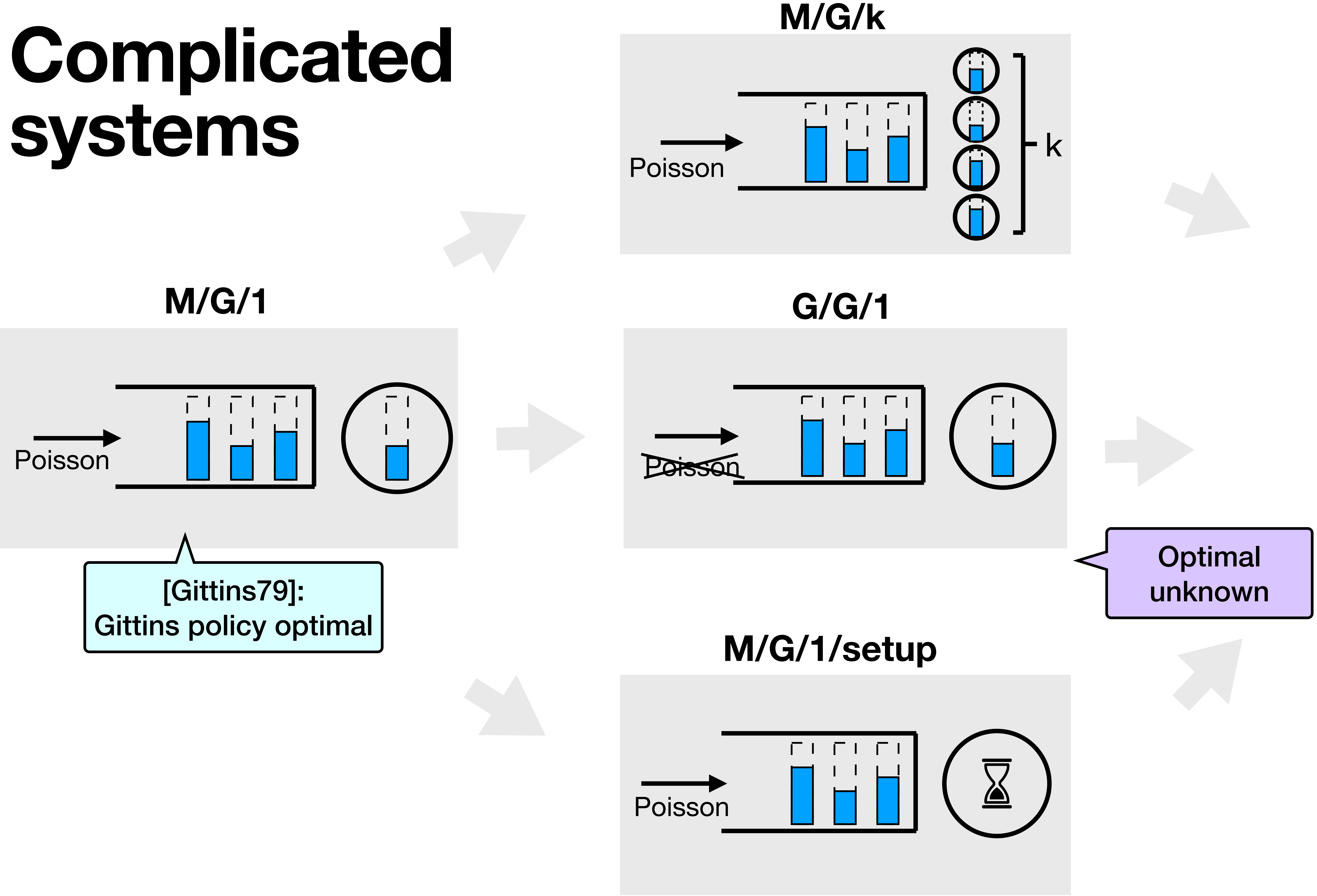


Optimal
unknown

M/G/1/setup



Complicated systems



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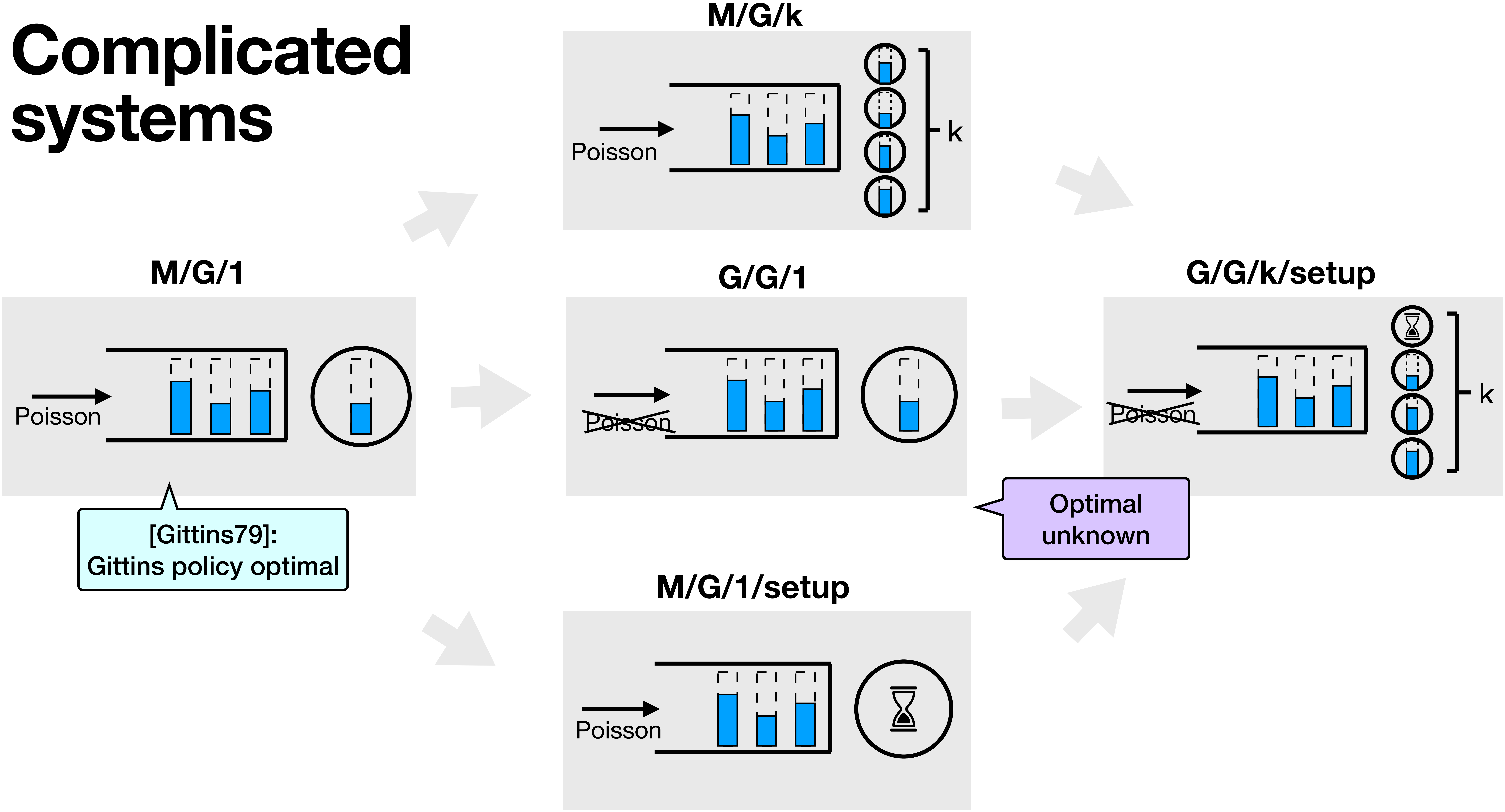


M/G/1

Poisson

[Gittins79]:
Gittins policy optimal

M/G/k

Poisson

k

G/G/1

~~Poisson~~

M/G/1/setup

Poisson

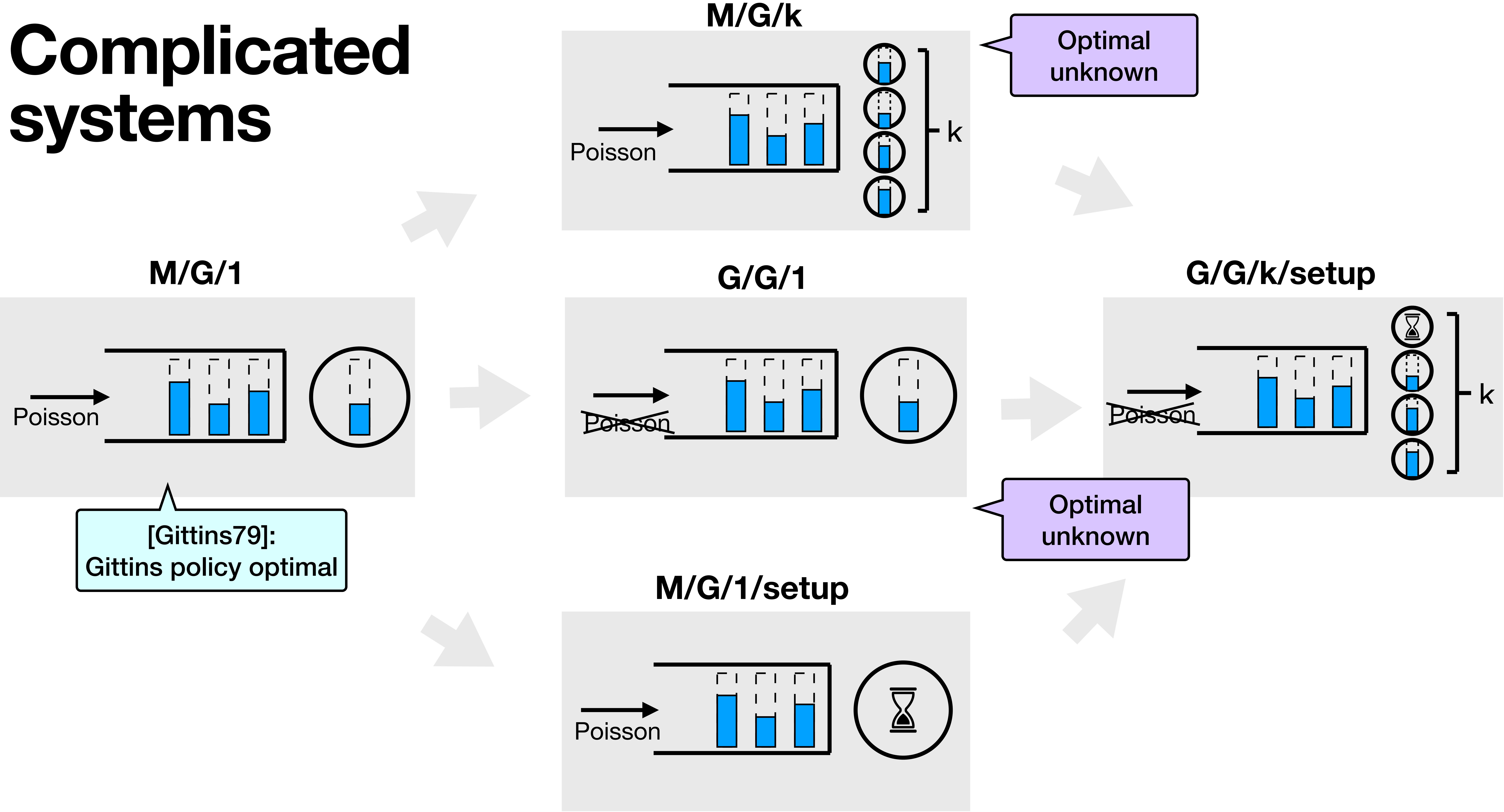
G/G/k/setup

~~Poisson~~

k

Optimal
unknown

Complicated systems



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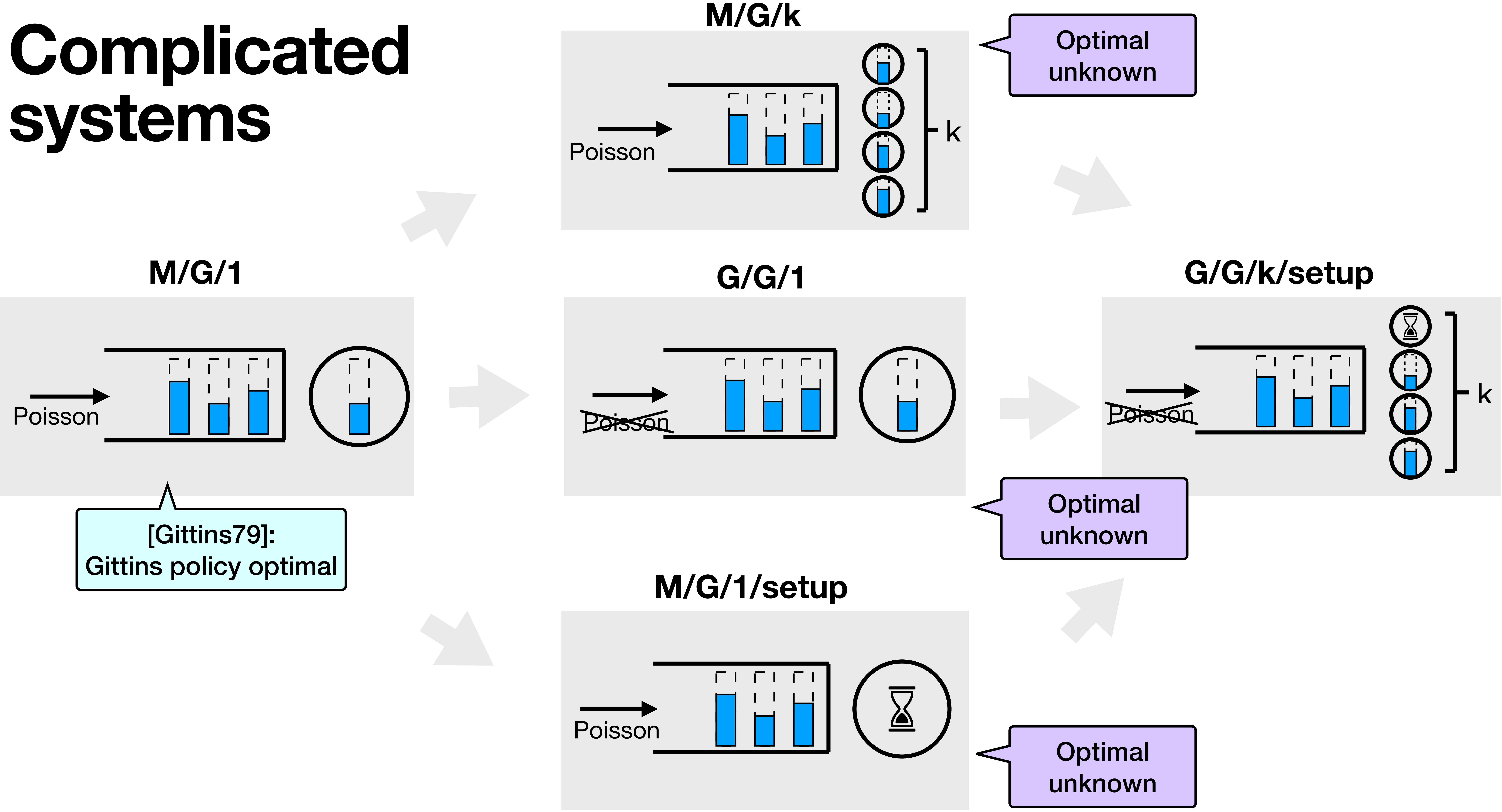

M/G/1

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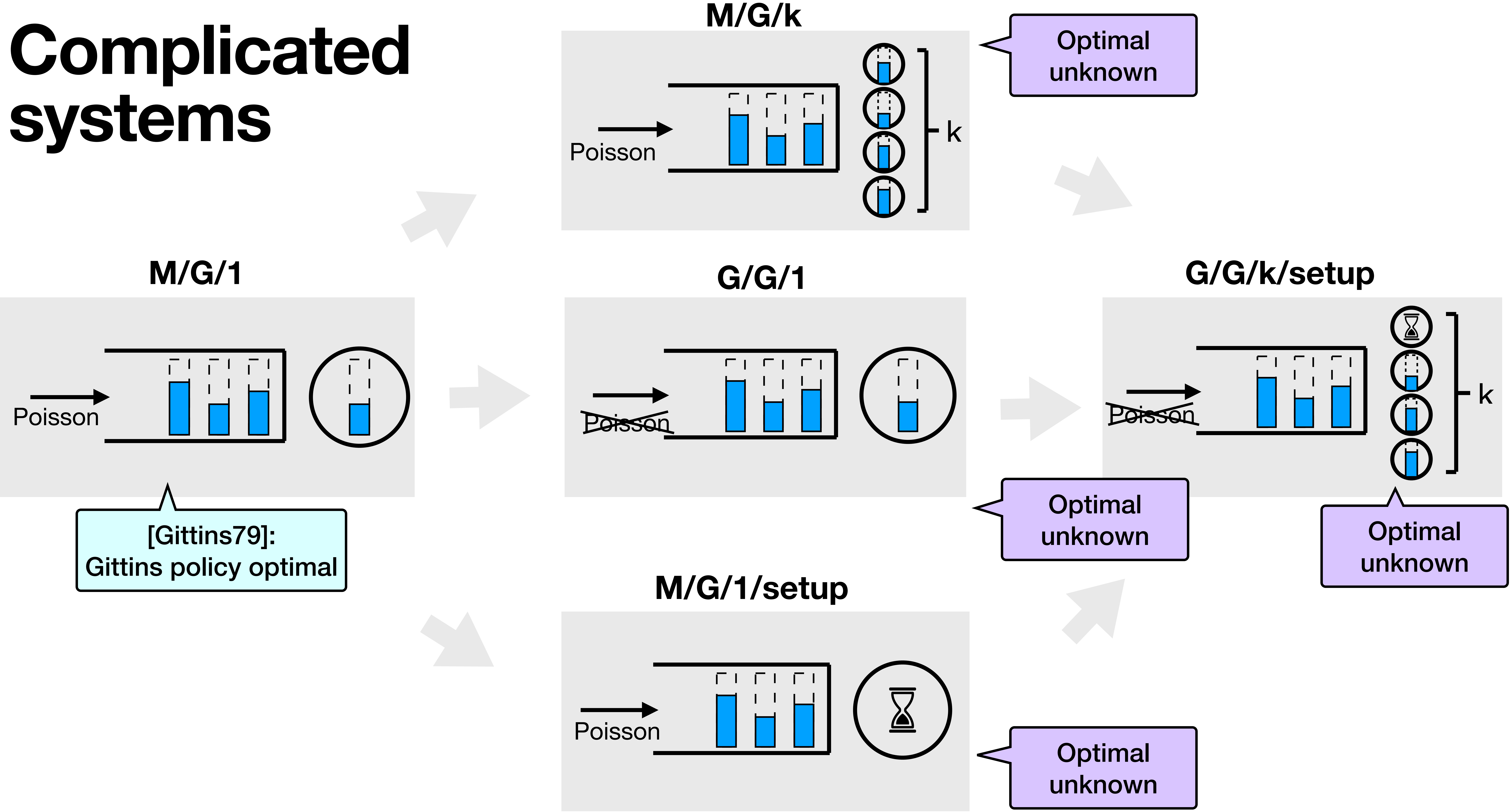

M/G/1

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M/G/k

Optimal unknown

G/G/1

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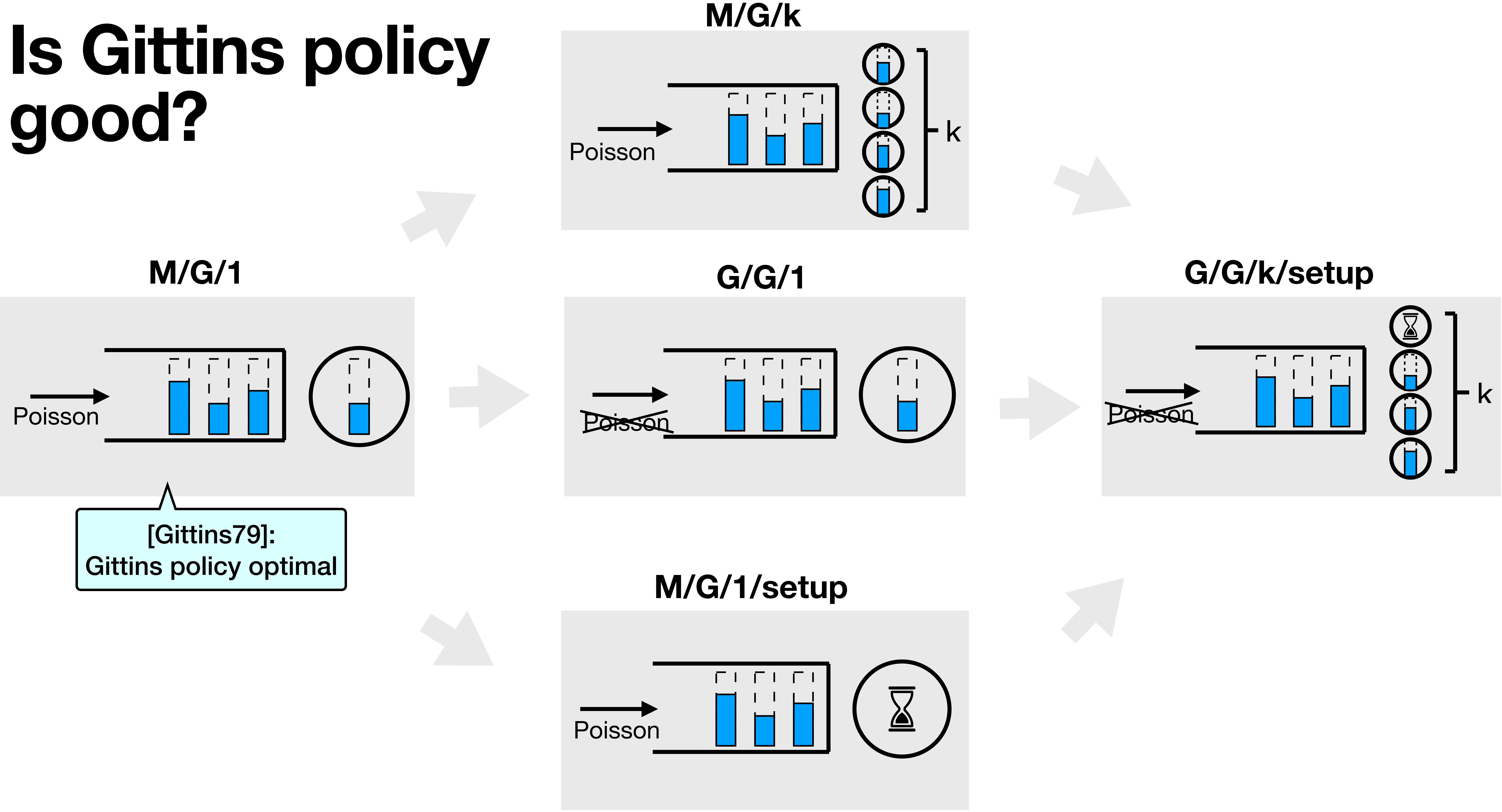
M/G/1/setup

Optimal unknown

G/G/k/setup

Optimal unknown

Is Gittins policy good?



M/G/1

Poisson

[Gittins79]:
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M/G/k

Poisson

k

G/G/1

~~Poisson~~

M/G/1/setup

Poisson

G/G/k/setup

~~Poisson~~

k

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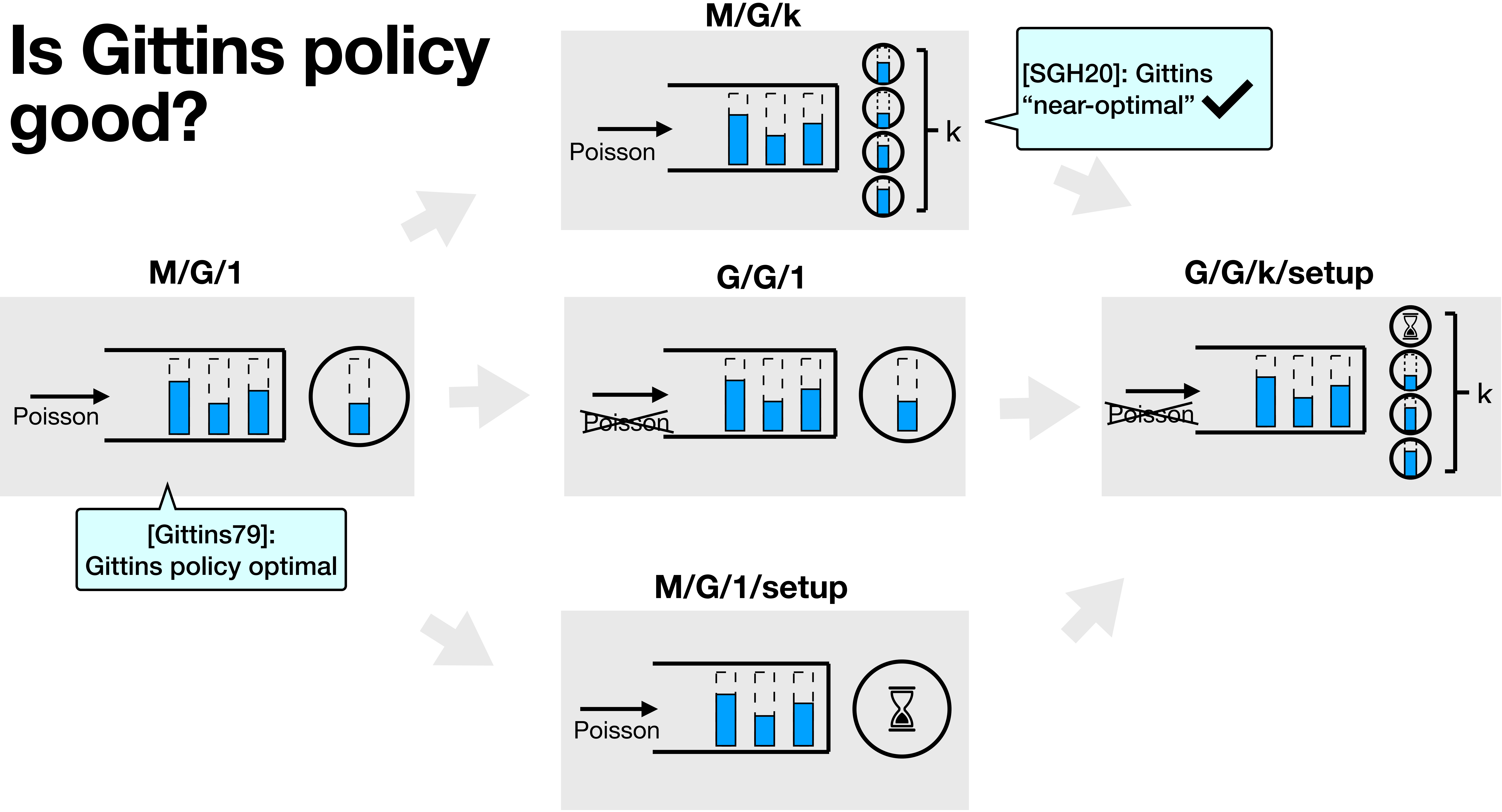

M/G/1

[Gittins79]:
Gittins policy optimal

M/G/k

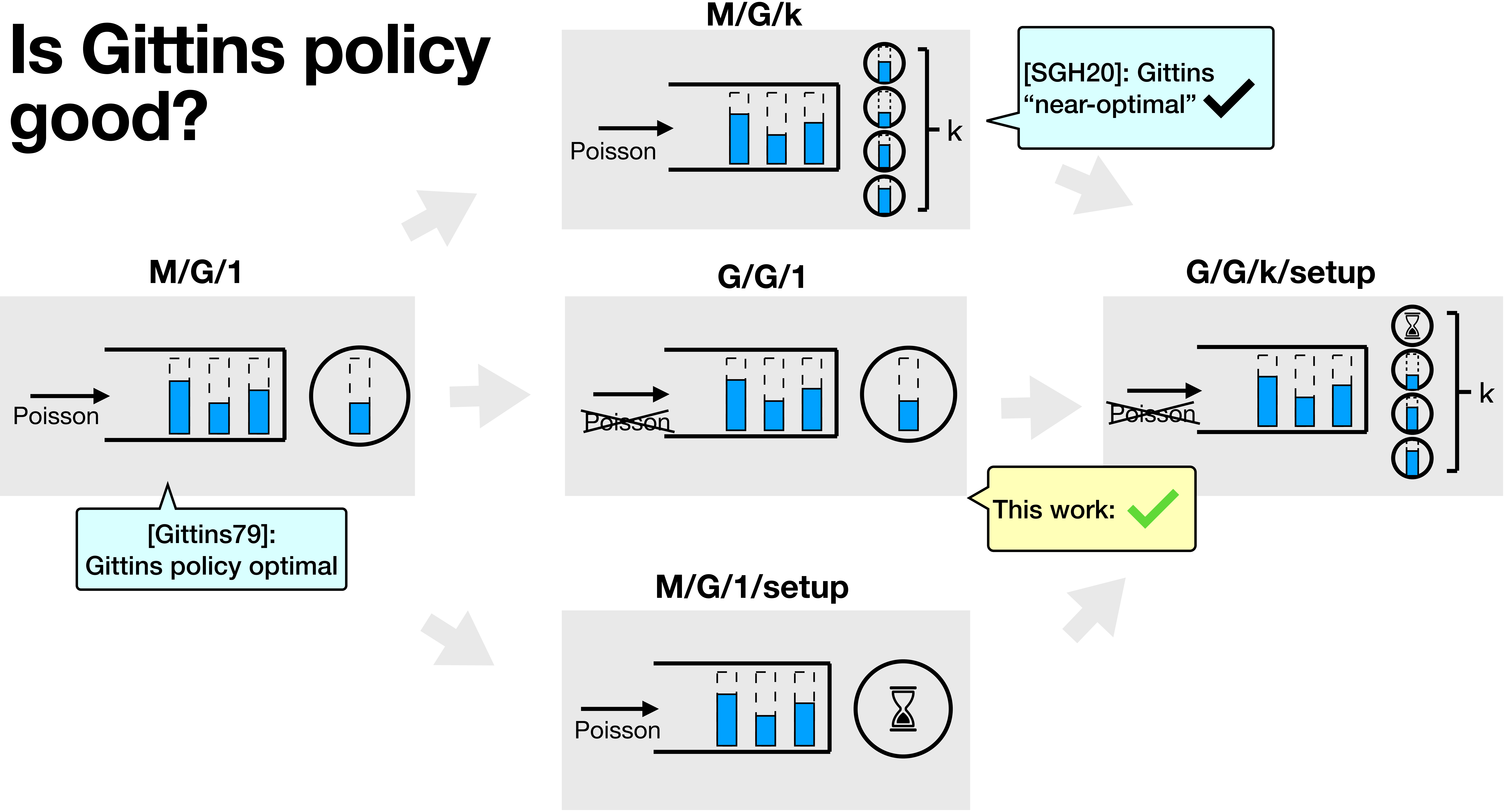
[SGH20]: Gittins
“near-optimal” ✓

G/G/1

M/G/1/setup

G/G/k/setup

Is Gittins policy good?



[Gittins79]:
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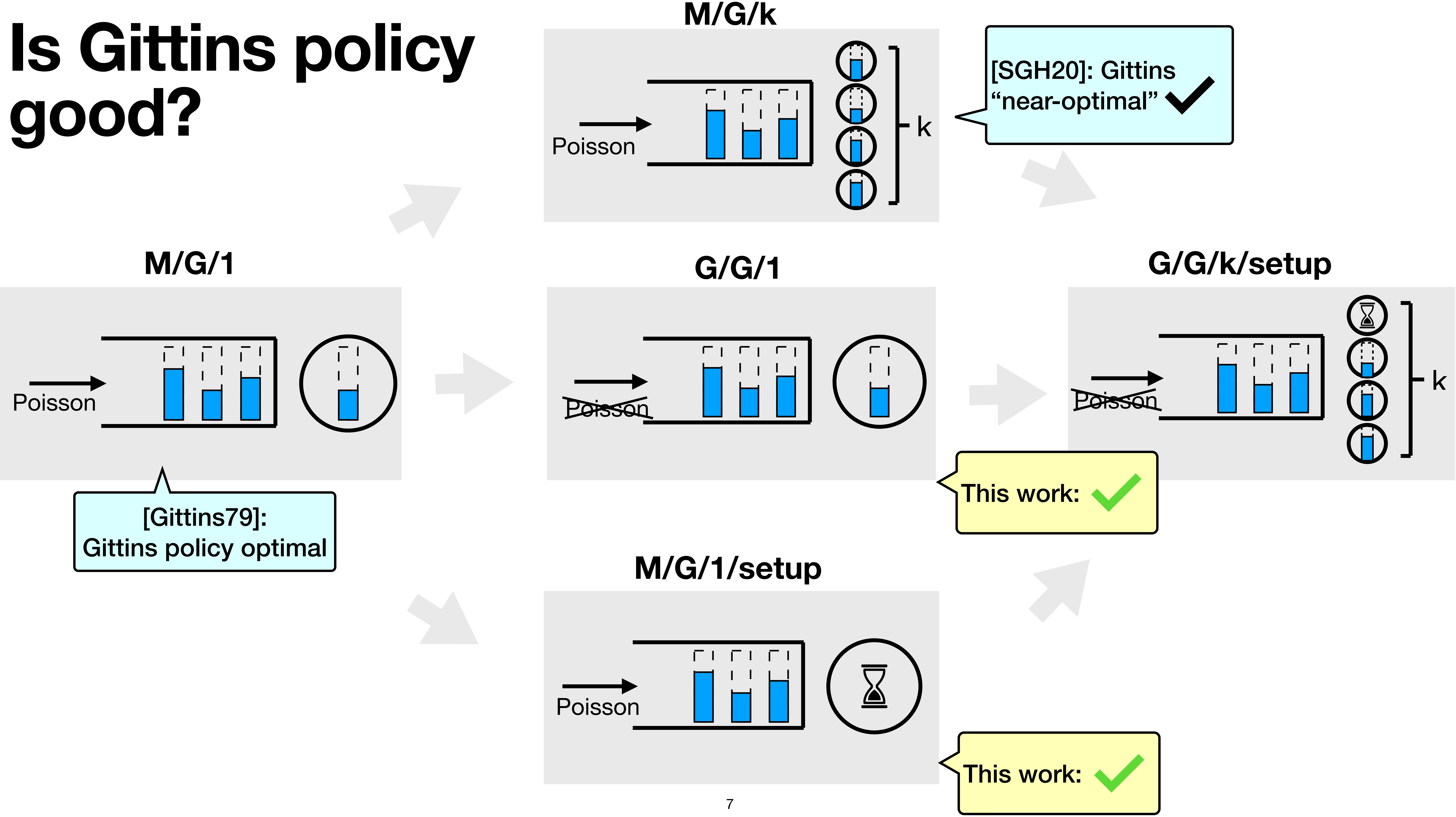
[SGH20]: Gittins
“near-optimal” ✓

This work: ✓

Is Gittins policy good?



[SGH20]: Gittins “near-optimal” ✓

[Gittins79]: Gittins policy optimal

This work: ✓

This work: ✓

Is Gittins policy good?

M/G/1

[Gittins79]: Gittins policy optimal

M/G/k

[SGH20]: Gittins "near-optimal" ✓

G/G/1

This work: ✓

M/G/1/setup

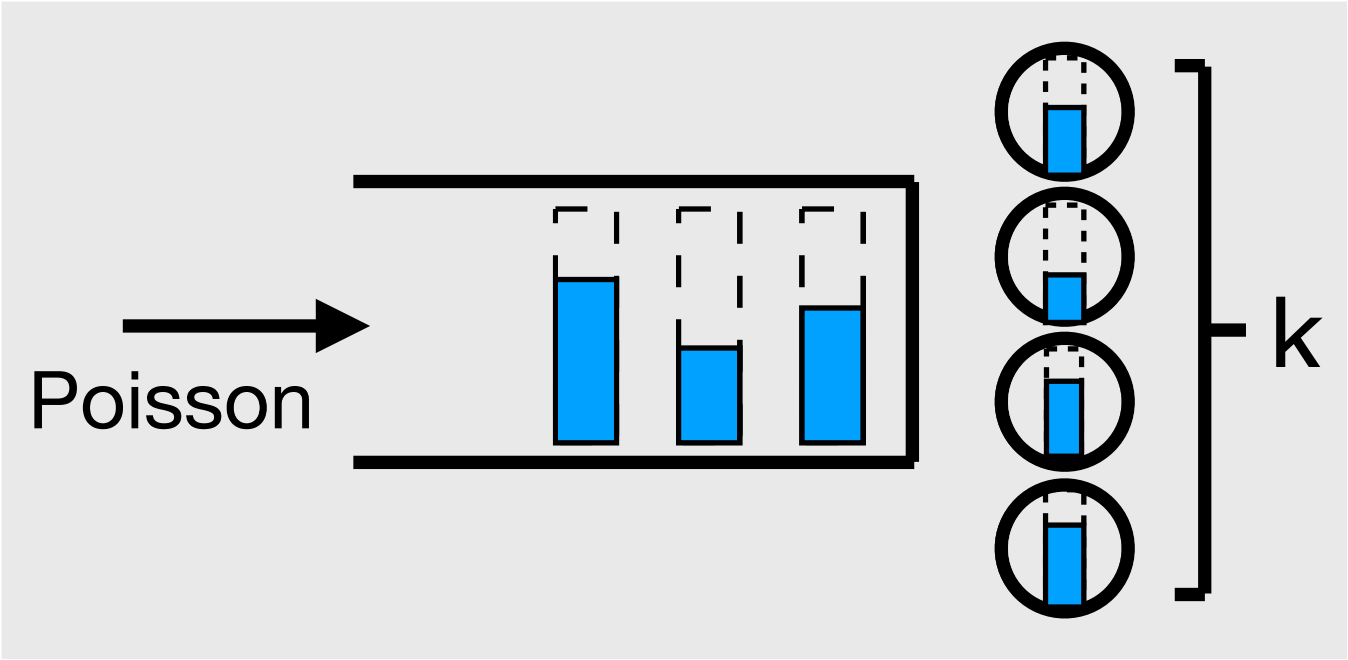
This work: ✓

G/G/k/setup

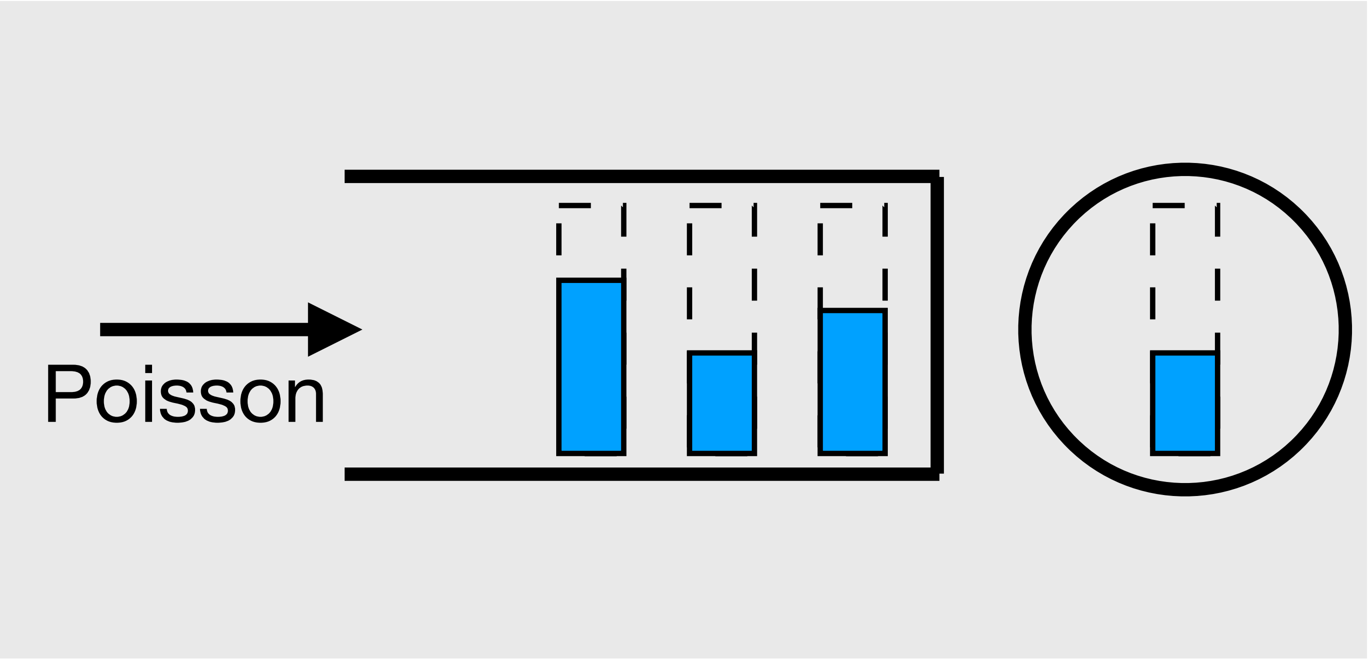
This work: ✓

Suboptimality gaps

M/G/k

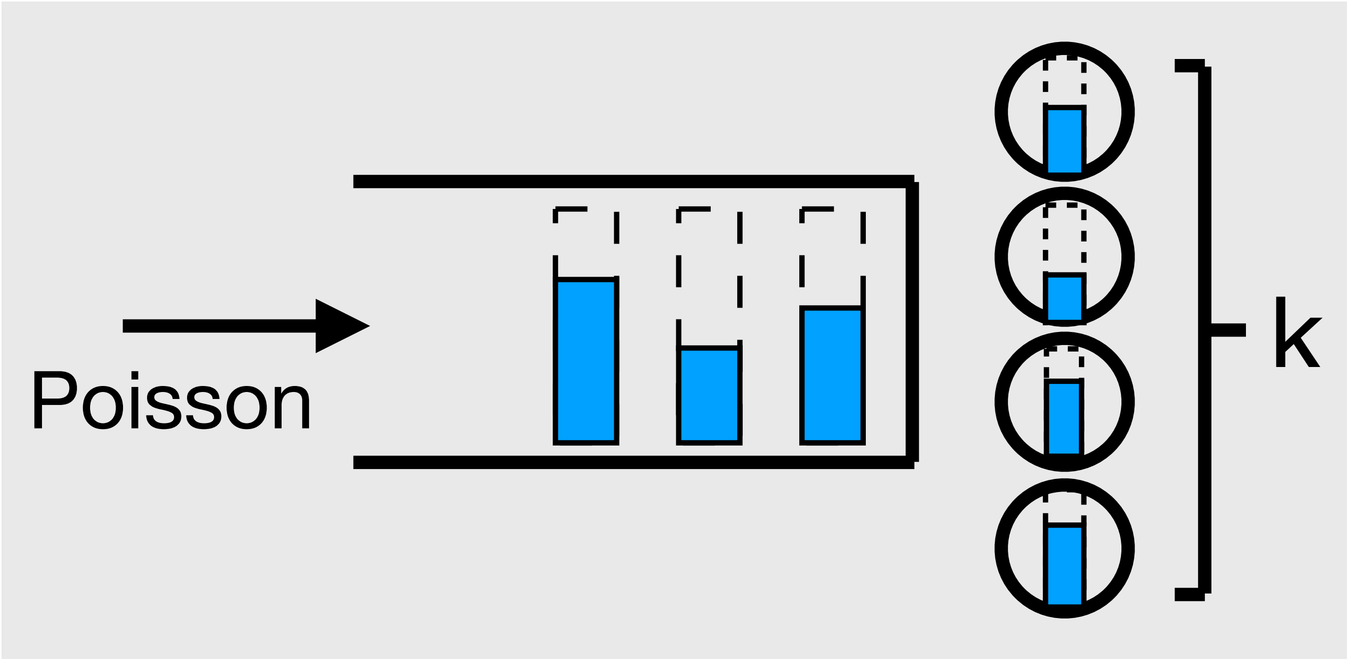


M/G/1

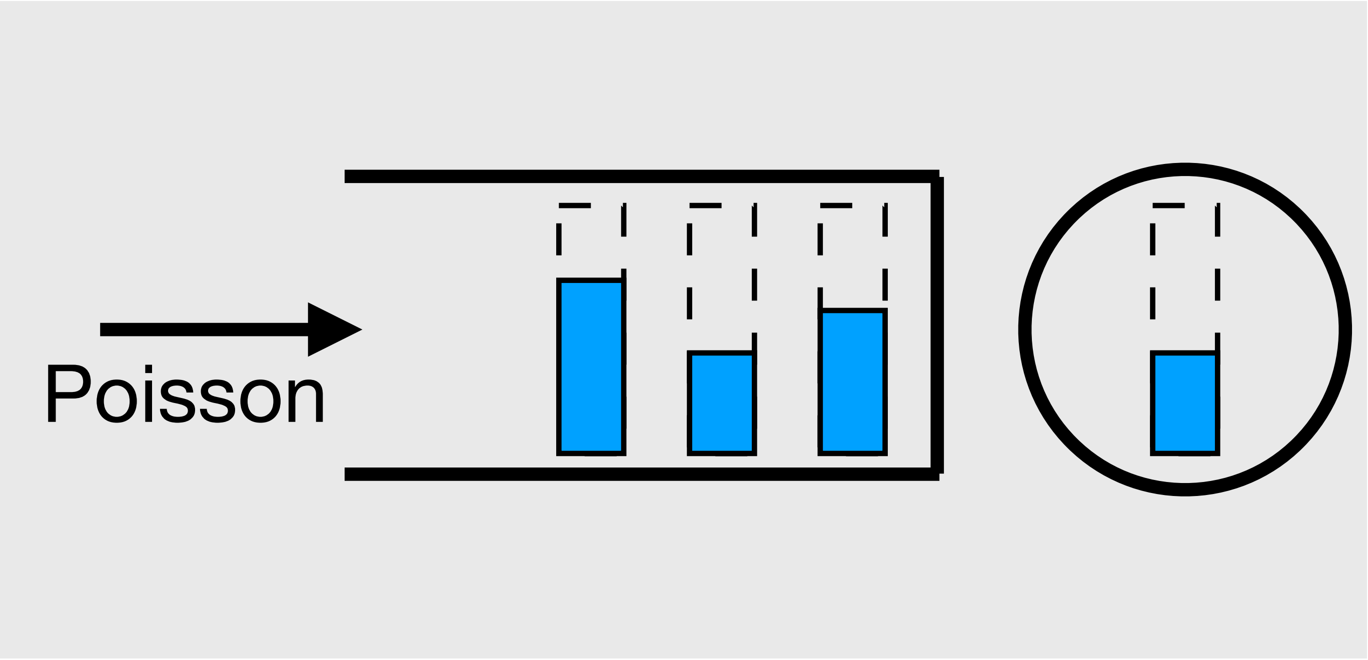


Suboptimality gaps

M/G/k



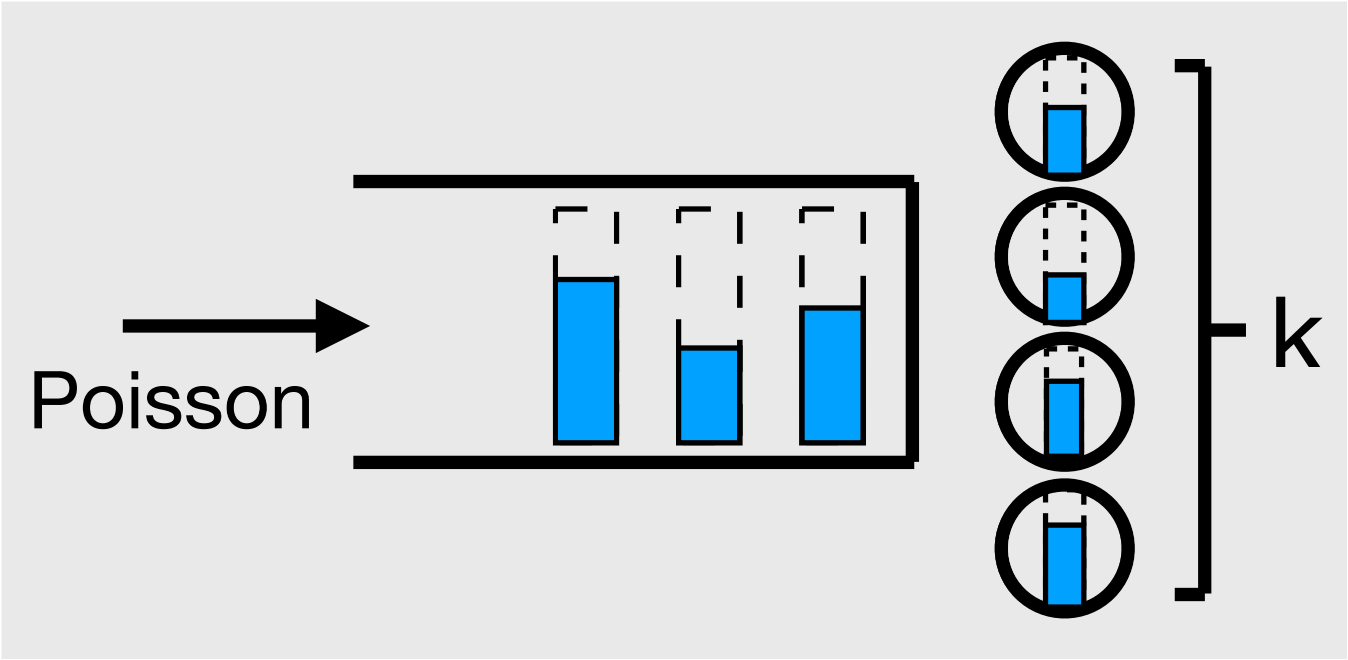
M/G/1



[Scully, Grosof, Harchol-Balter 20; Scully 22]

Suboptimality gaps

M/G/k



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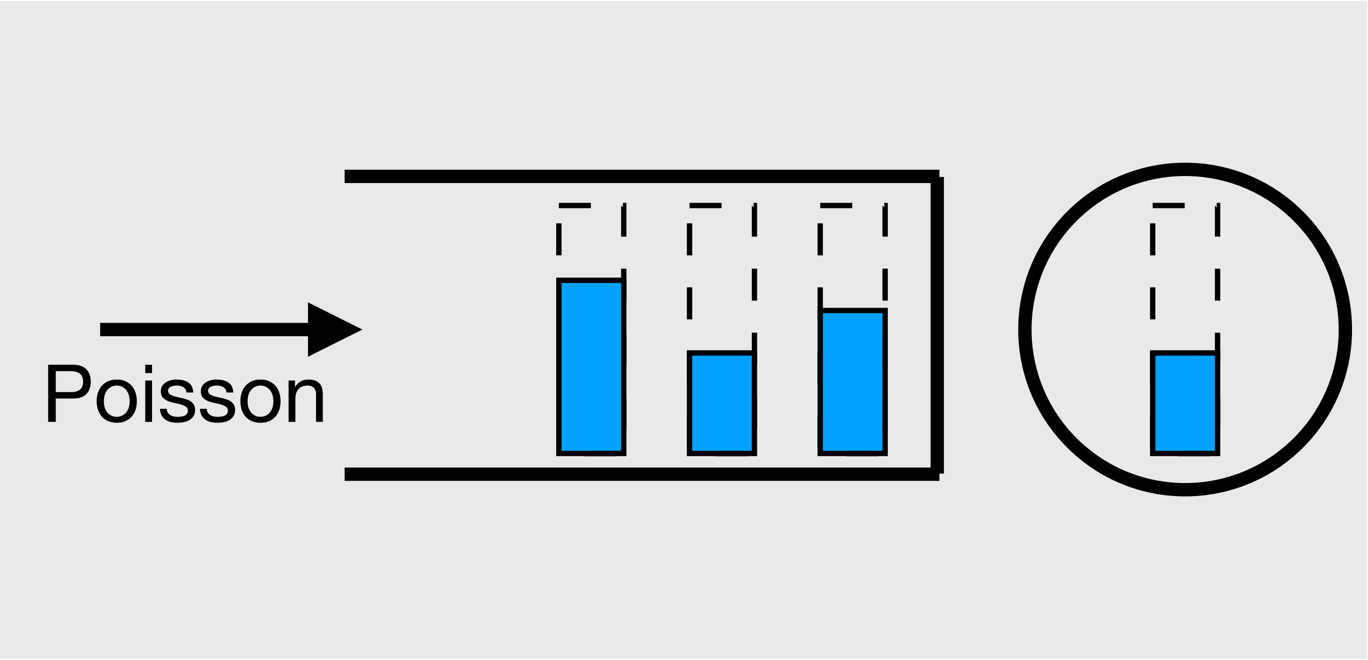
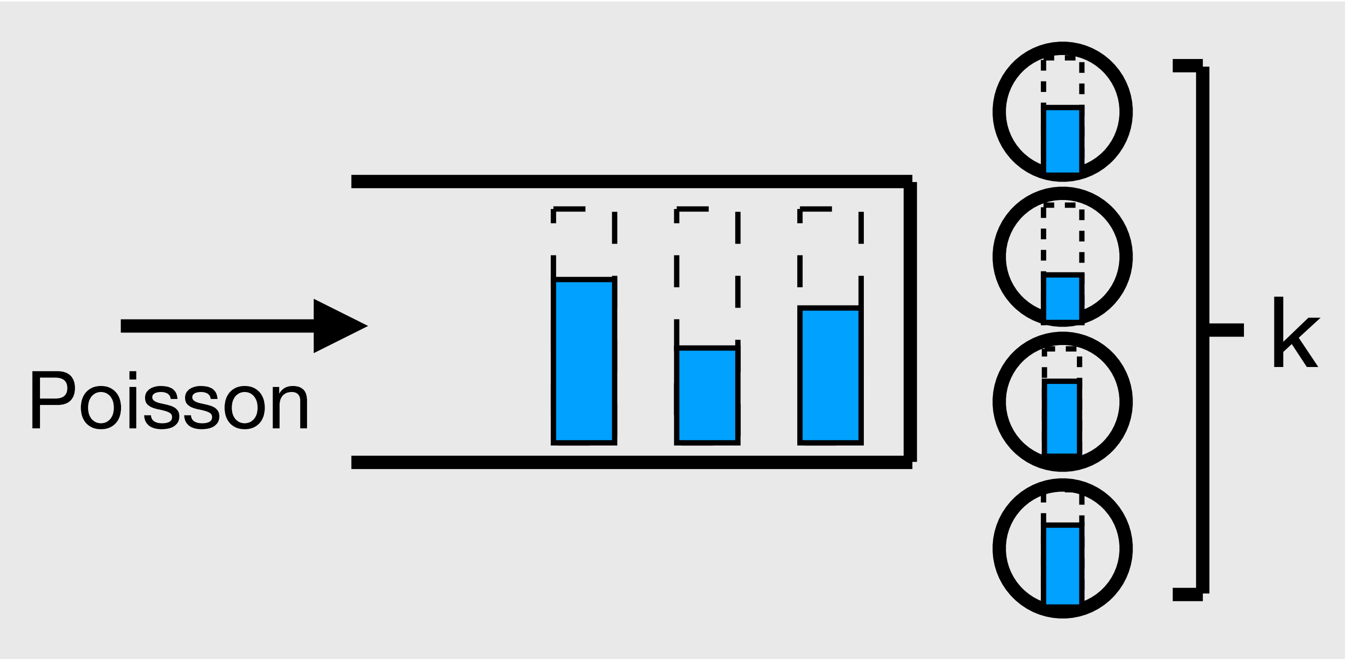


[Scully, Grosof, Harchol-Balter 20; Scully 22]

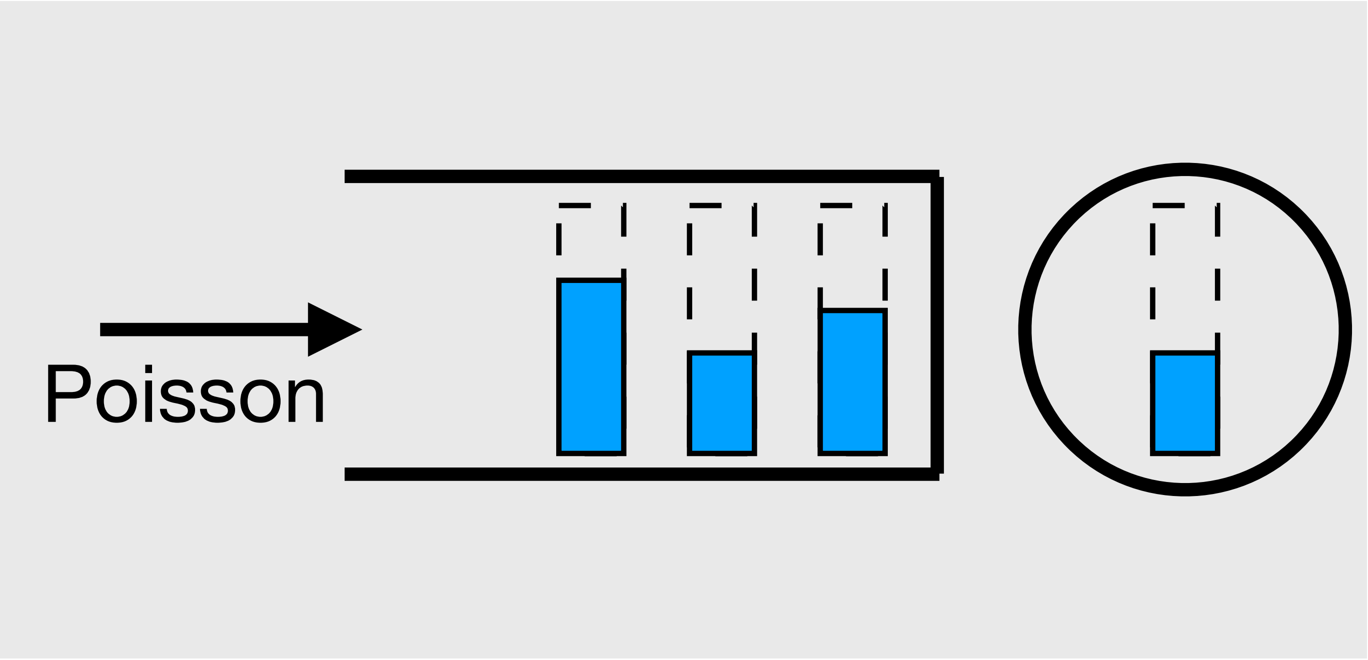
1. $\mathbb{E}[N]^{\text{Gittins}} \leq \mathbb{E}[N]^{\text{OPT}} + \ell_{(a)}, \ \ell_{(a)} = 3.8(k-1)\log\frac{1}{1-\rho}$

Suboptimality gaps

M/G/k



M/G/1



[Scully, Grosof, Harchol-Balter 20; Scully 22]

1. $\mathbb{E}[N]^{\text{Gittins}} \leq \mathbb{E}[N]^{\text{OPT}} + \ell_{(a)},\ \ \ell_{(a)} = 3.8(k-1)\log\frac{1}{1-\rho}$

(load $\rho \triangleq \lambda\mathbb{E}[S]$ = expected fraction of busy servers)

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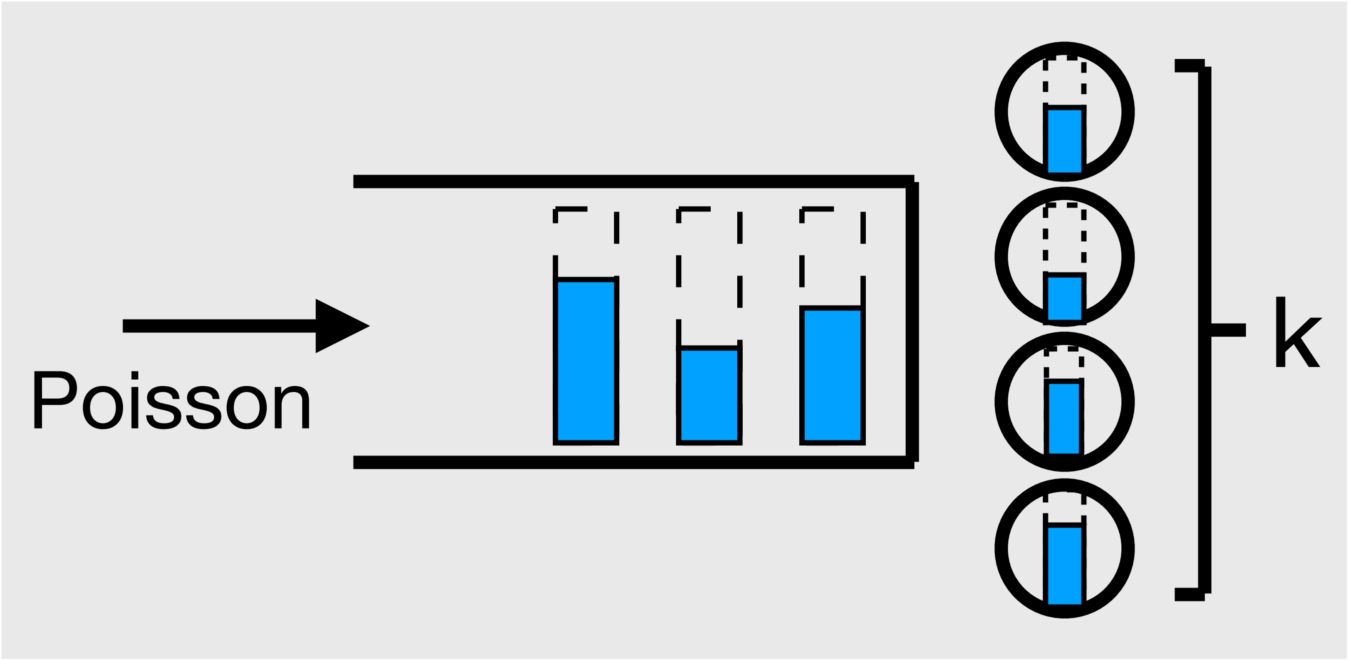
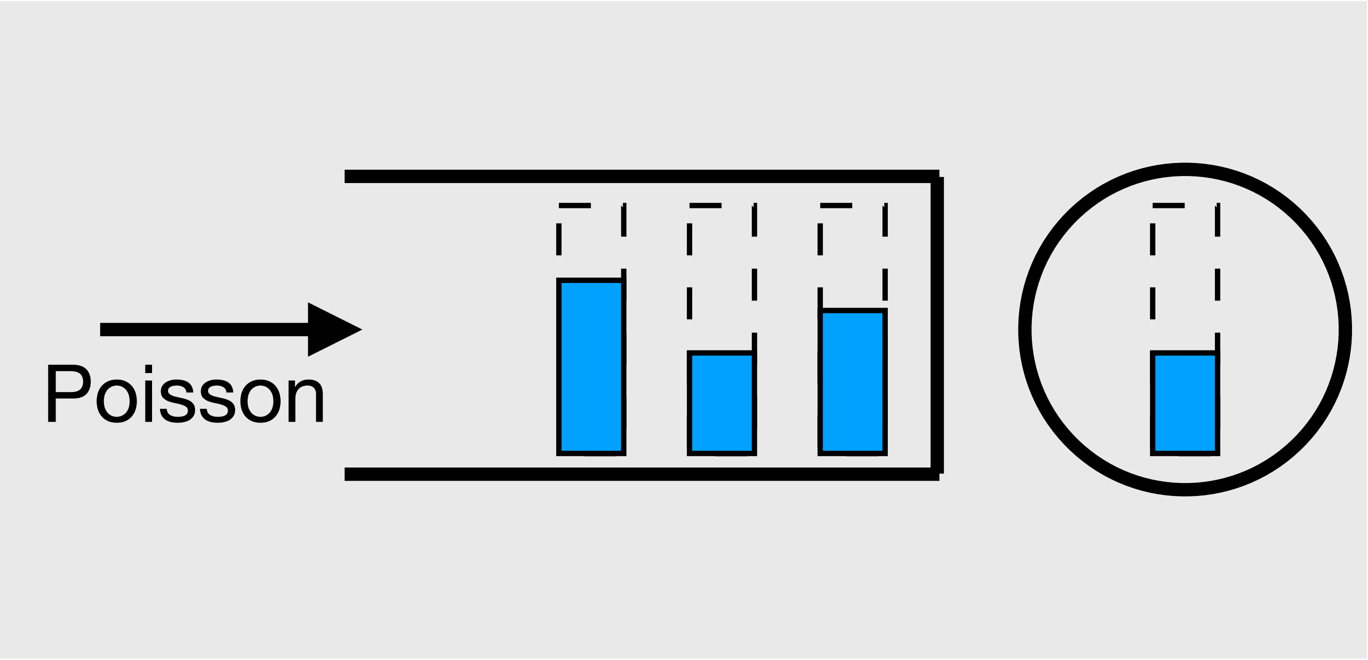


M/G/1



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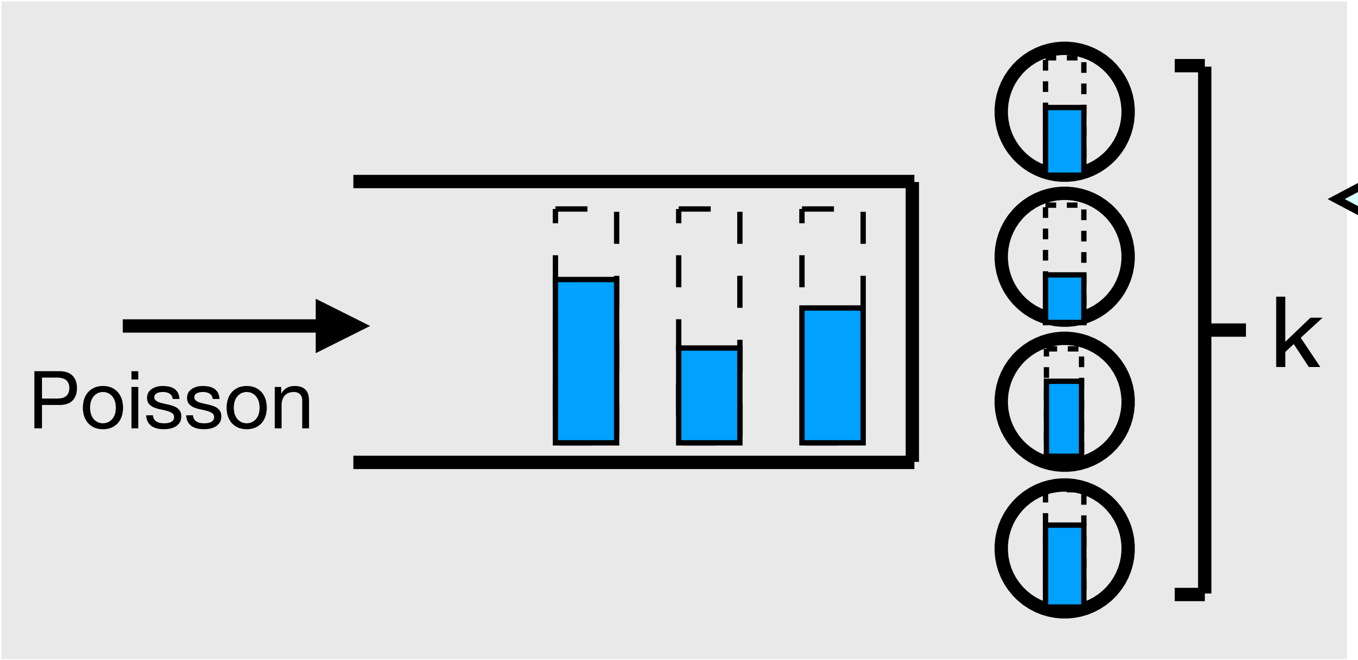
Suboptimality gaps

M/G/k



[SGH20]:
gap $\leq \ell_{(a)}$
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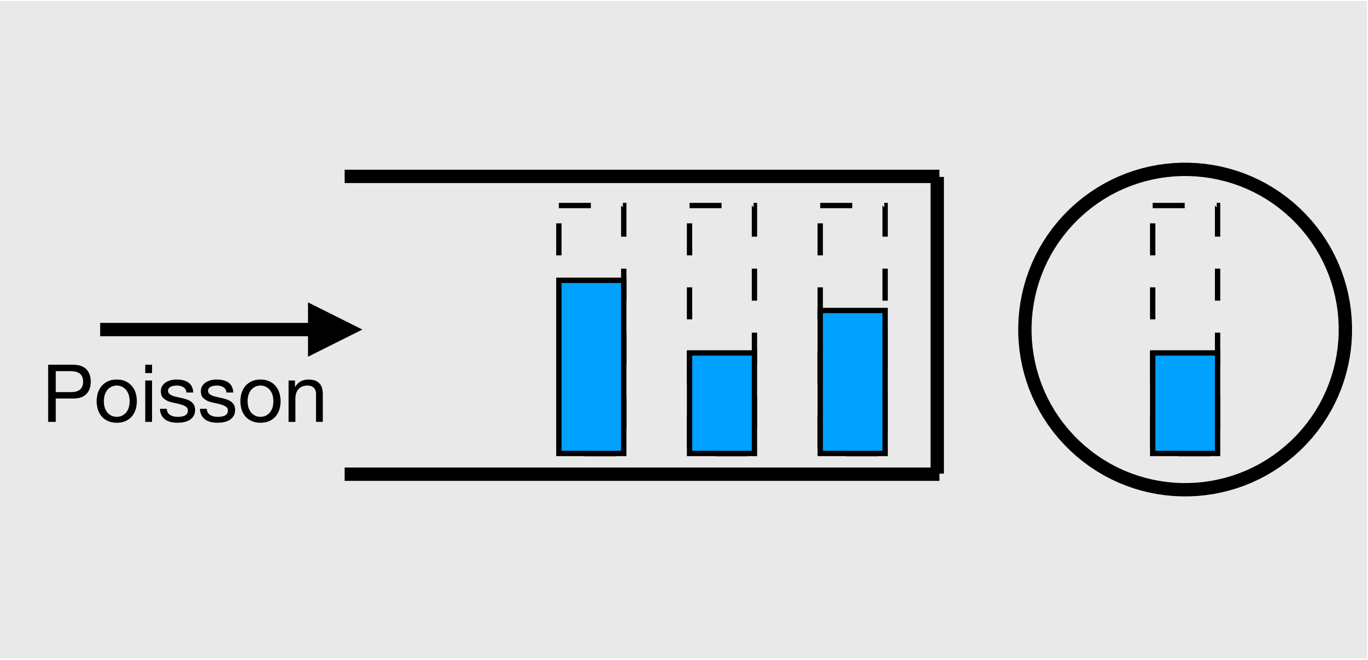

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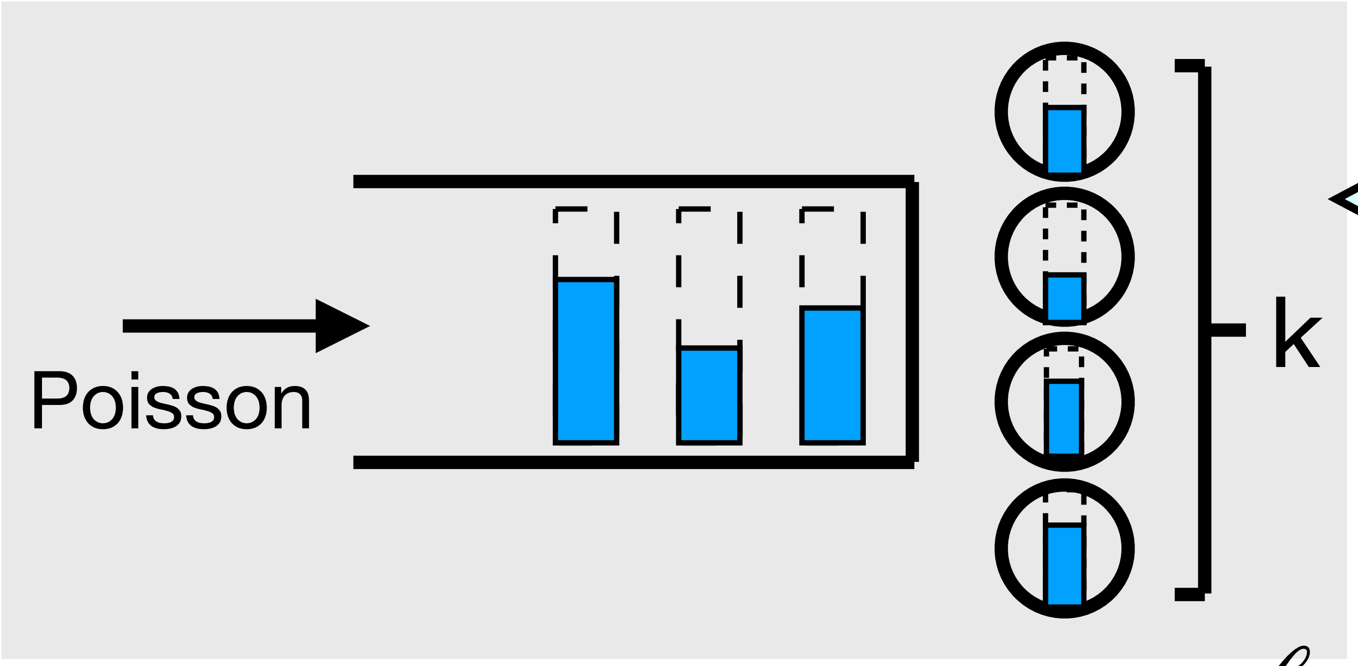

[SGH20]:
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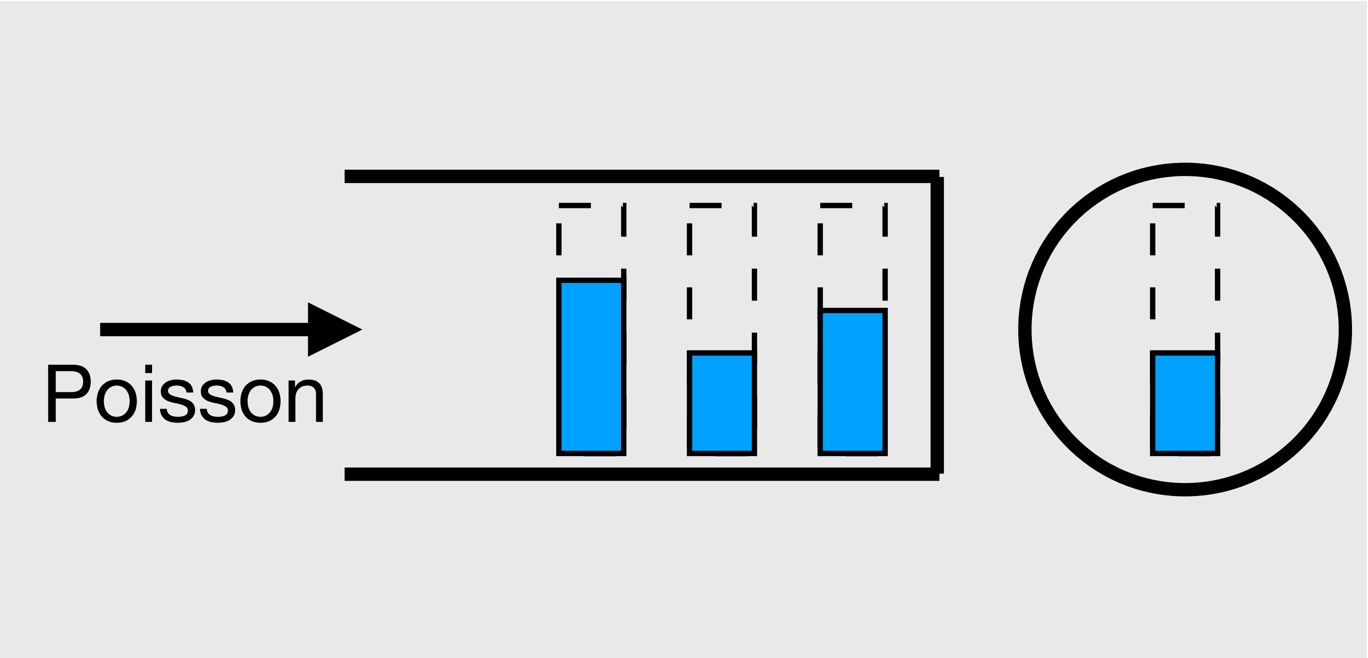
M/G/1



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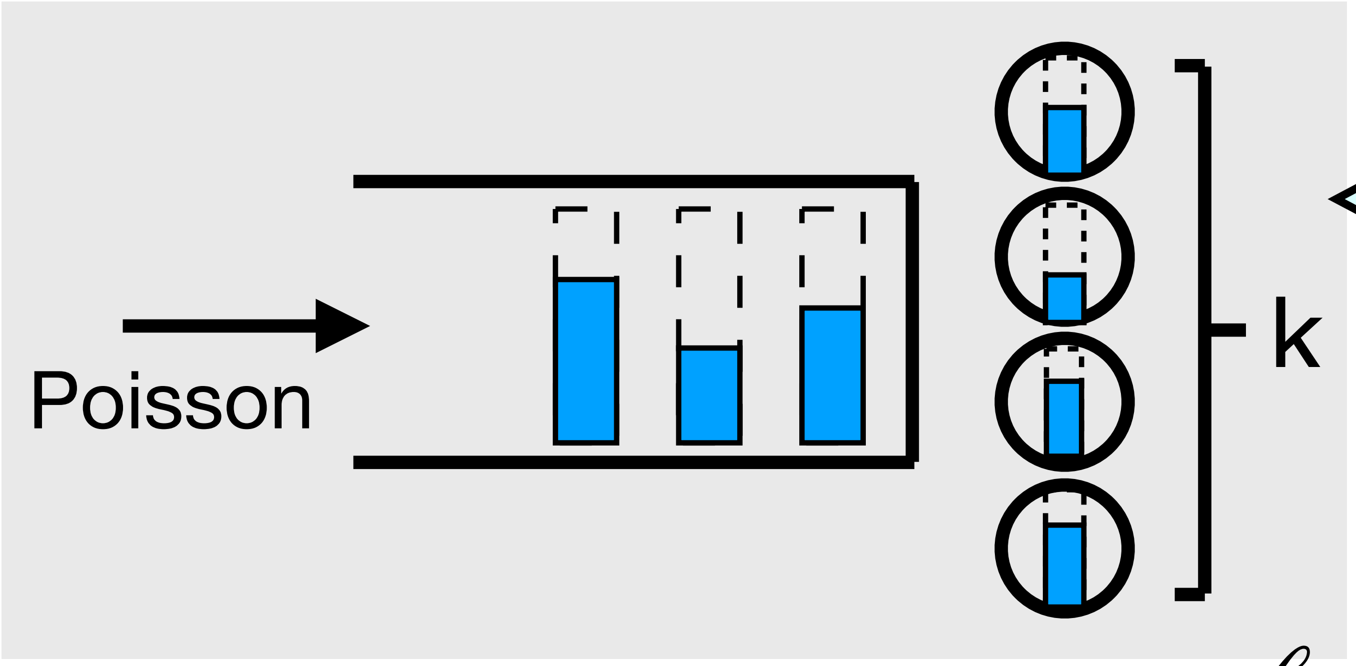
Suboptimality gaps

M/G/k



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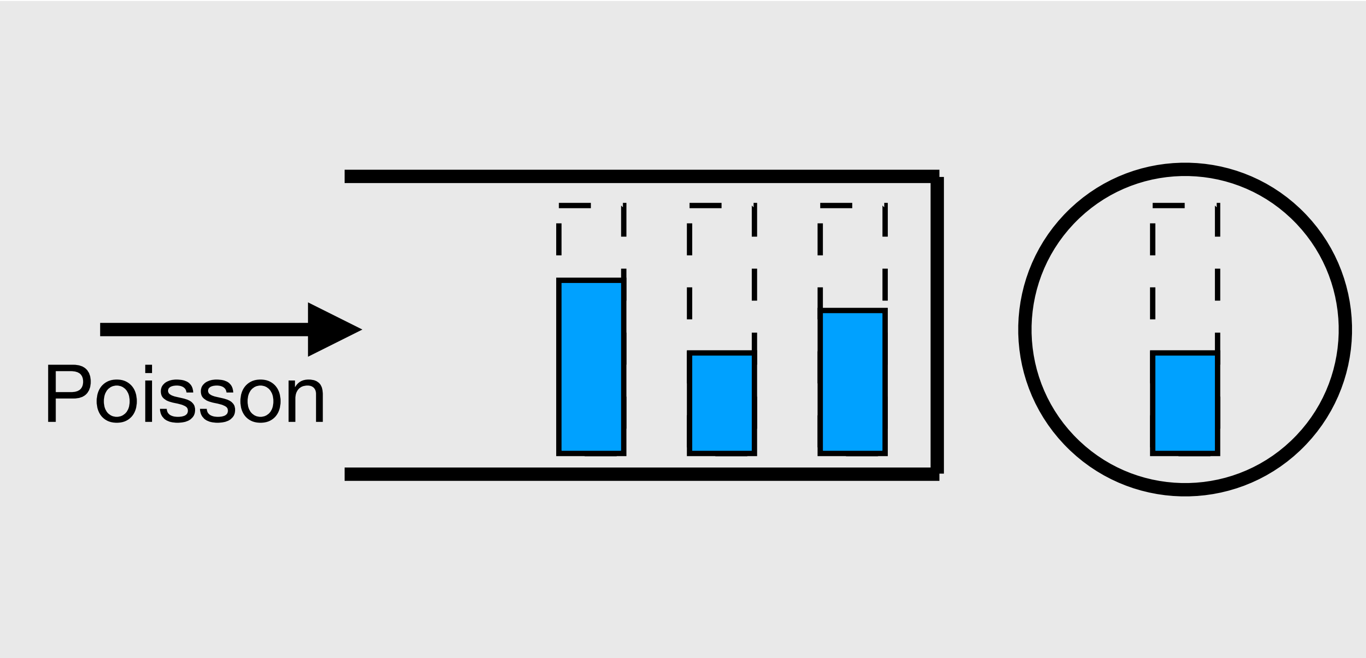
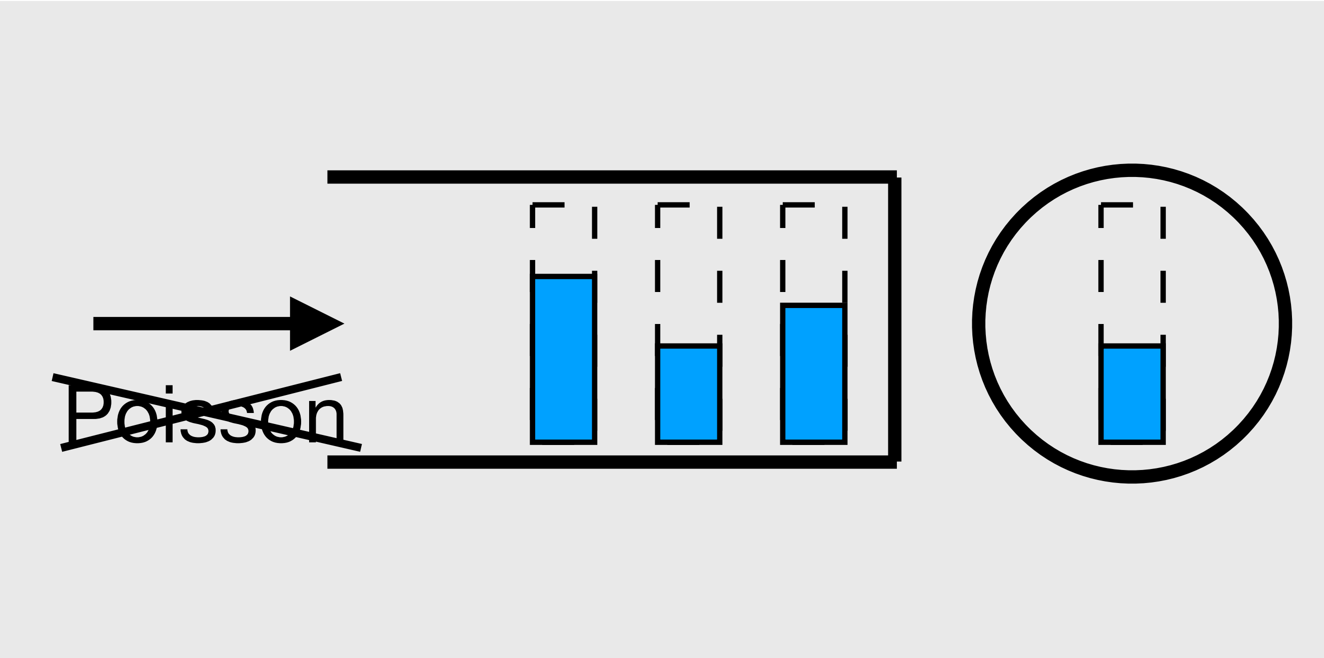
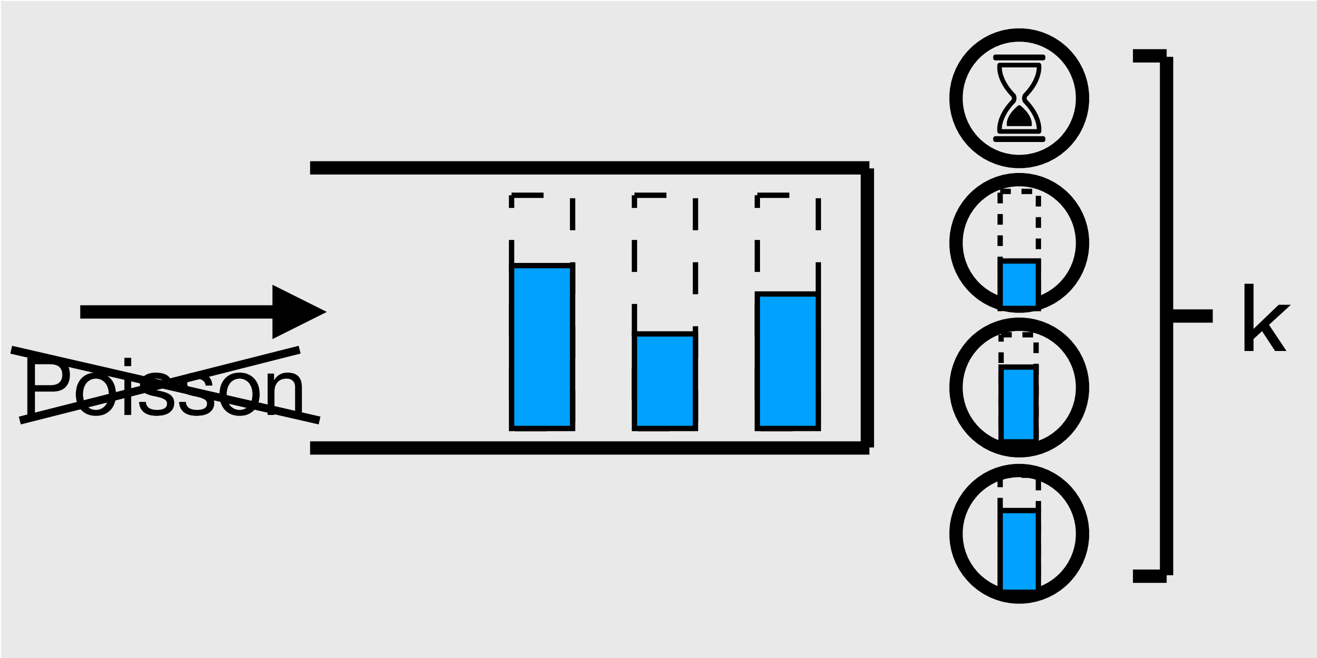


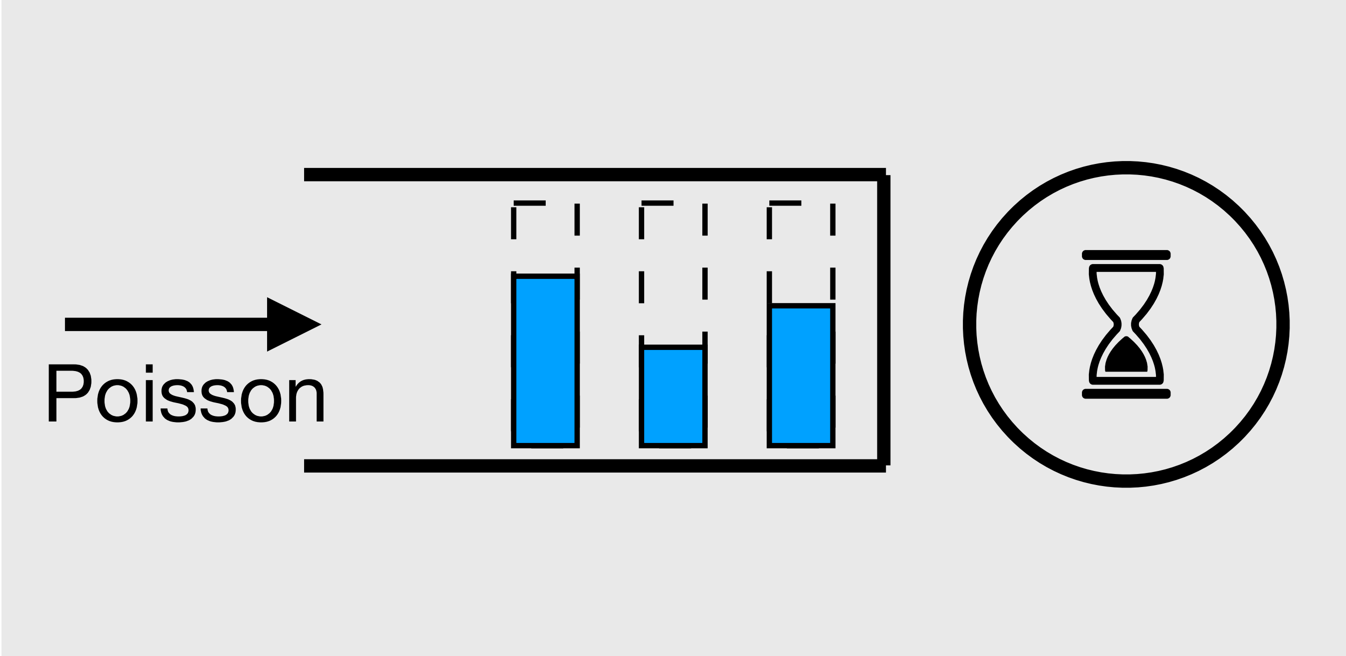
G/G/1



G/G/k/setup

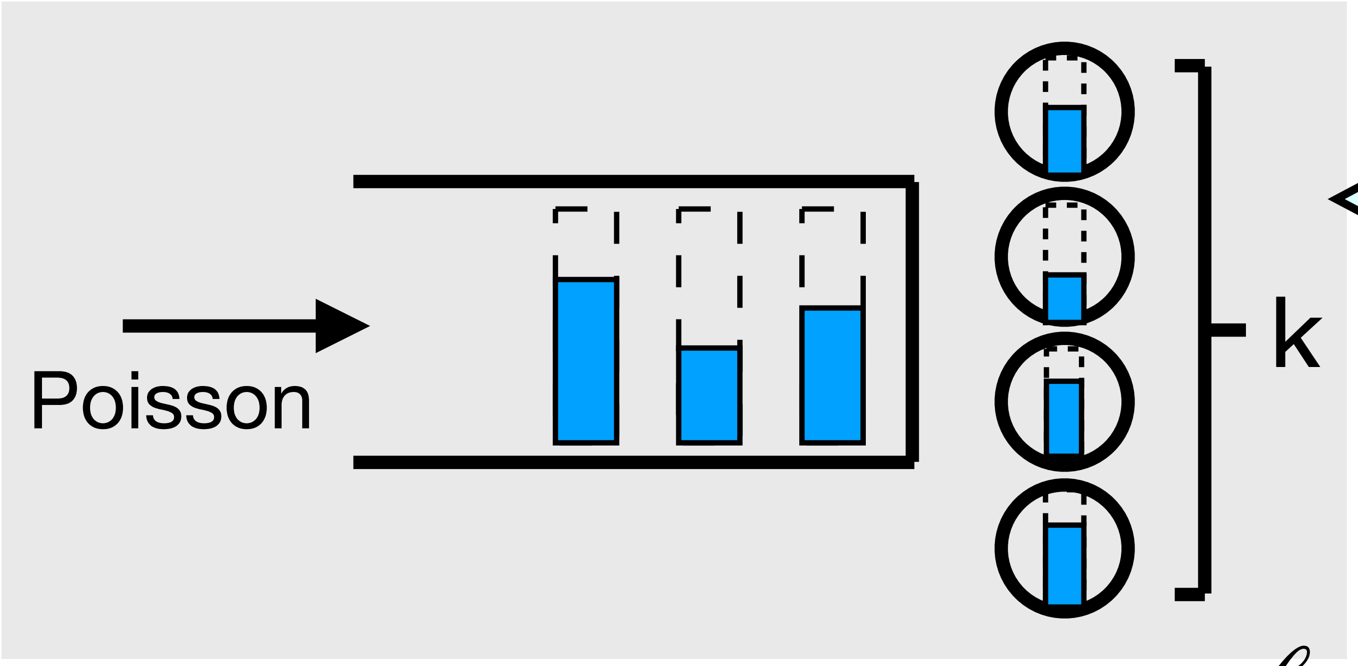


M/G/1/setup



Suboptimality gaps

M/G/k



[SGH20]:
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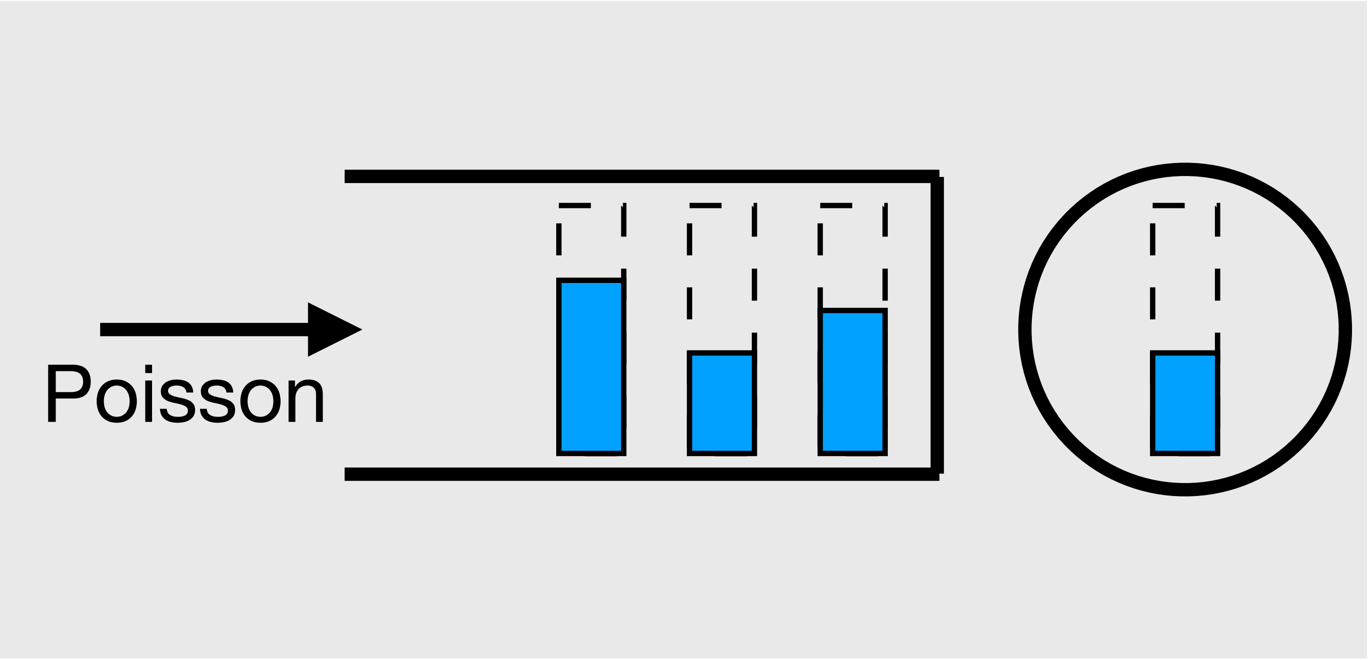
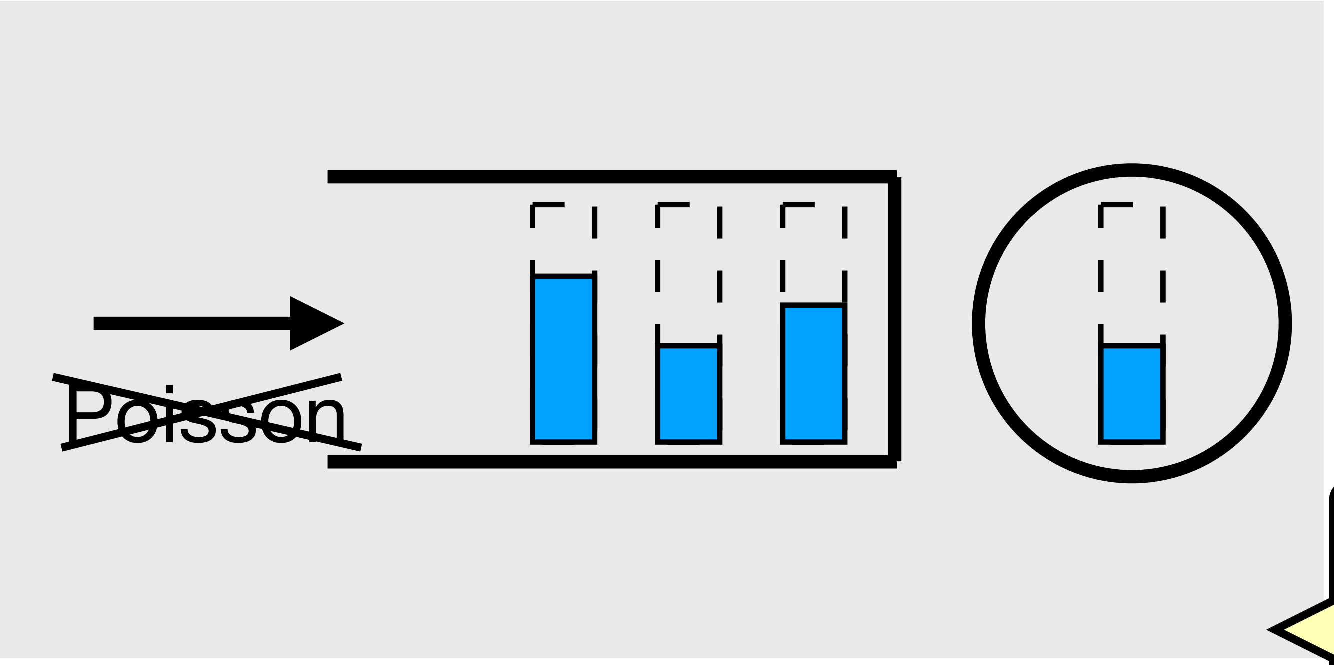
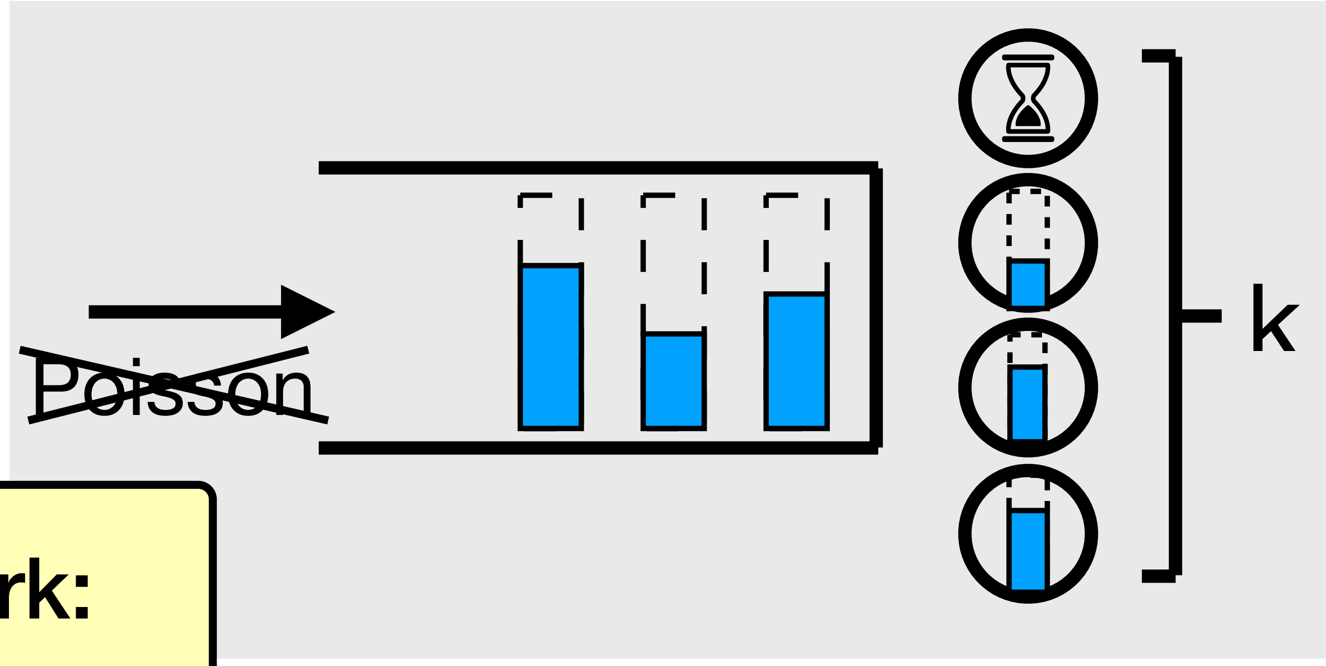


G/G/1

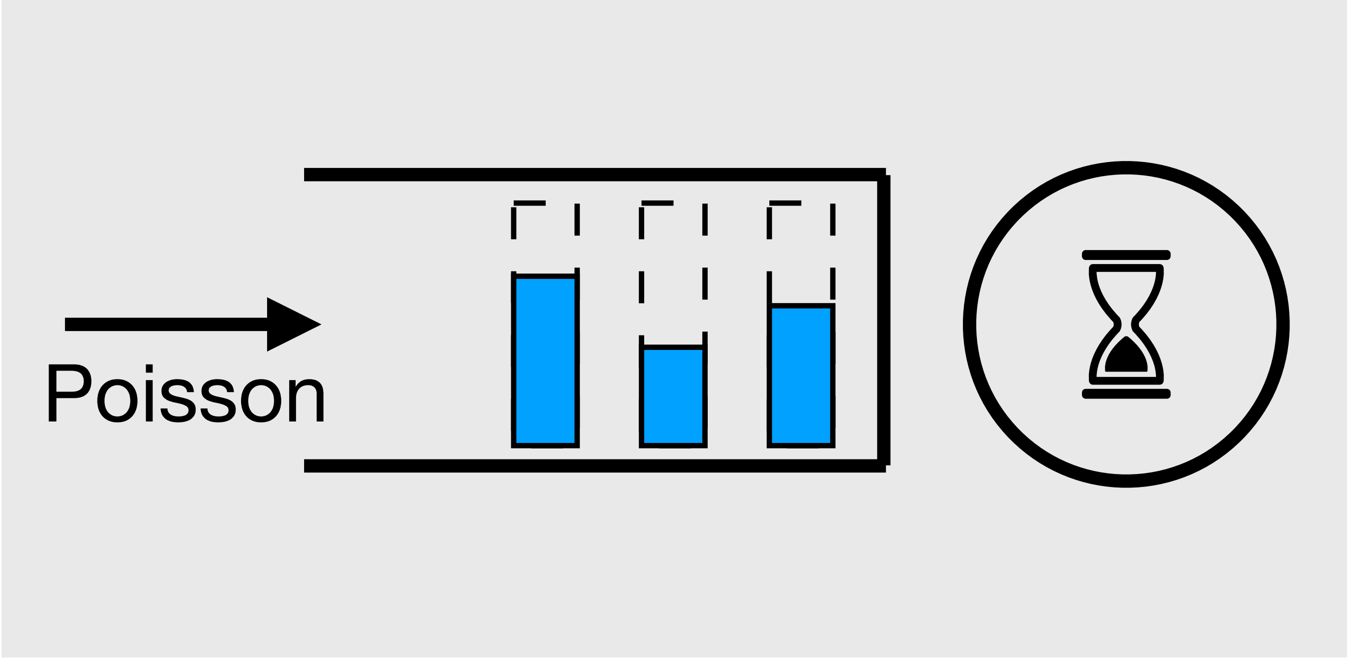


G/G/k/setup



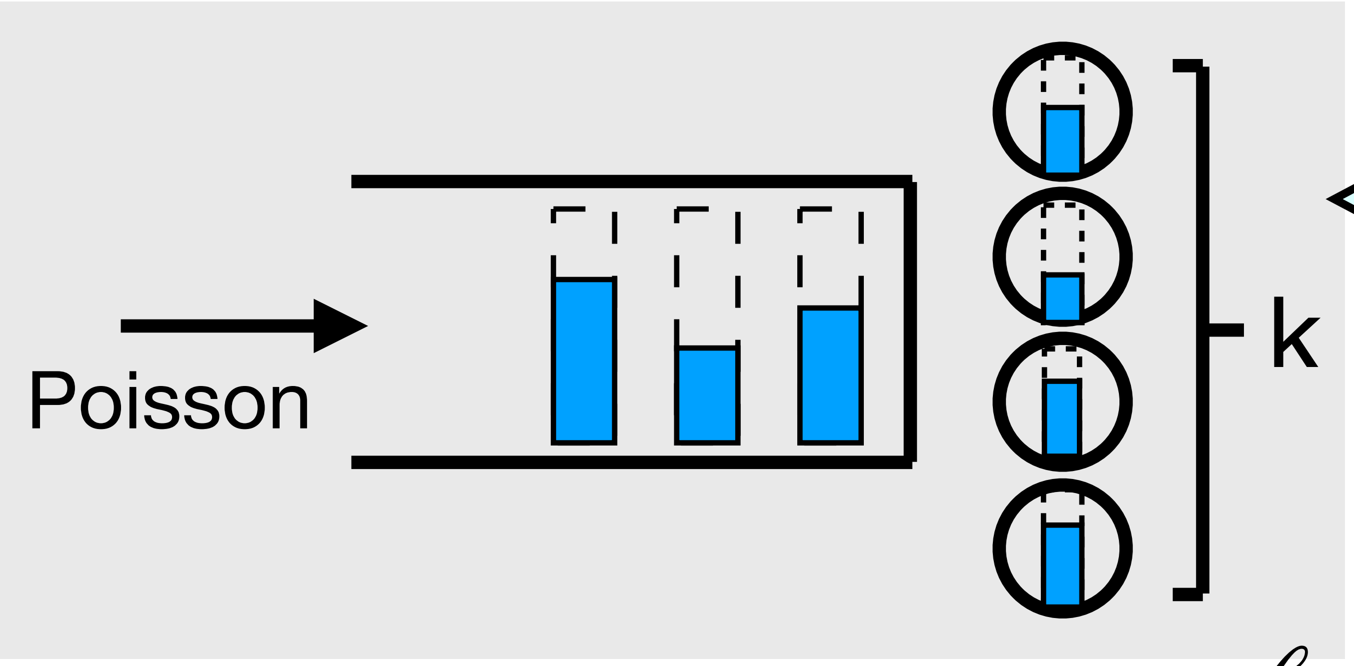
This work:
gap $\le \ell_{(b)}$

M/G/1/setup



Suboptimality gaps

M/G/k



[SGH20]:
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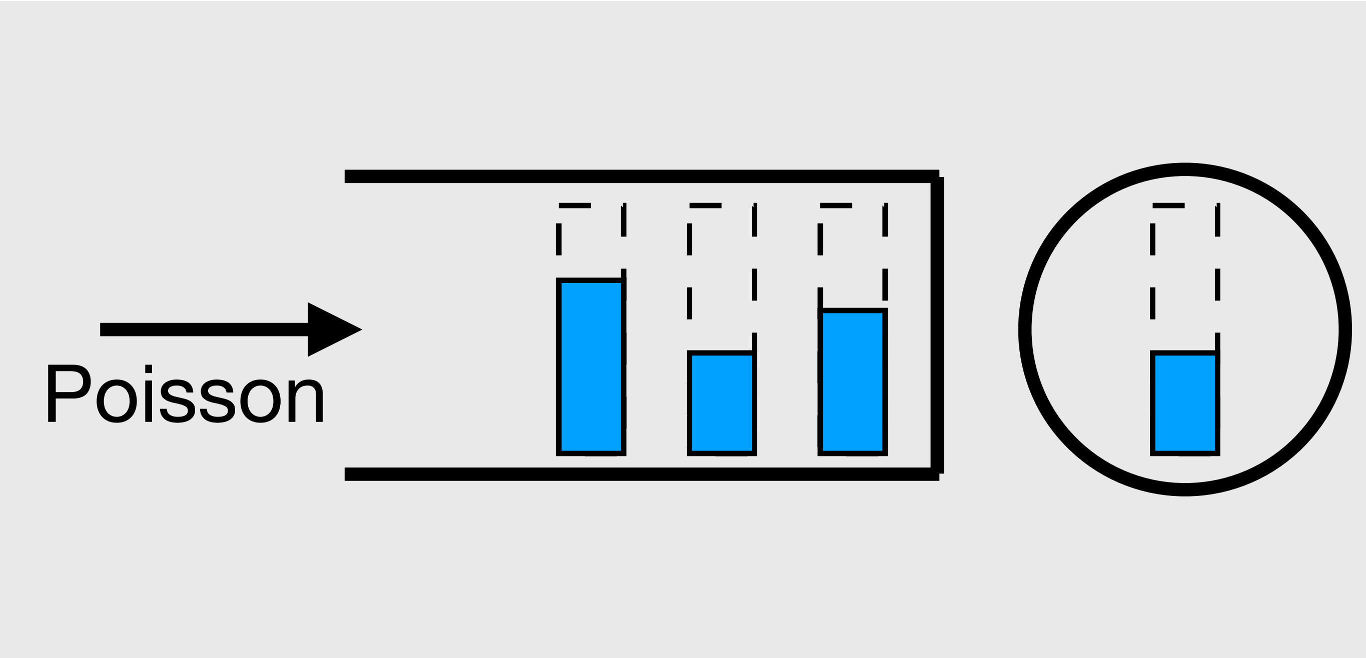
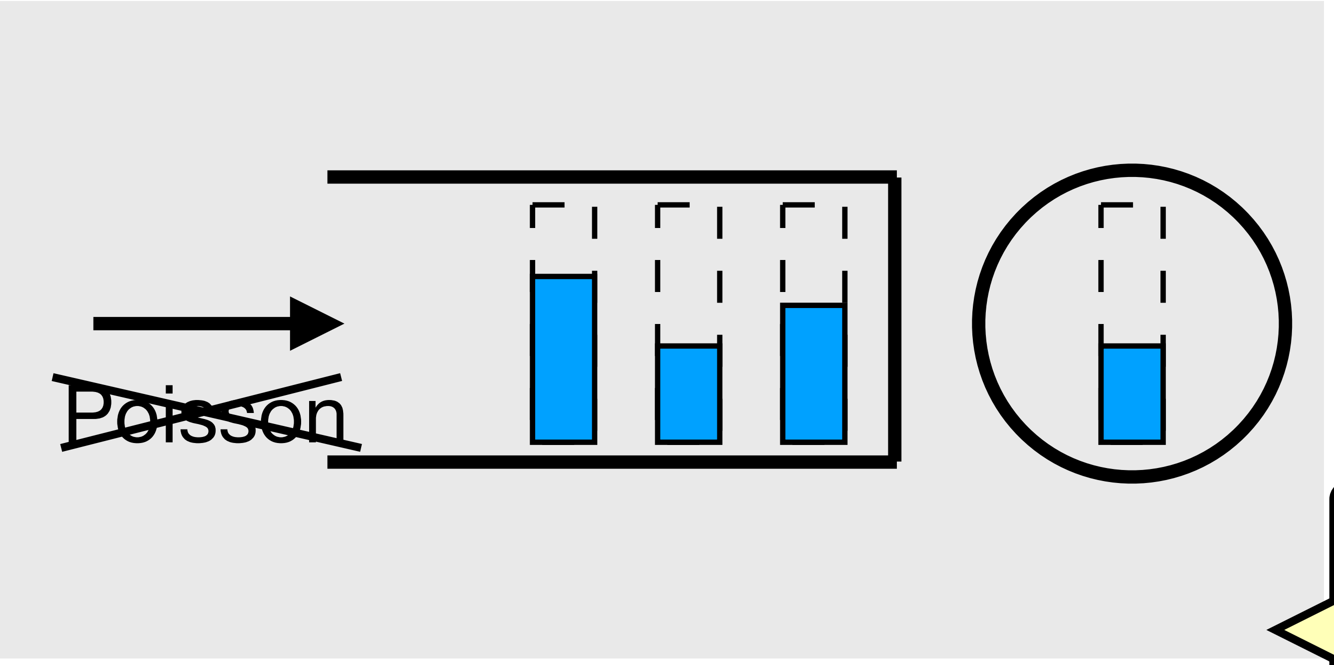
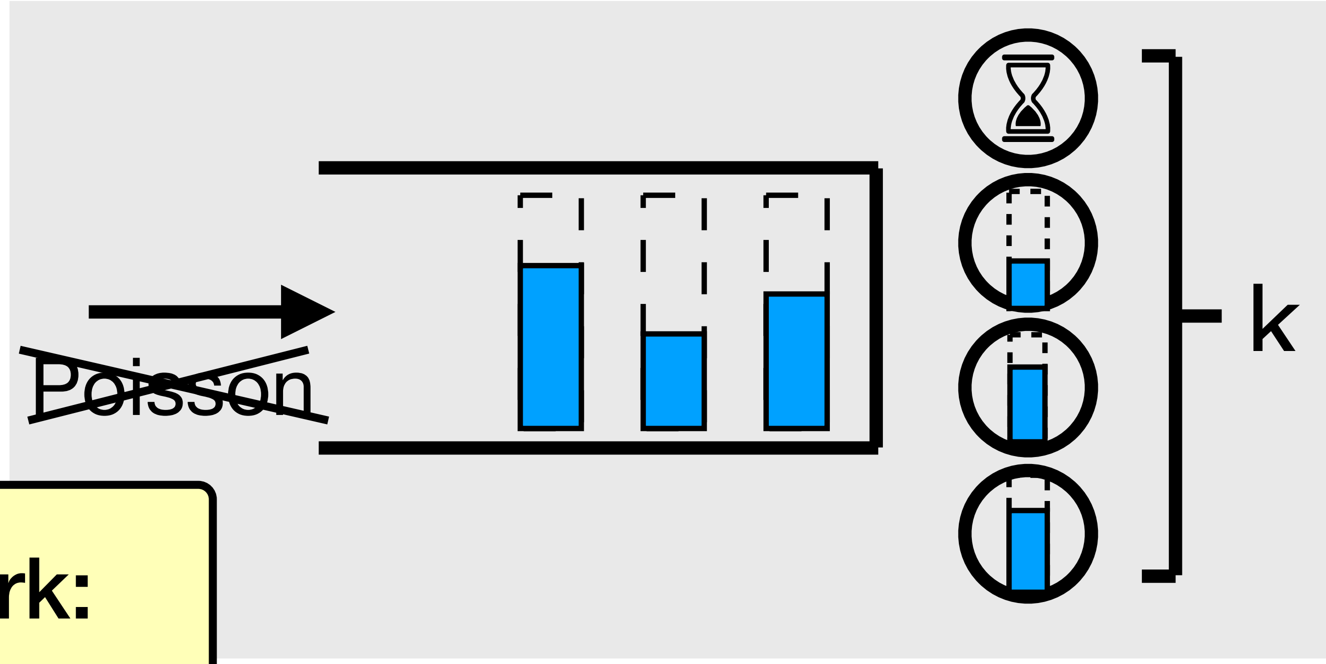
M/G/1



G/G/1



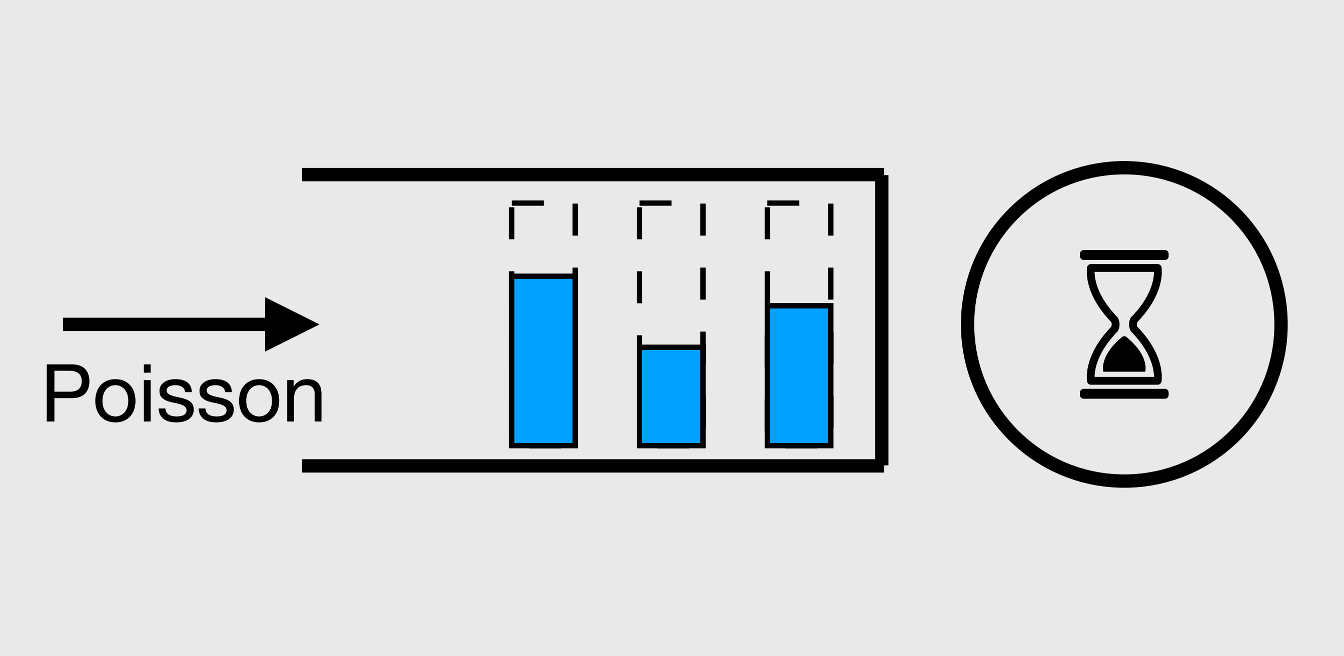
G/G/k/setup



This work:
gap $\leq \ell_{(b)}$

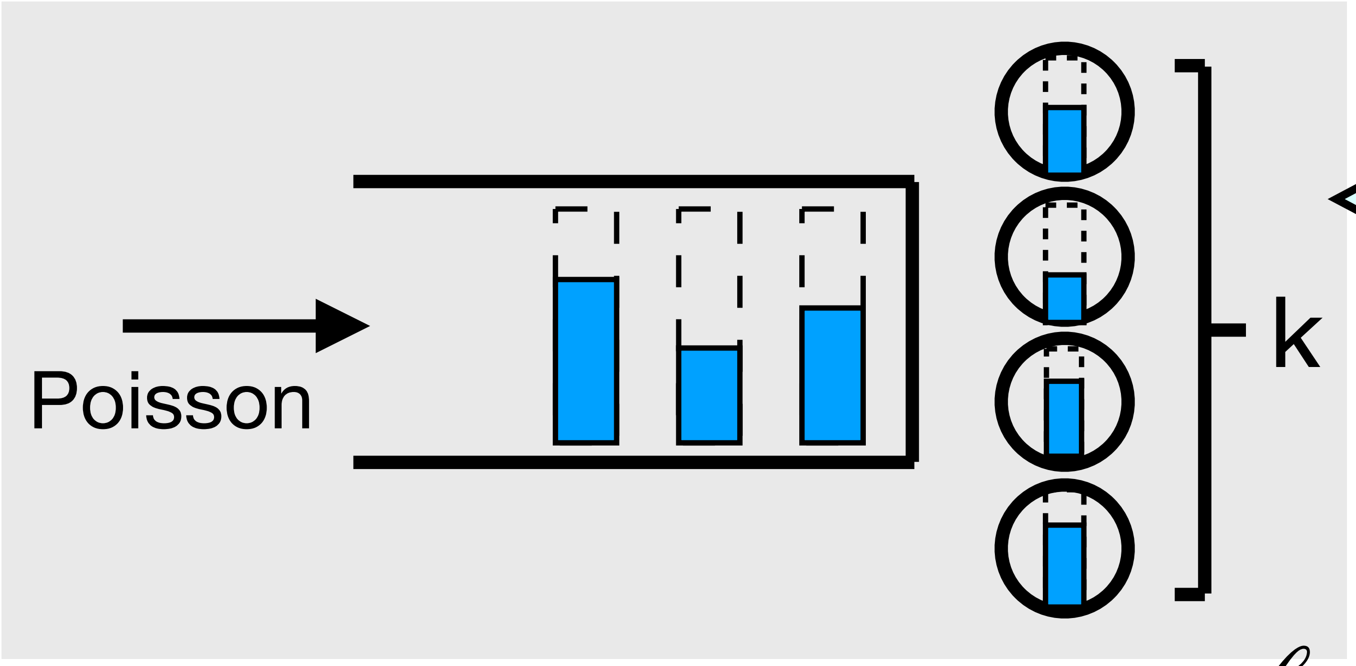
$$\ell_{(b)} = O(1)$$

M/G/1/setup



Suboptimality gaps

M/G/k



[SGH20]:
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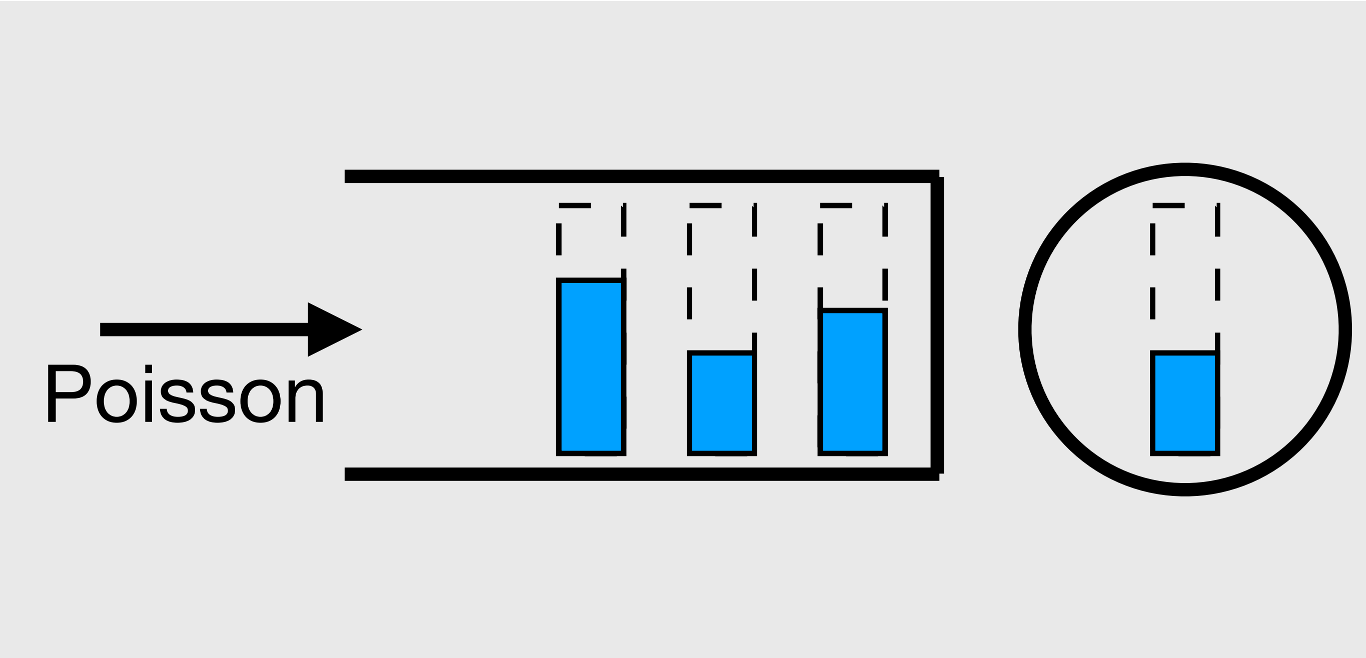

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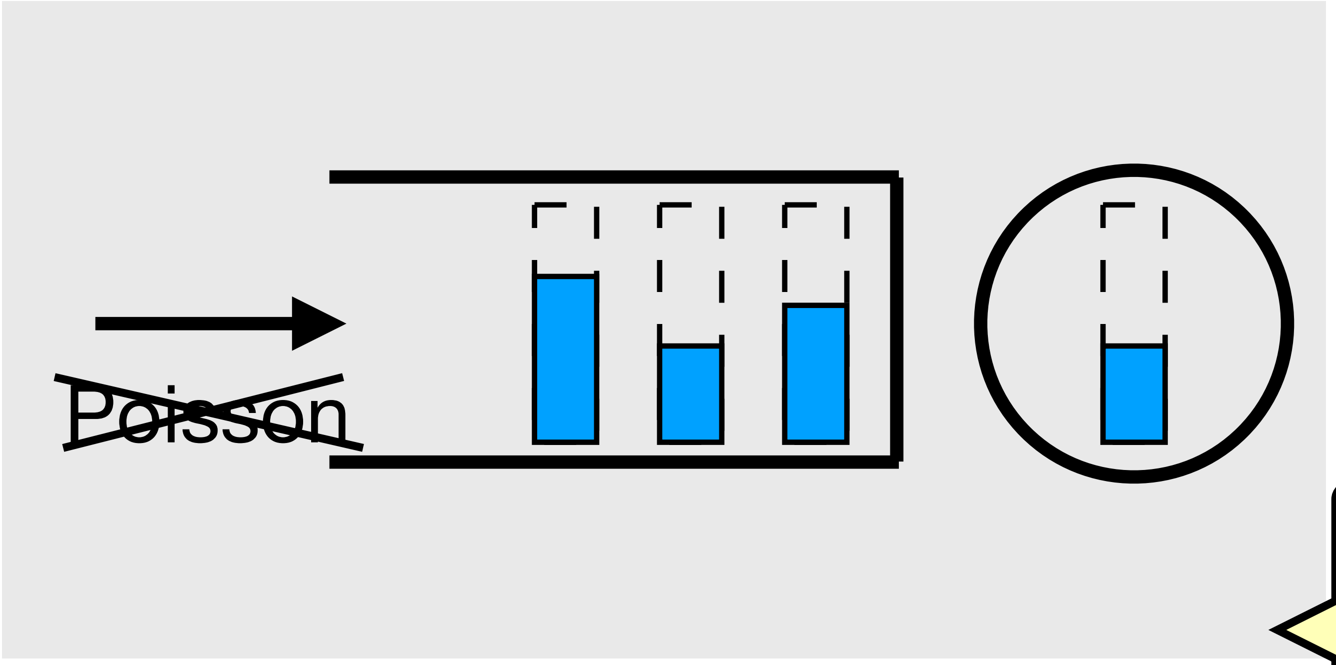

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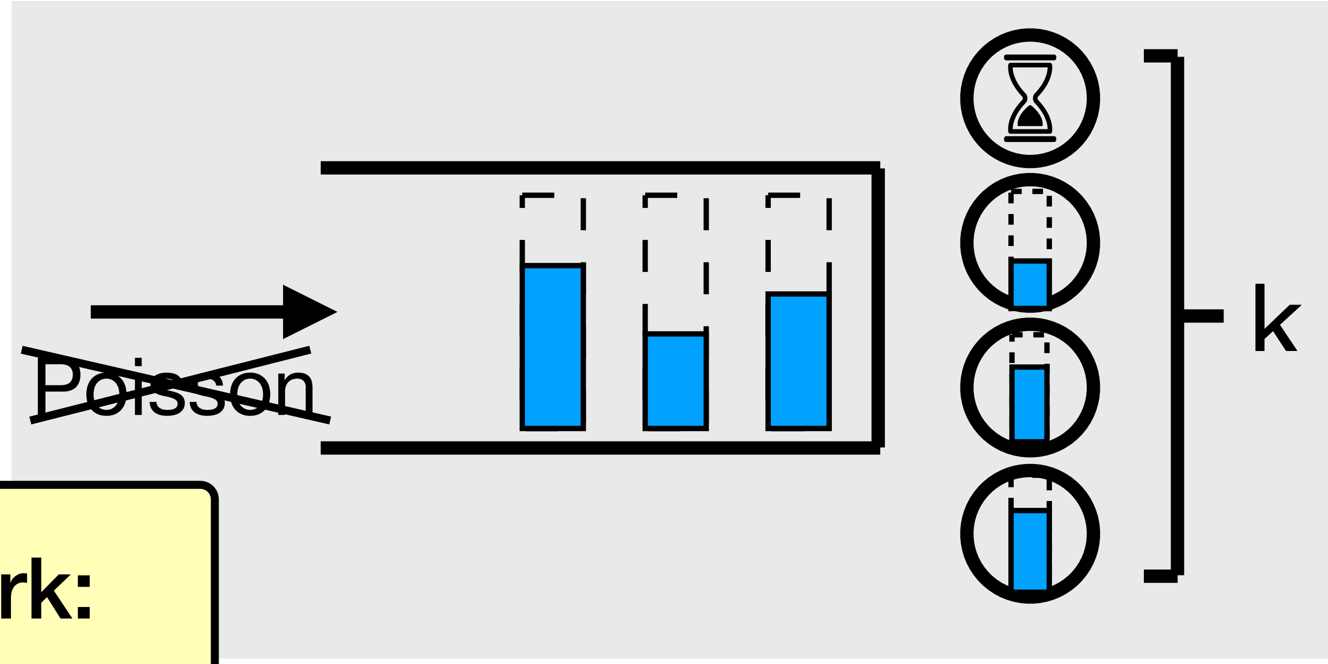
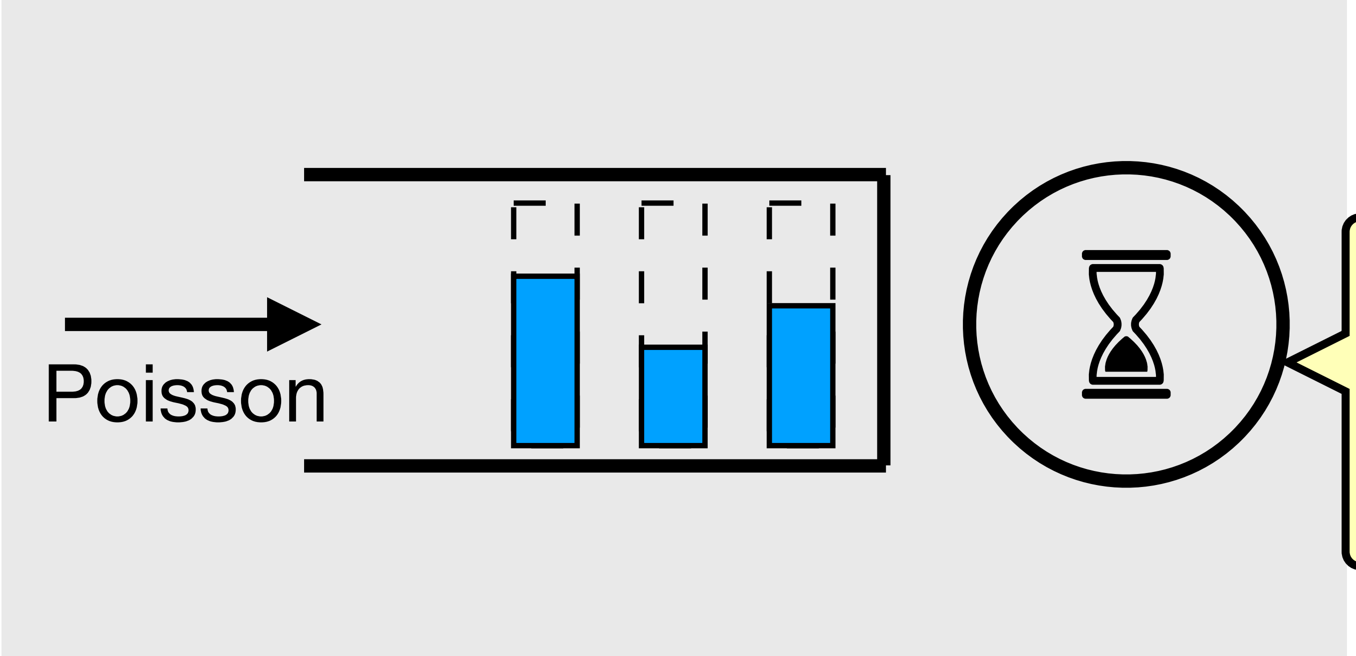


$$\ell_{(b)} = O(1)$$

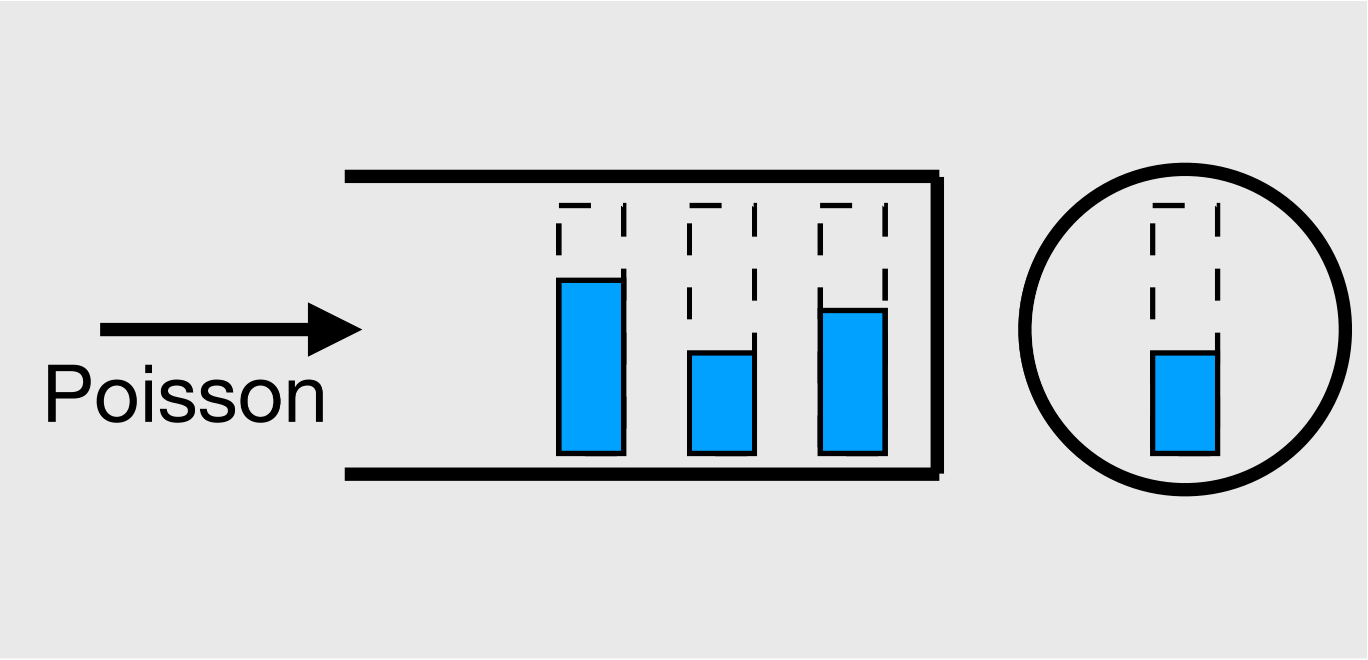
M/G/1/setup



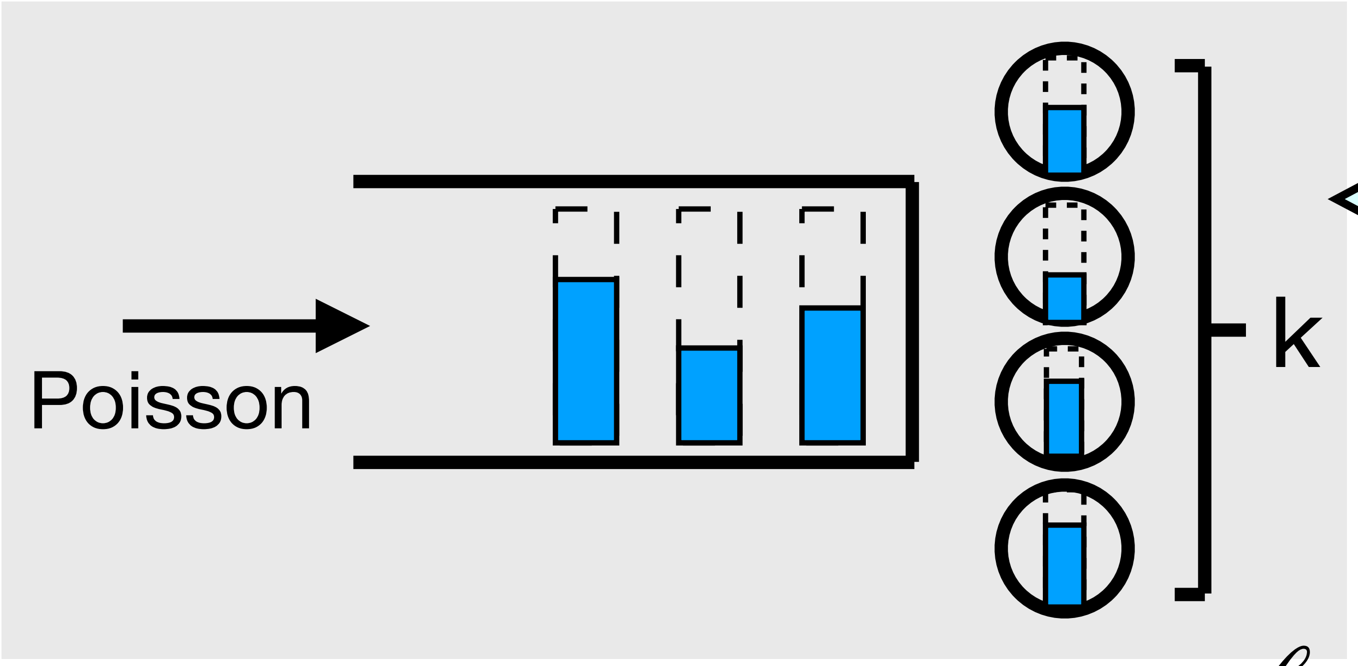
This work:
gap $\leq \ell_{(c)}$

Suboptimality gaps

M/G/1



M/G/k



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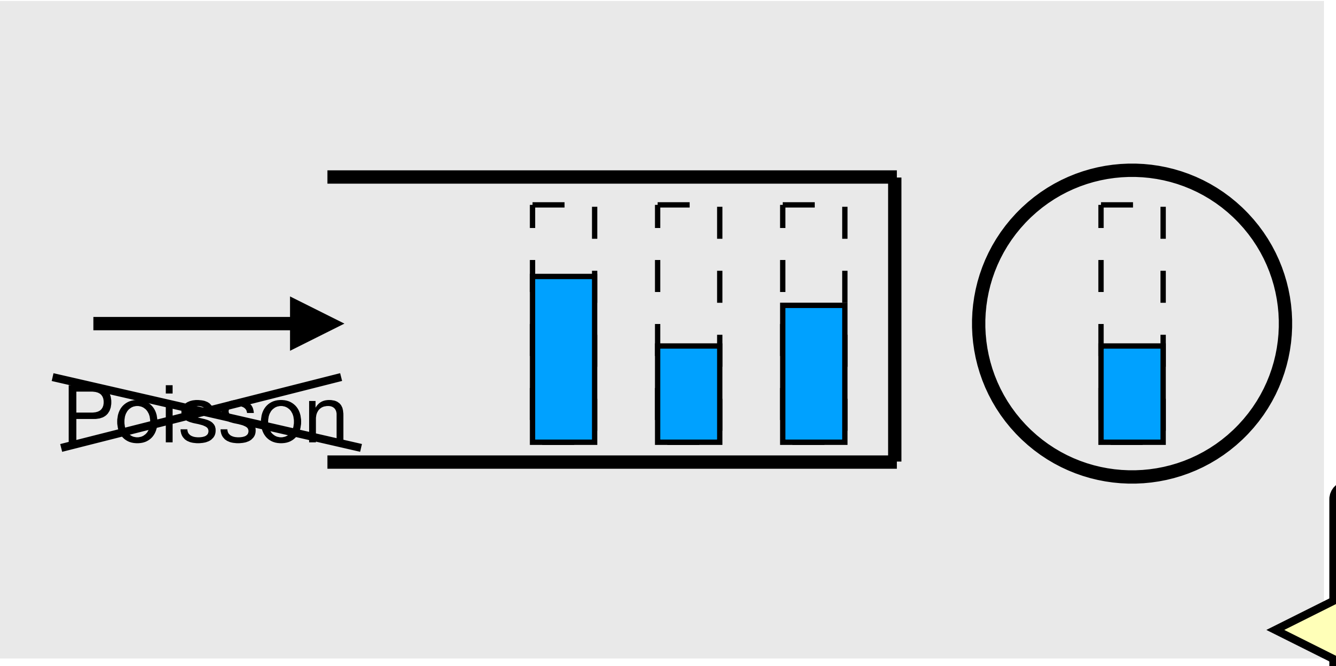

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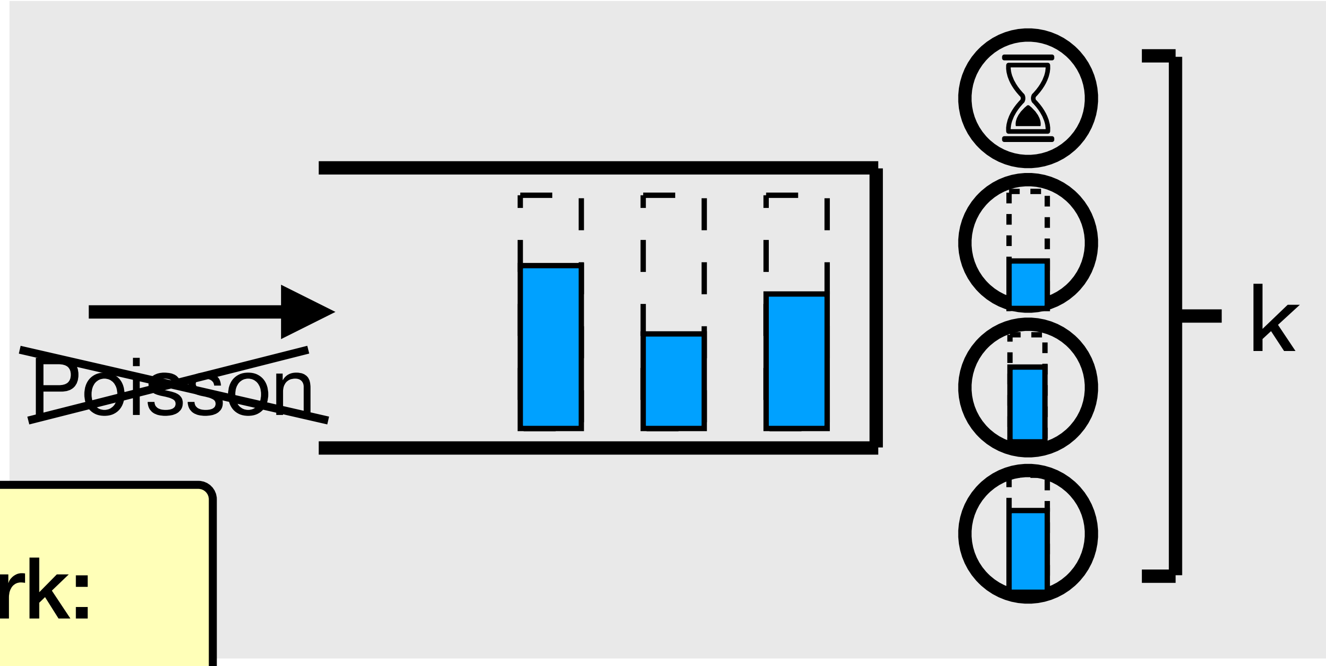

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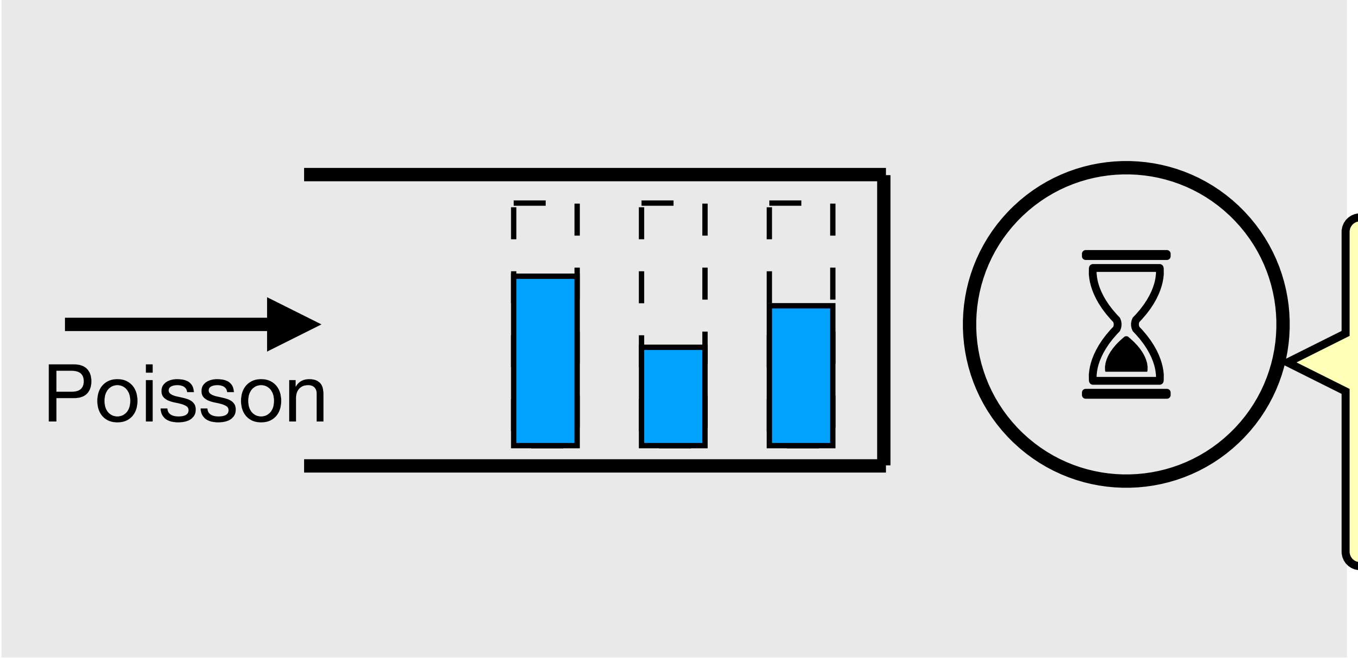

This work:
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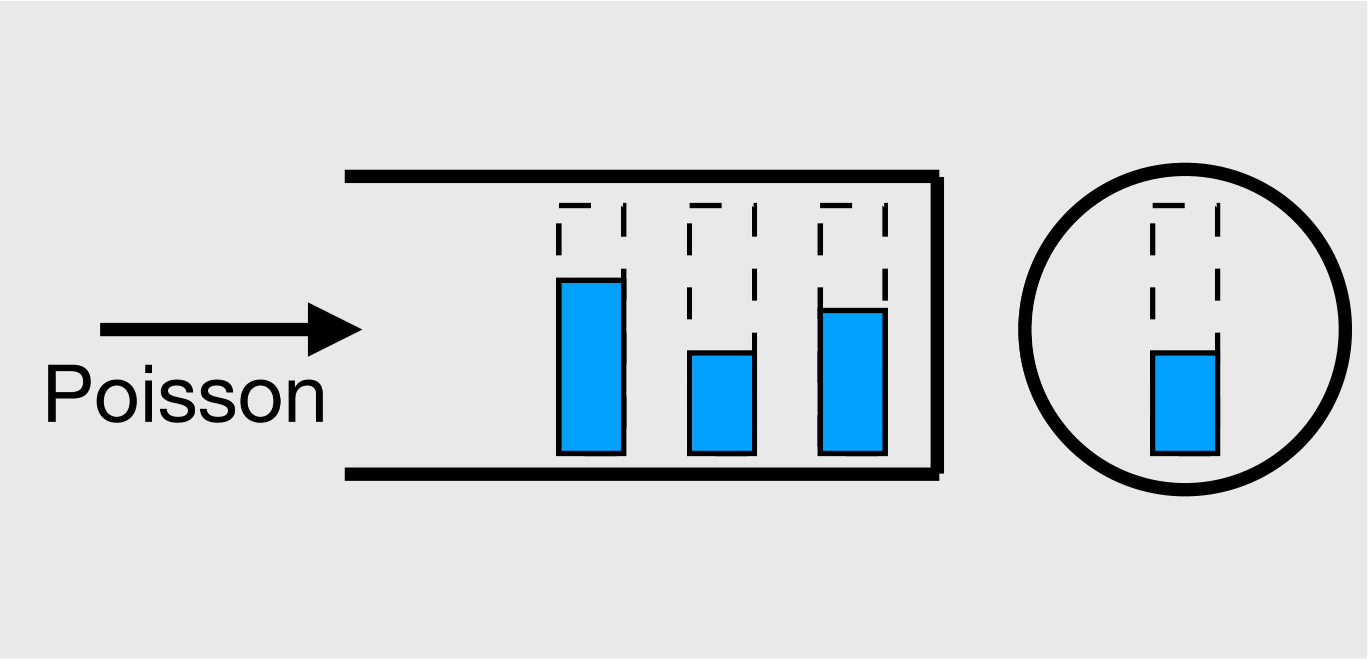
Suboptimality gaps

M/G/1



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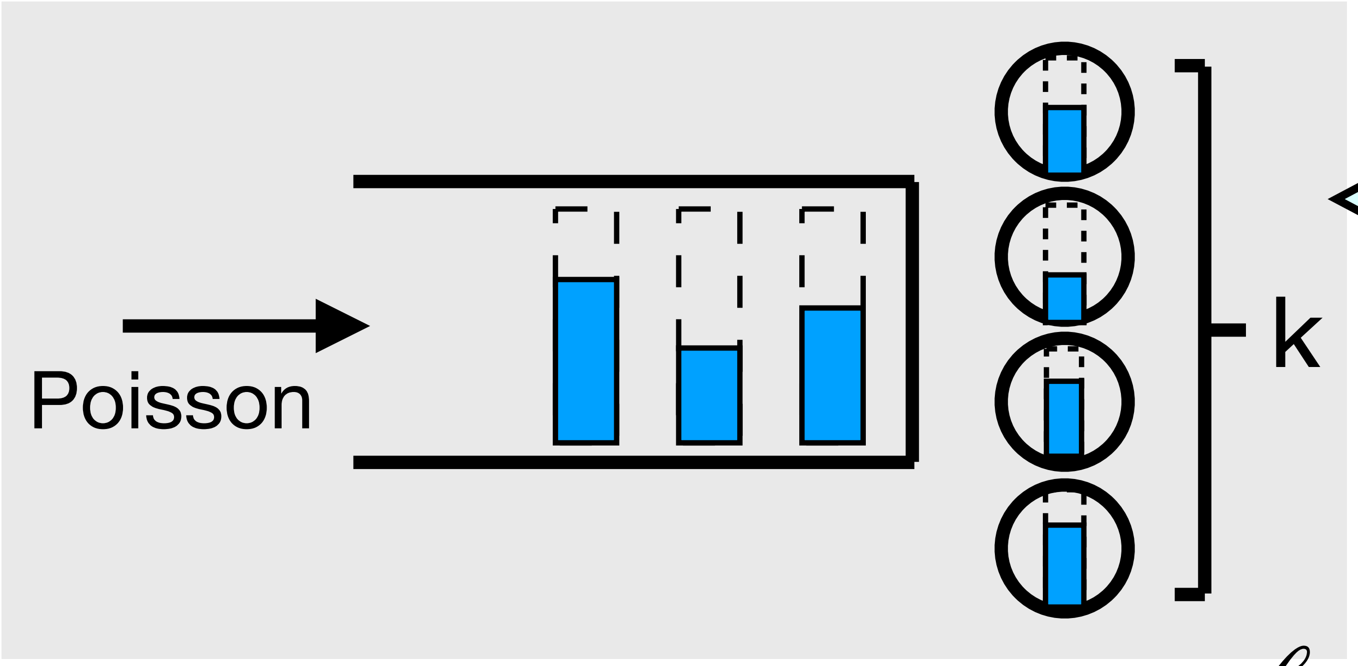

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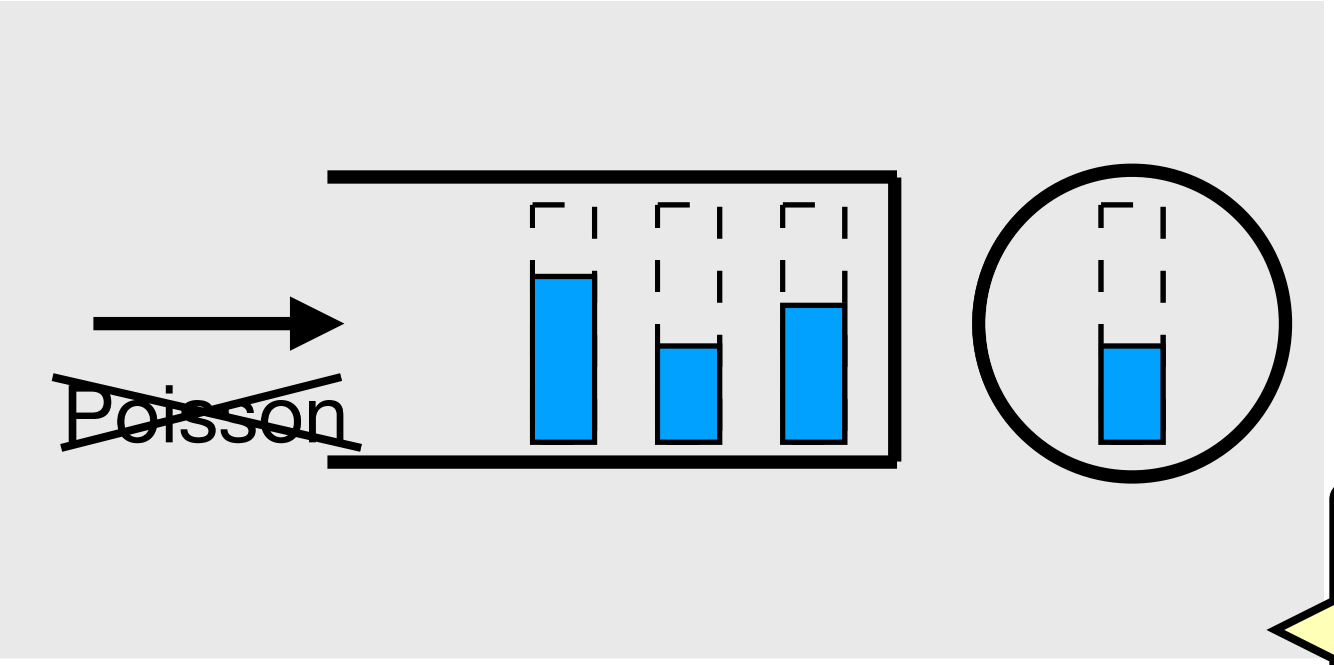
G/G/1



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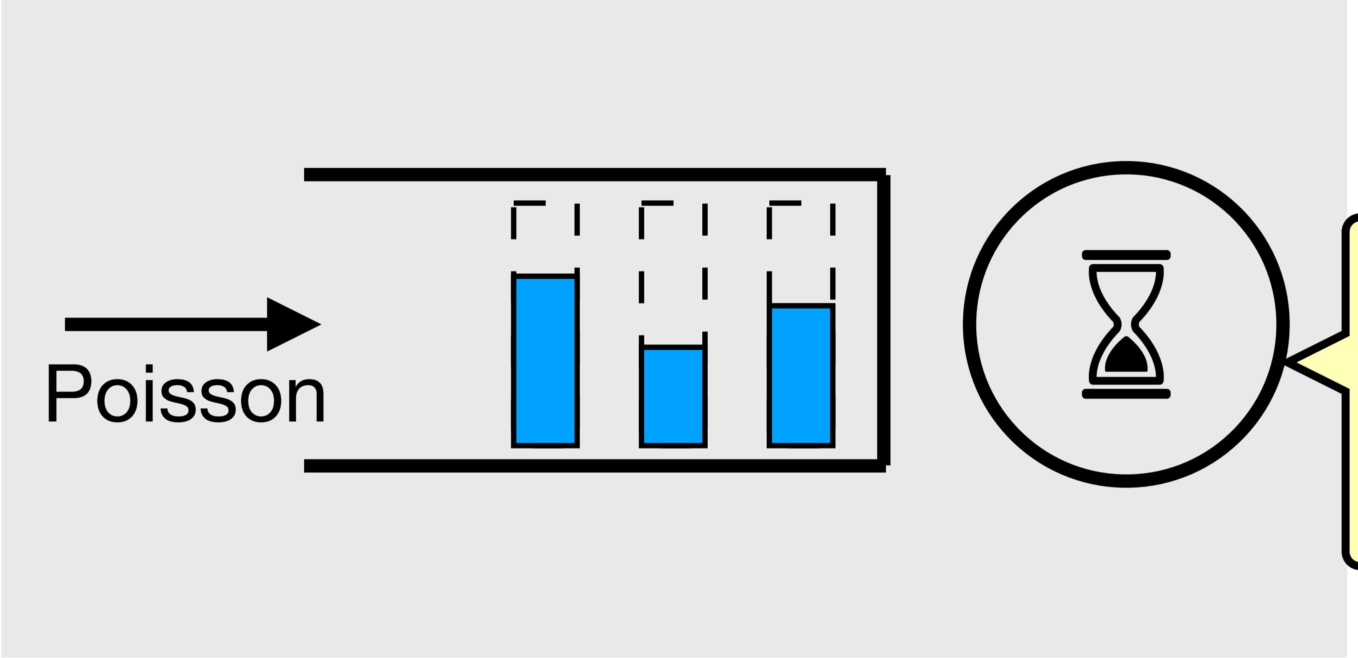

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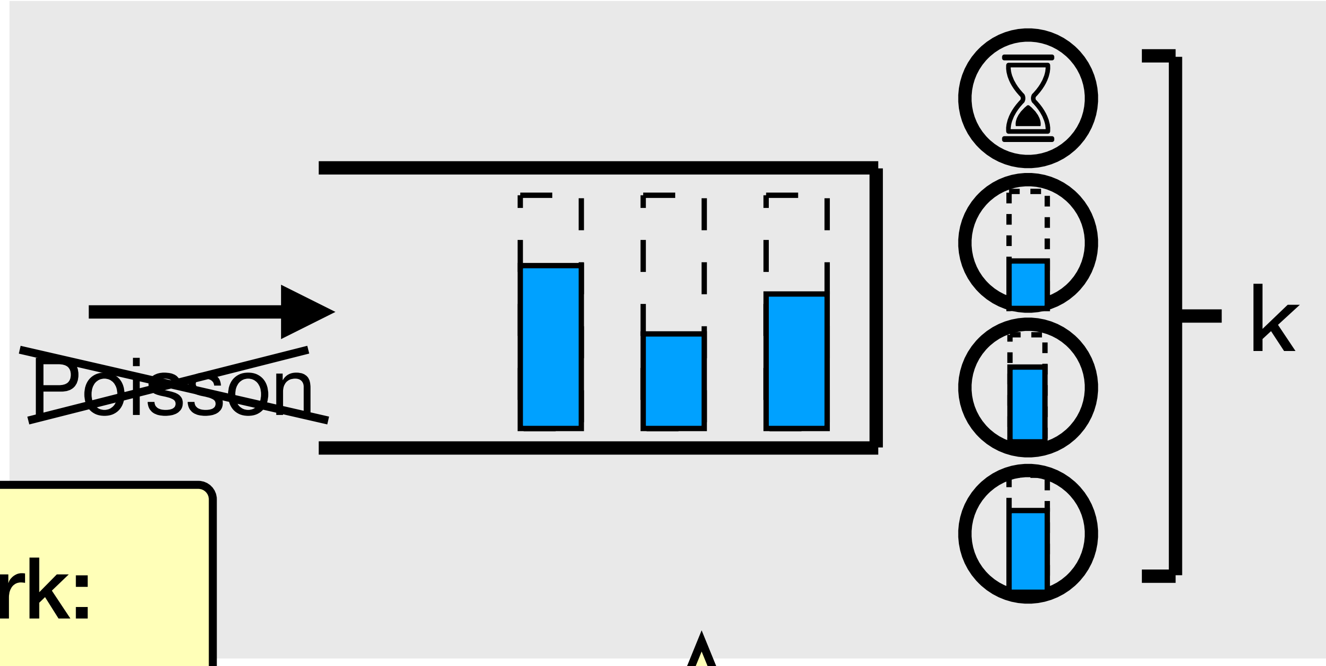
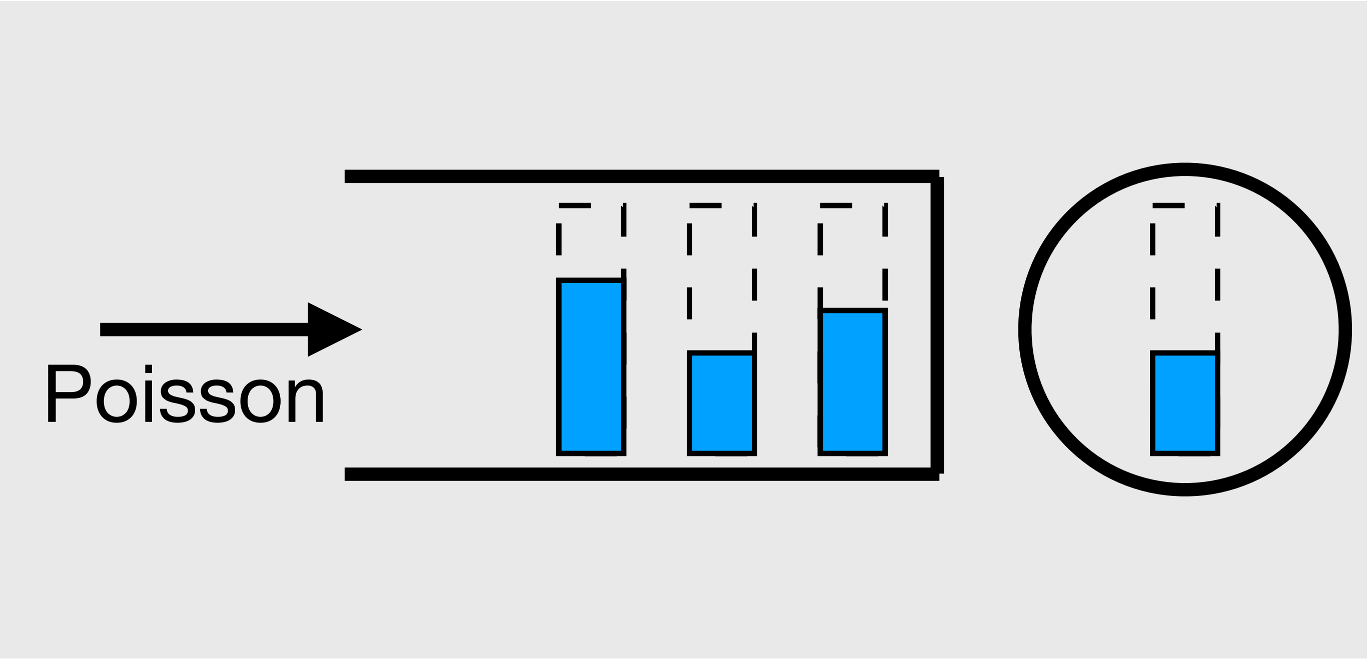
G/G/k/setup



This work:
gap $\leq \ell_{(a)} + \ell_{(b)} + \ell_{(c)}$

Suboptimality gaps

M/G/1



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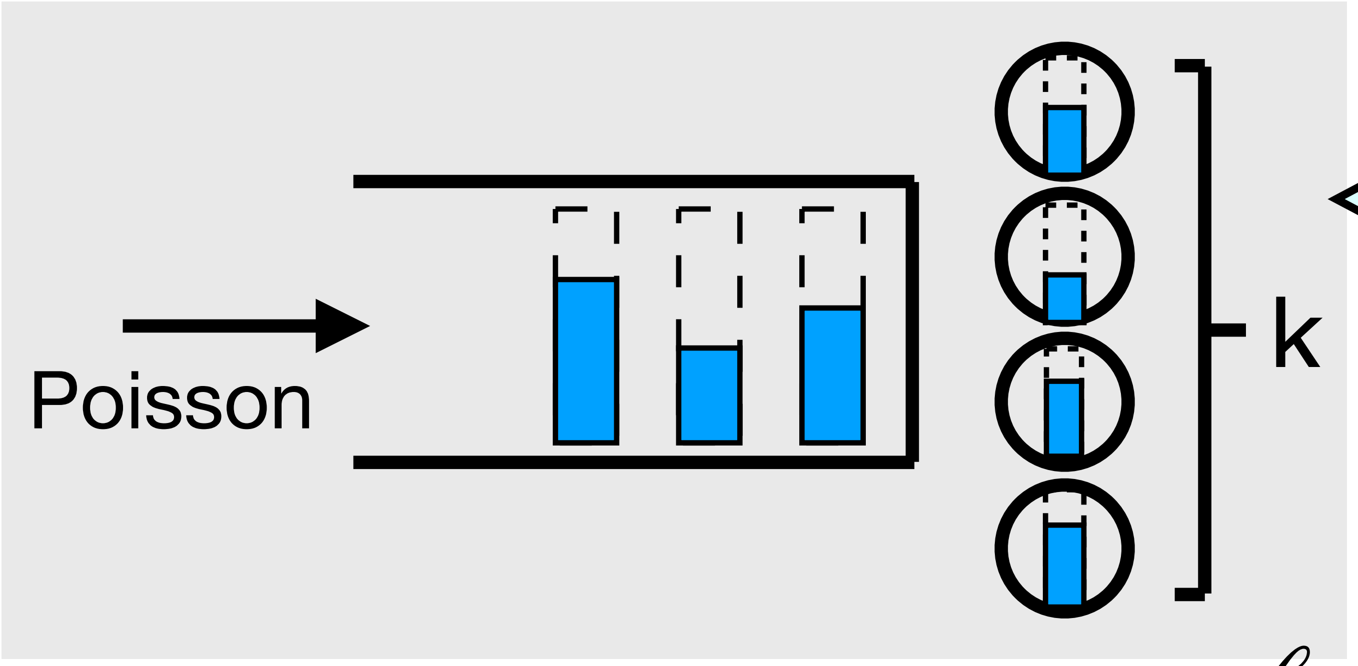

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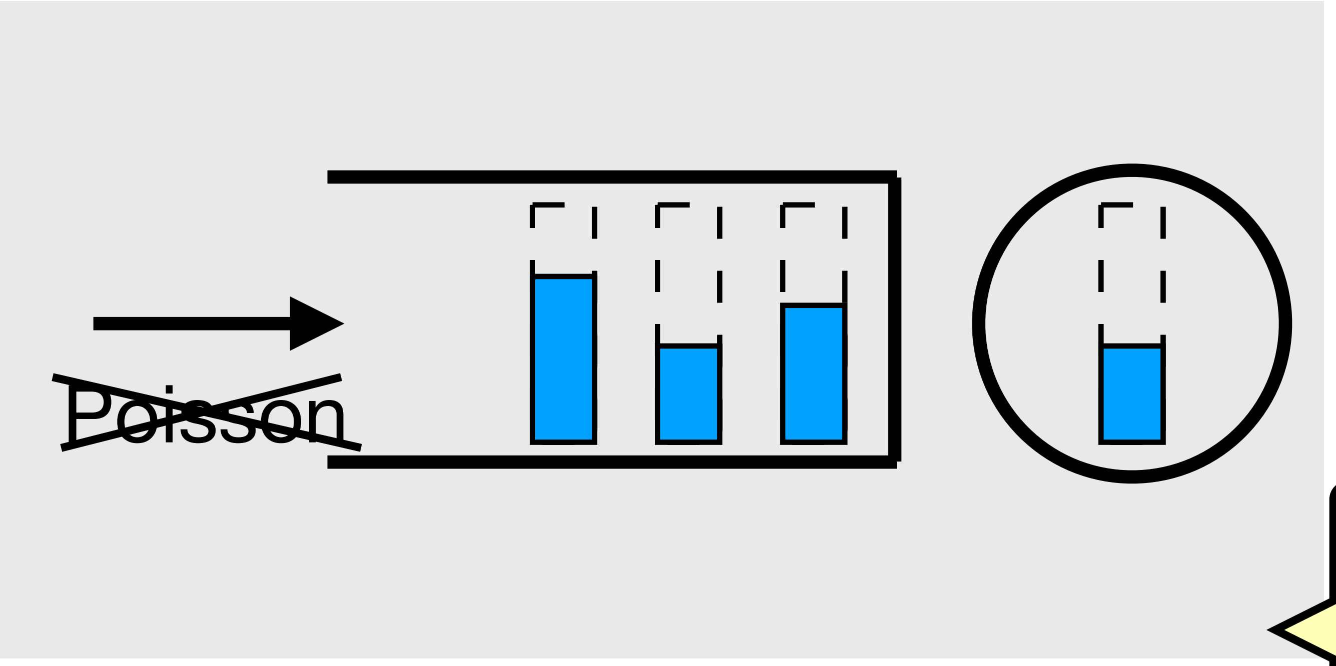
G/G/1



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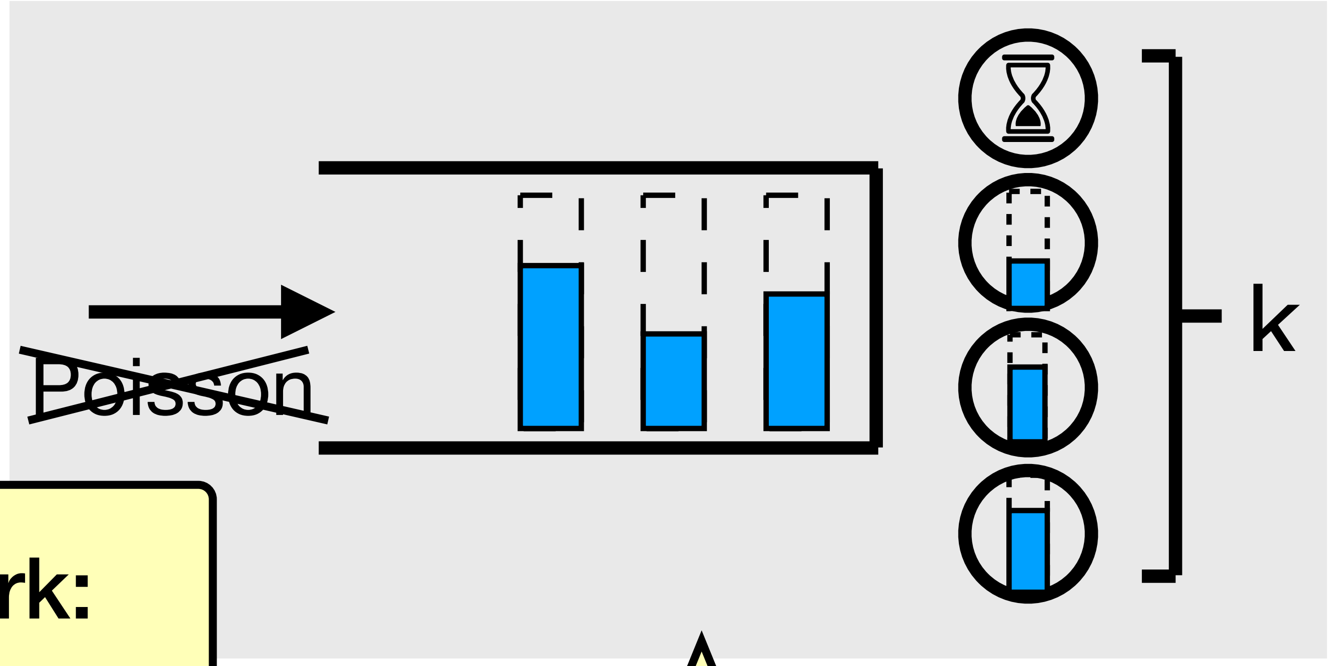
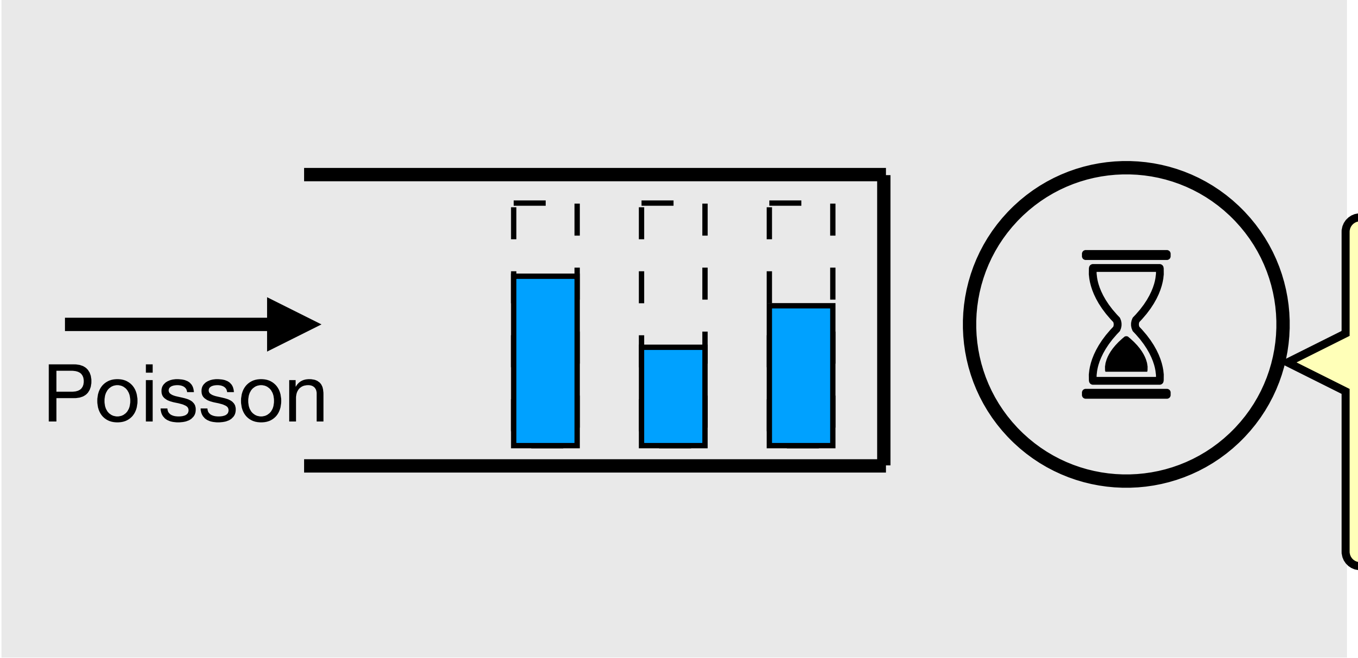
G/G/k/setup



This work:
gap $\leq \ell_{(a)} + \ell_{(b)} + \ell_{(c)}$

M/G/1/setup



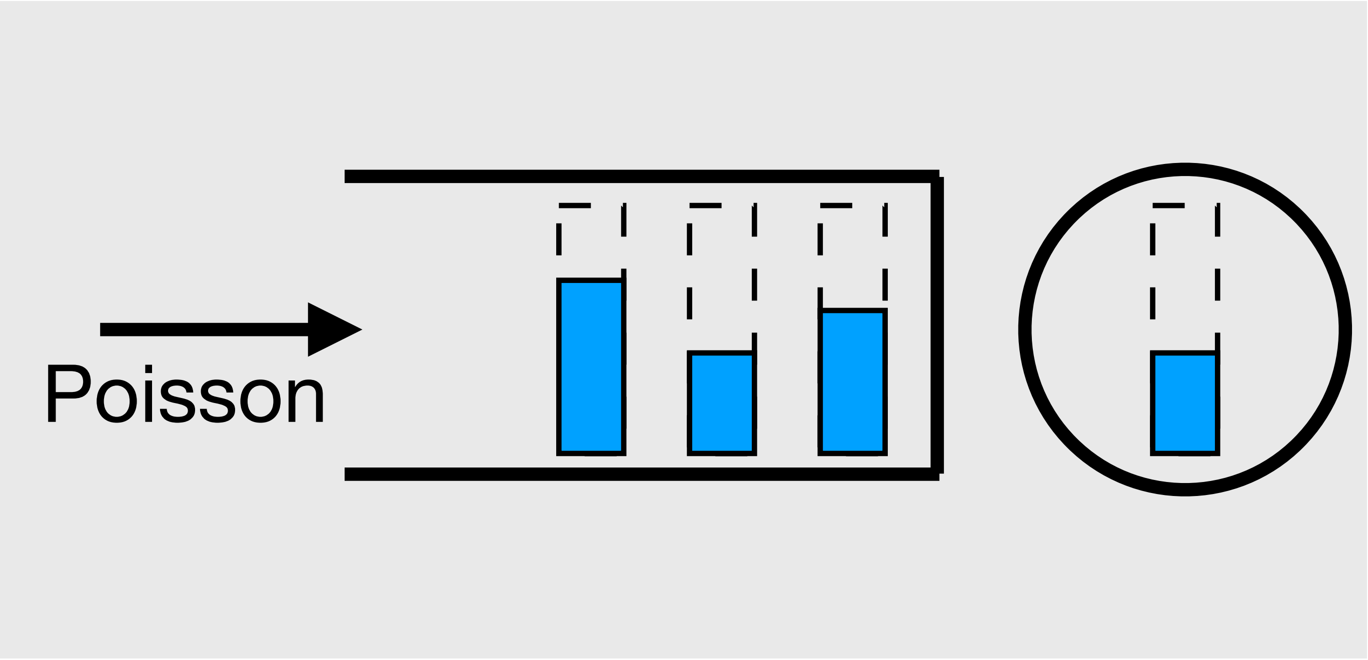
This work:
gap $\leq \ell_{(c)}$

$$\ell_{(c)} = O(1)$$

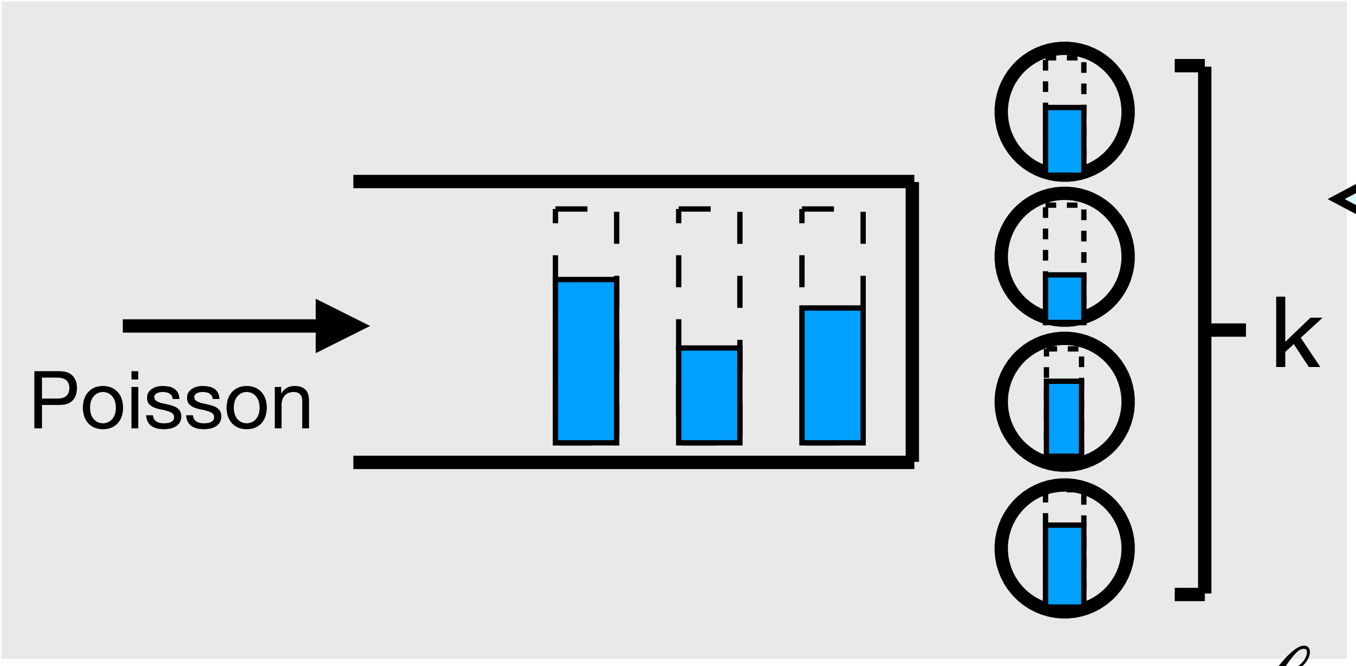
All cases:
heavy-traffic
opt

Suboptimality gaps

M/G/1



M/G/k

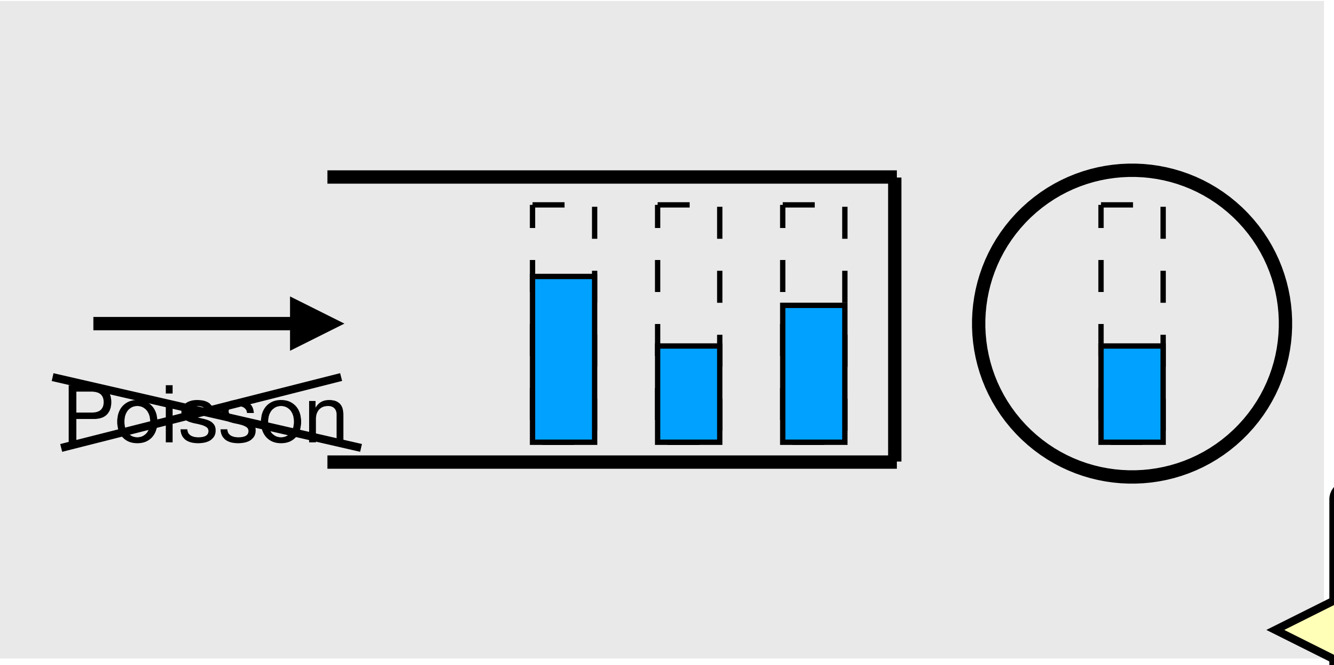


[SGH20]:
gap $\leq \ell_{(a)}$
heavy-traffic opt

Assumption:
interarrival times
light-tailed

$$\ell_{(a)} = O\left(\log \frac{1}{1-\rho}\right)$$

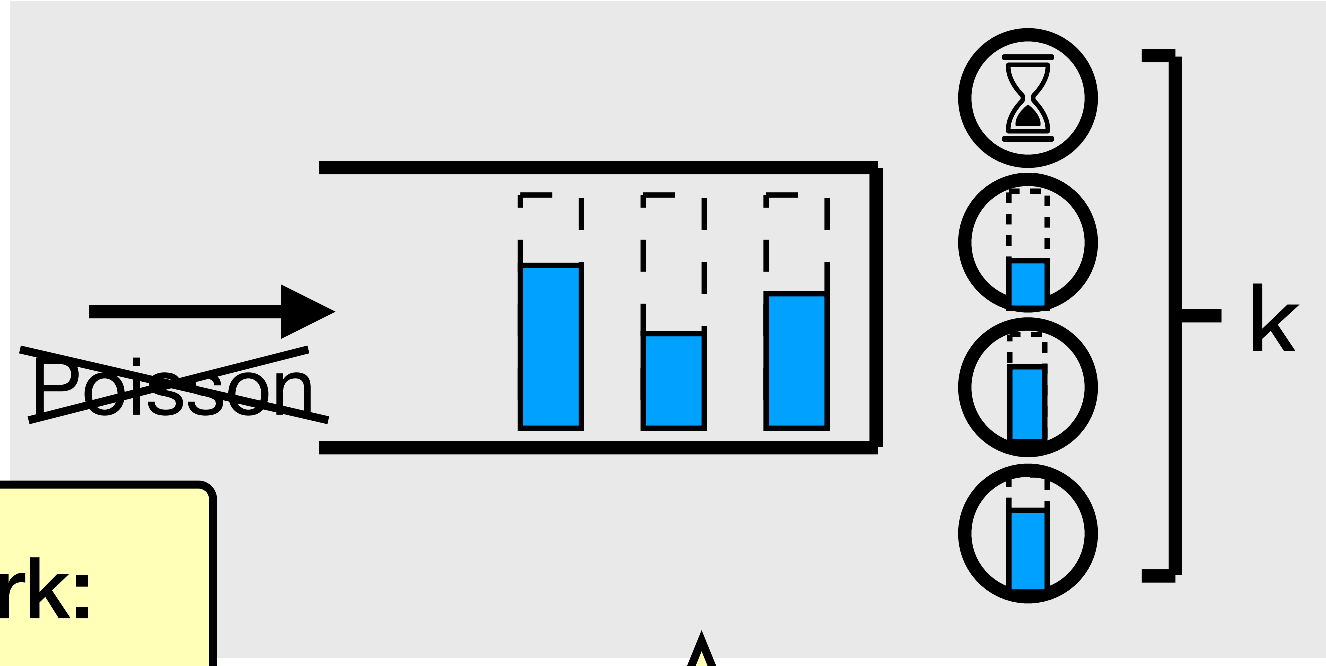
G/G/1



This work:
gap $\leq \ell_{(b)}$

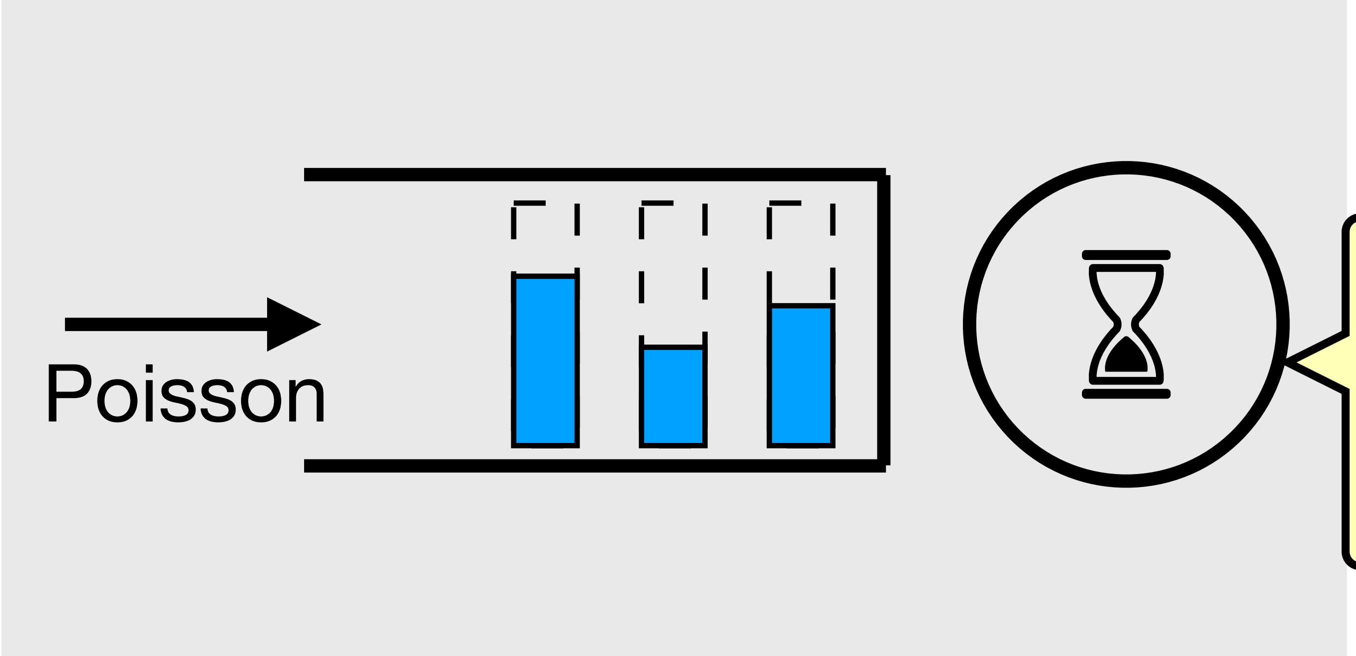
$$\ell_{(b)} = O(1)$$

G/G/k/setup



This work:
gap $\leq \ell_{(a)} + \ell_{(b)} + \ell_{(c)}$

M/G/1/setup



This work:
gap $\leq \ell_{(c)}$

$$\ell_{(c)} = O(1)$$

All cases:
heavy-traffic
opt

Outline for the rest of the talk

- What is our problem and result? ✓

Outline for the rest of the talk

G/G/k/setup-Gittins

- What is our problem and result? ✓

Outline for the rest of the talk

G/G/k/setup-Gittins

suboptimality gaps
+ heavy-traffic opt

- What is our problem and result? ✓

Outline for the rest of the talk

G/G/k/setup-Gittins

suboptimality gaps
+ heavy-traffic opt

- What is our problem and result? ✓
- How does our G/G/k/setup analysis work?

Outline for the rest of the talk

G/G/k/setup-Gittins

suboptimality gaps
+ heavy-traffic opt

- What is our problem and result? ✓
- How does our G/G/k/setup analysis work?
- What is the main obstacle and how do we solve it?

Analysis roadmap of G/G/k/setup

Analysis roadmap of G/G/k/setup

based on [SGH20]

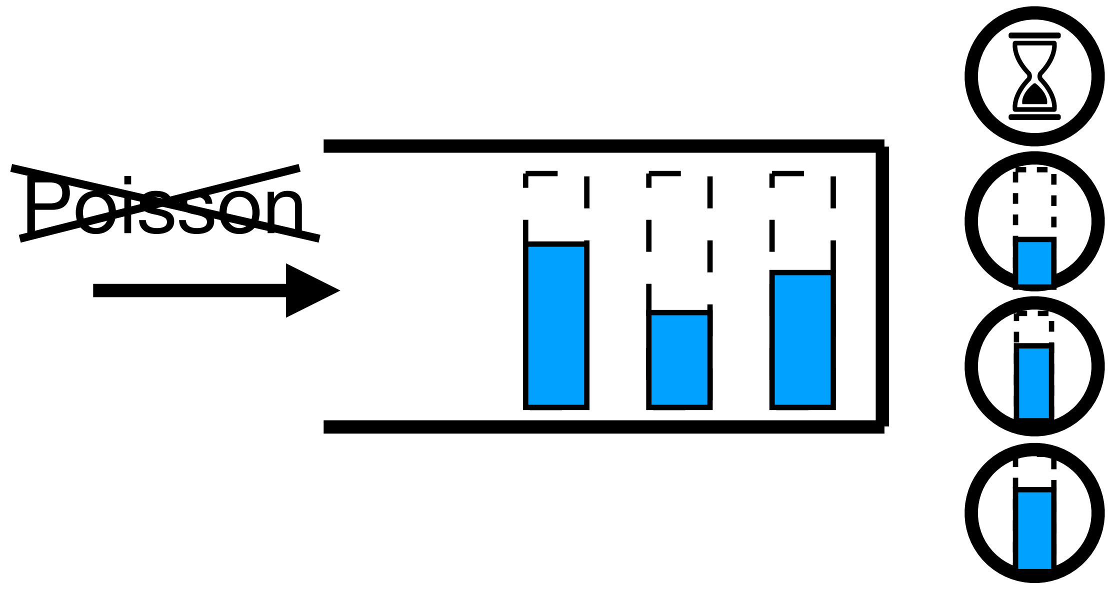
Analysis roadmap of G/G/k/setup

based on [SGH20]

number in

Analysis roadmap of G/G/k/setup

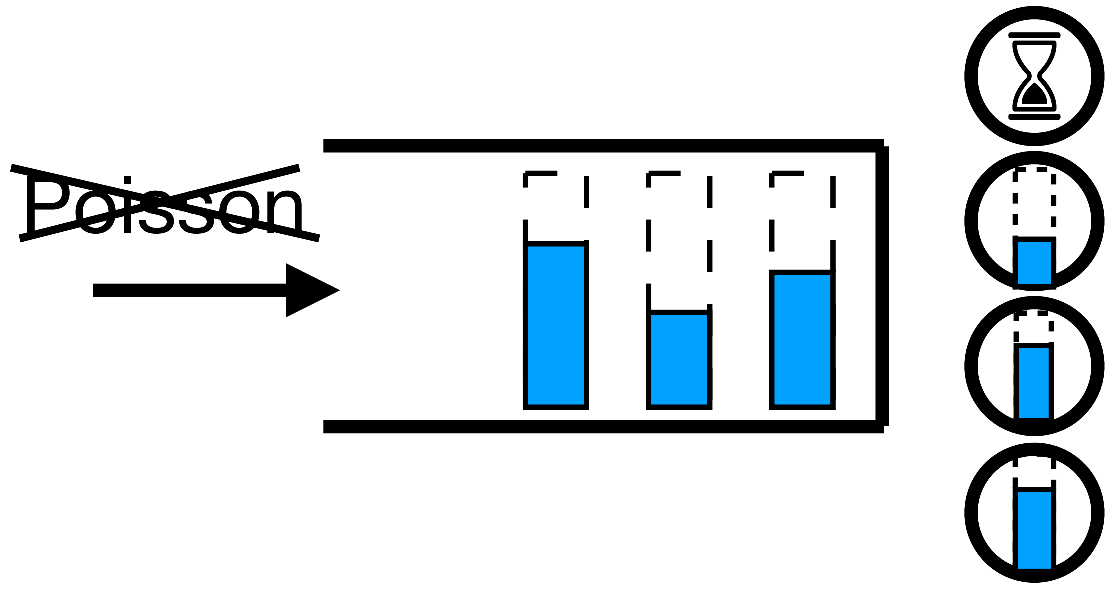
based on [SGH20]



number in **G/G/k/setup-Gittins**

Analysis roadmap of G/G/k/setup

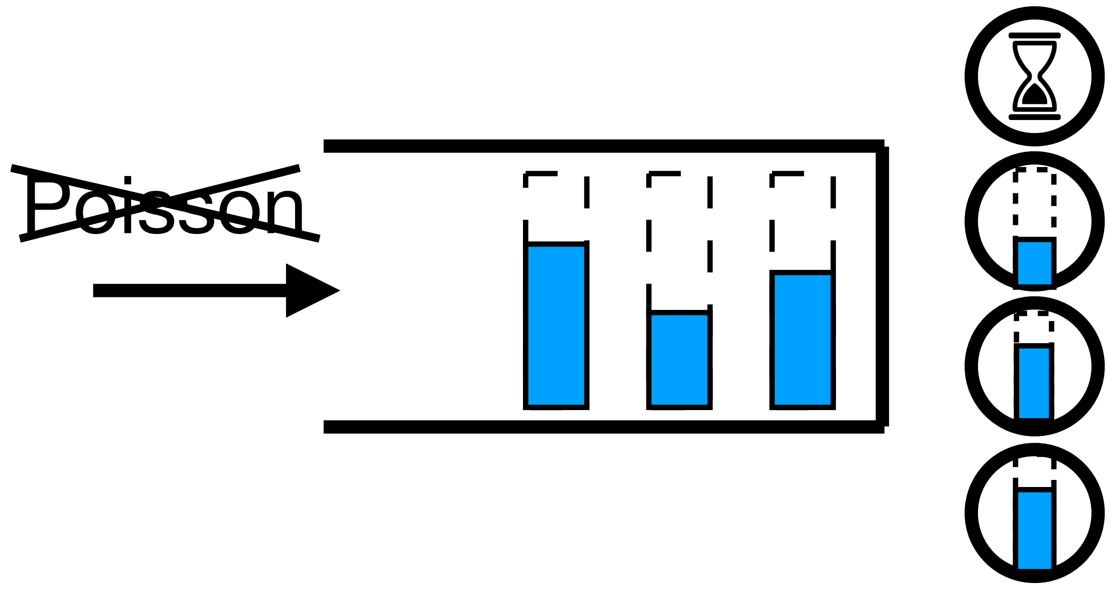
based on [SGH20]

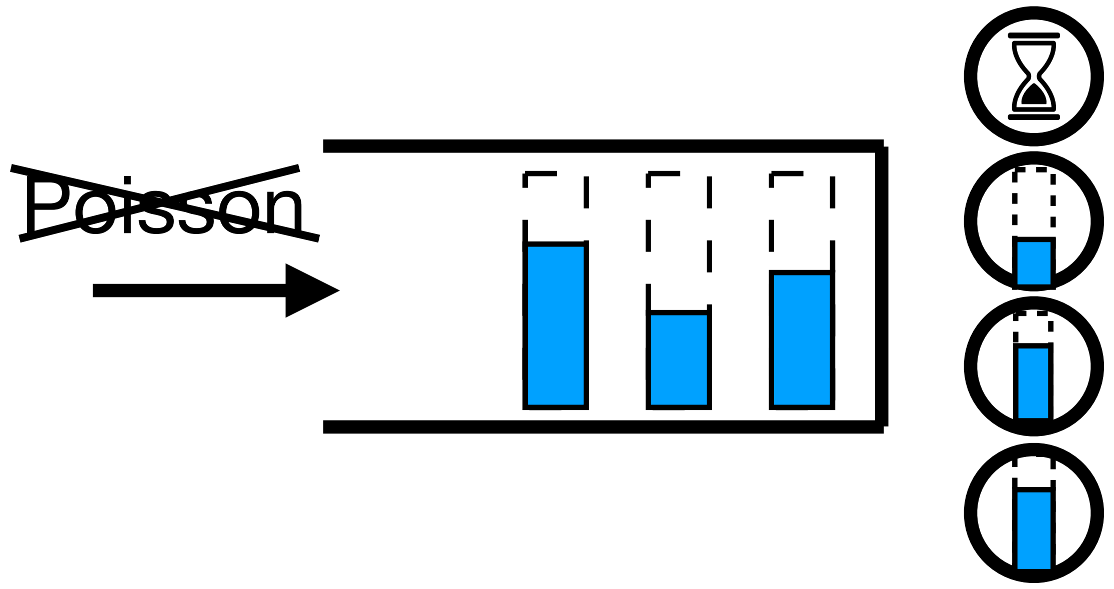


number in **G/G/k/setup-Gittins** $\geq$

Analysis roadmap of G/G/k/setup

based on [SGH20]

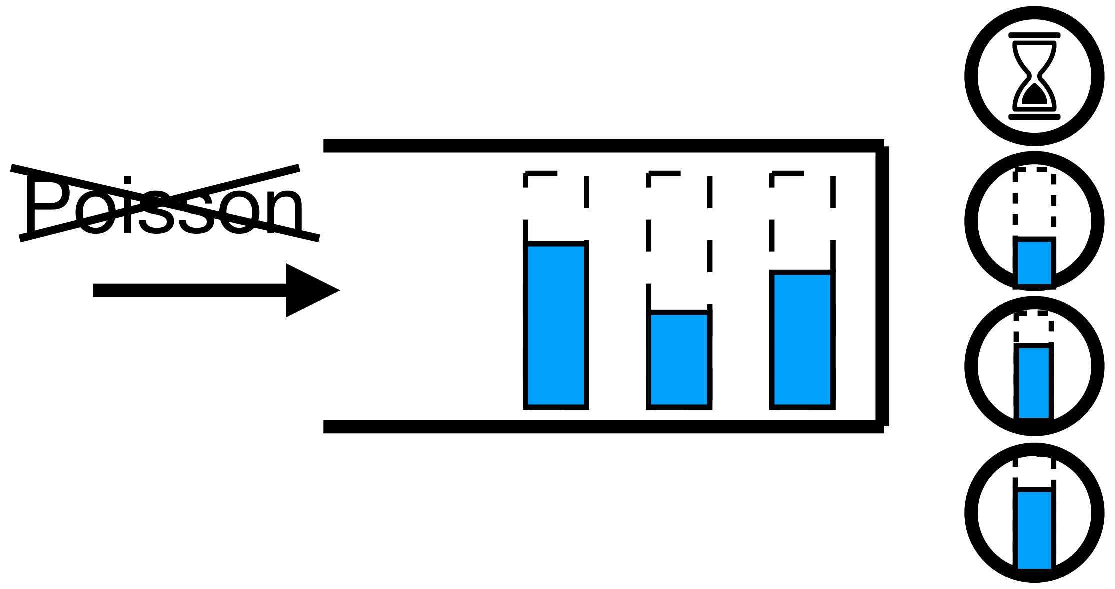


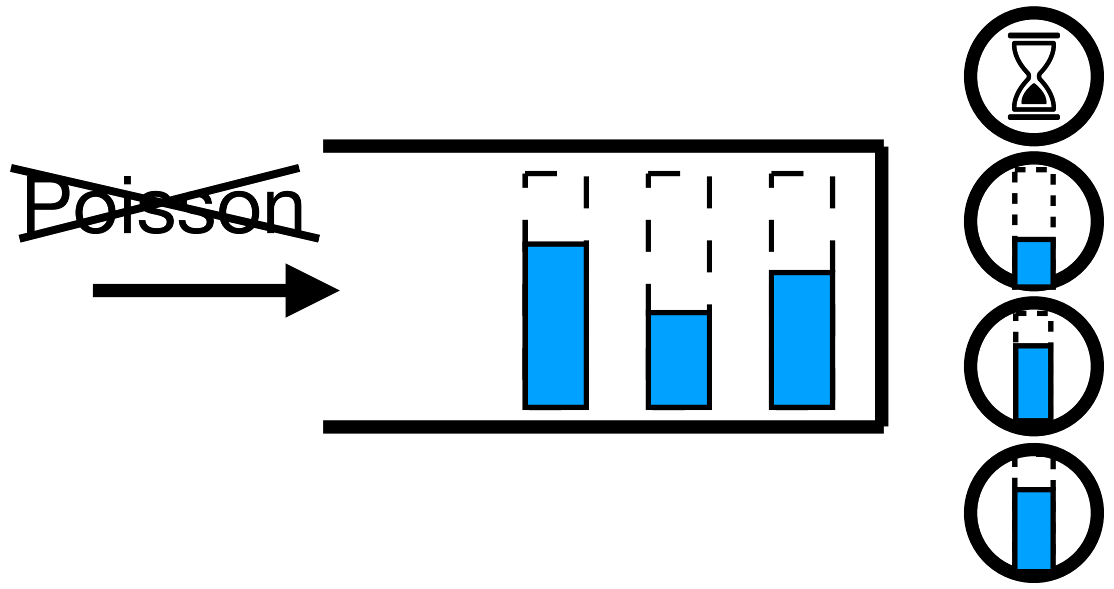


number in **G/G/k/setup-Gittins** $\geq$ **G/G/k/setup-OPT**

Analysis roadmap of G/G/k/setup

based on [SGH20]



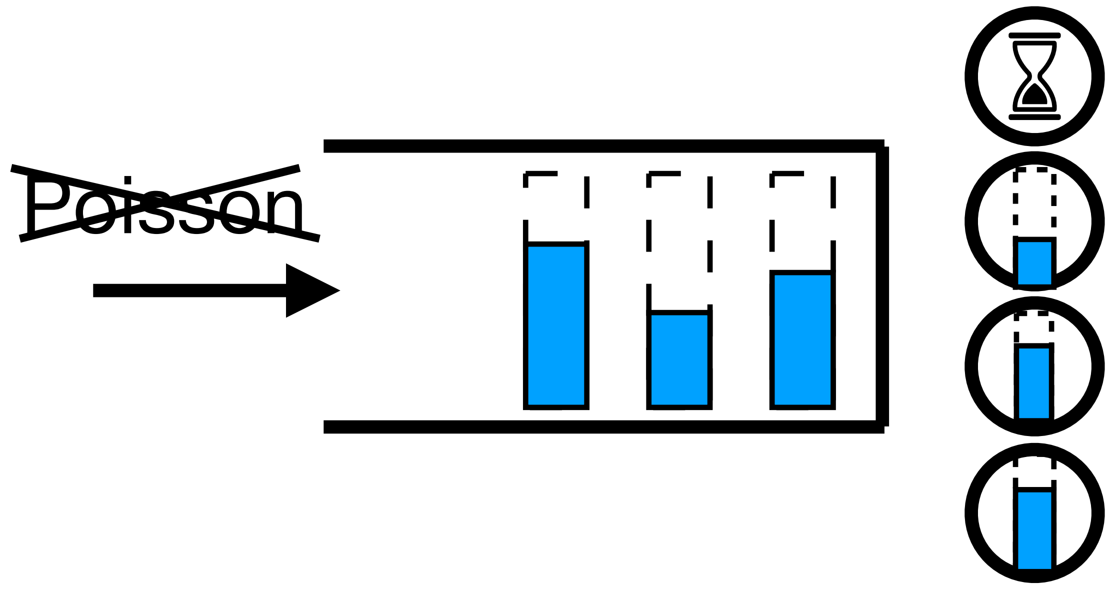


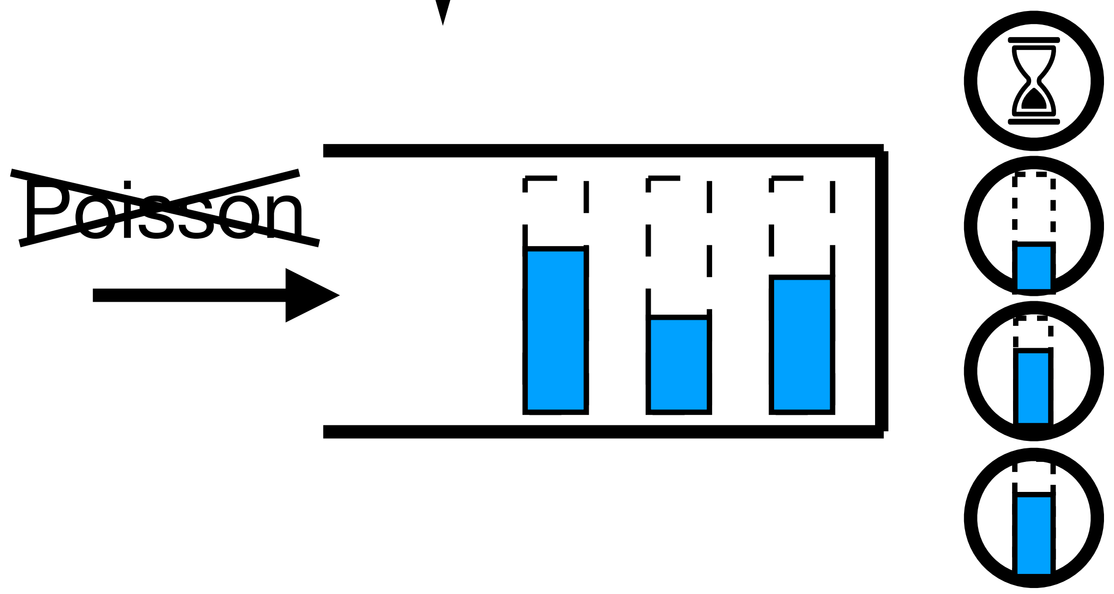
number in $\underbrace{\textbf{G/G/k/setup-Gittins} \geq \textbf{G/G/k/setup-OPT}}_{\text{Gap}}$

Analysis roadmap of G/G/k/setup

based on [SGH20]

Unknown and complicated



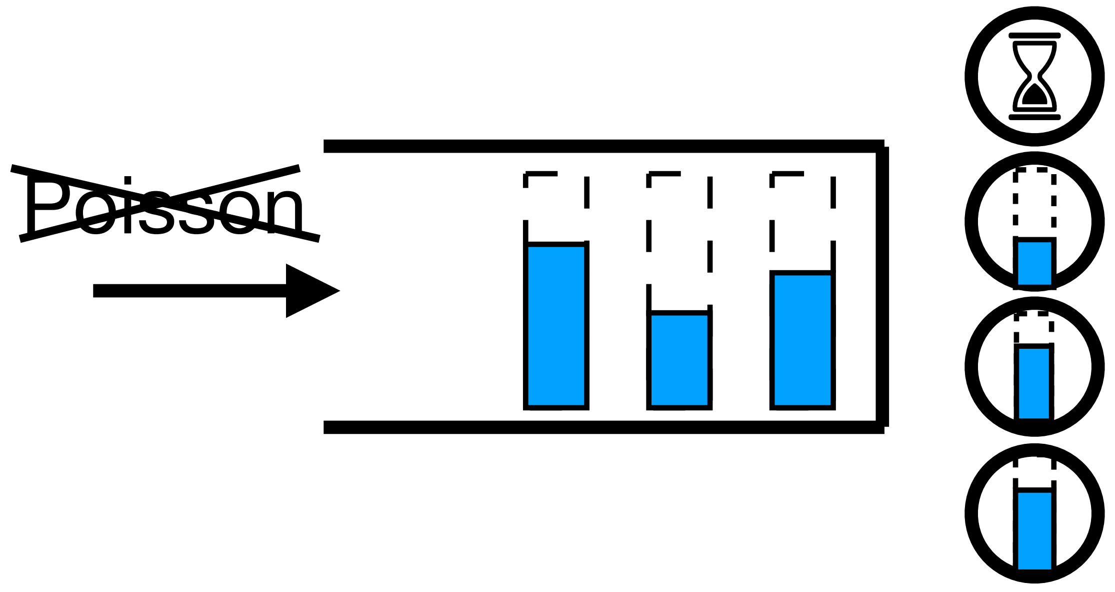


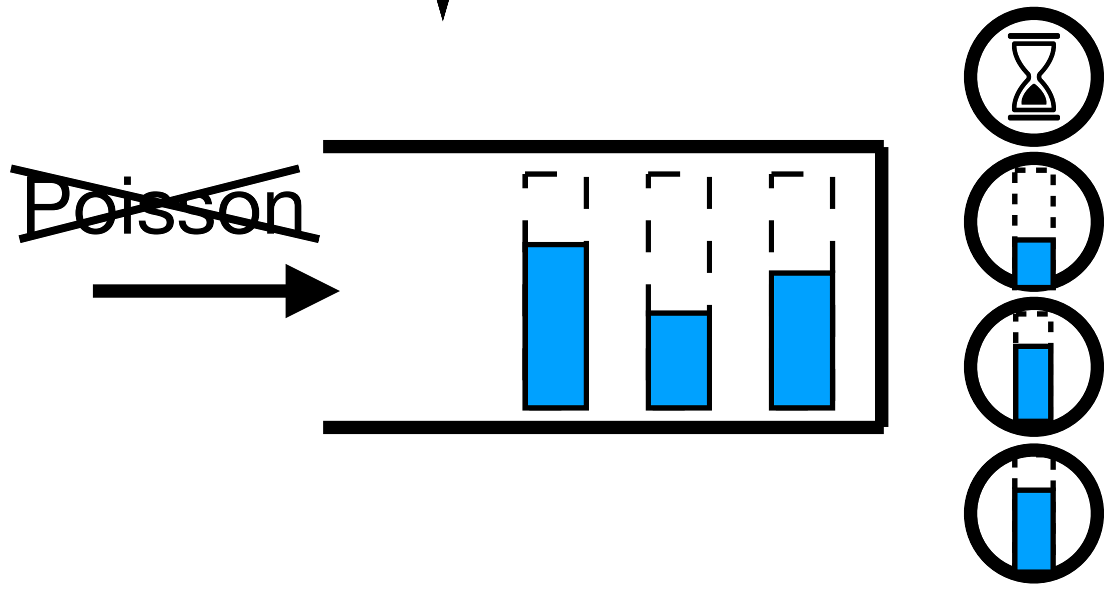
number in $\underbrace{\textbf{G/G/k/setup-Gittins} \geq \textbf{G/G/k/setup-OPT}}_{\text{Gap}}$

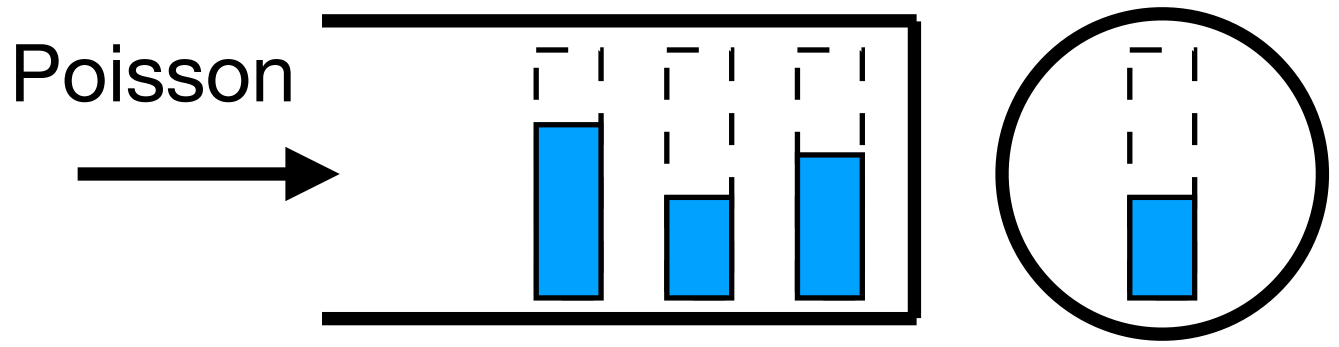
Analysis roadmap of G/G/k/setup

based on [SGH20]

Unknown and complicated







number in **G/G/k/setup-Gittins** ≥ **G/G/k/setup-OPT** ≥ **G/G/1-OPT**

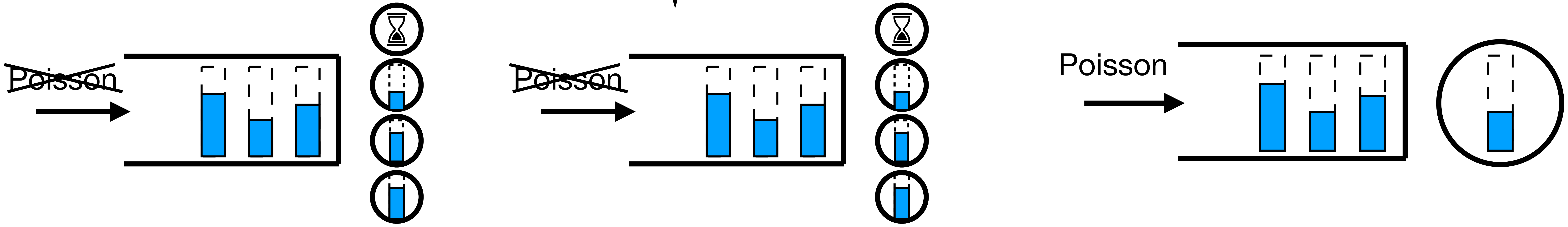
Gap

Analysis roadmap of G/G/k/setup

based on [SGH20]

Unknown and complicated

Simpler

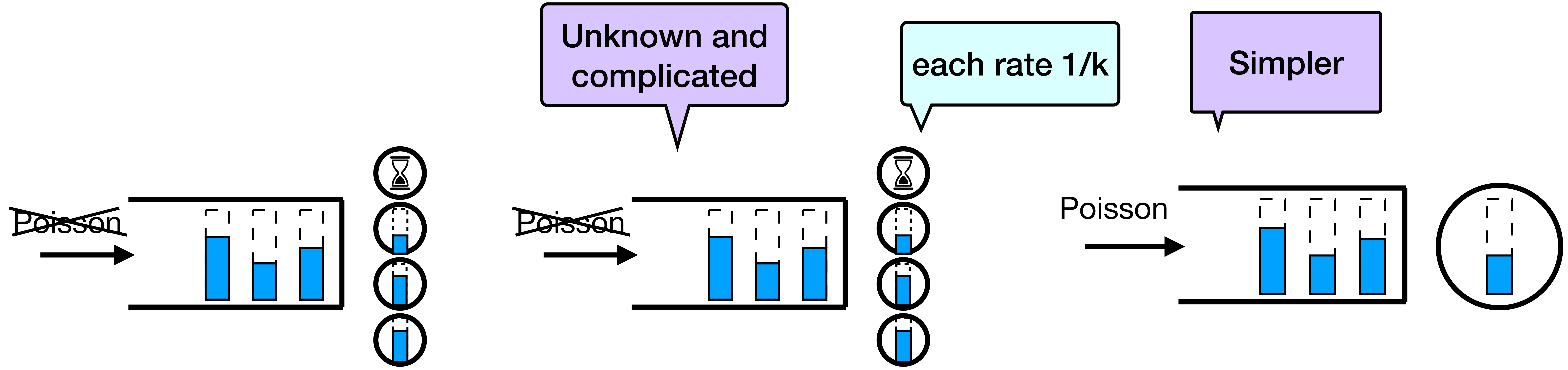


number in **G/G/k/setup-Gittins** $\geq$ **G/G/k/setup-OPT** $\geq$ **G/G/1-OPT**

$$\underbrace{\text{G/G/k/setup-Gittins} \geq \text{G/G/k/setup-OPT}}_{\text{Gap}}$$

Analysis roadmap of G/G/k/setup

based on [SGH20]

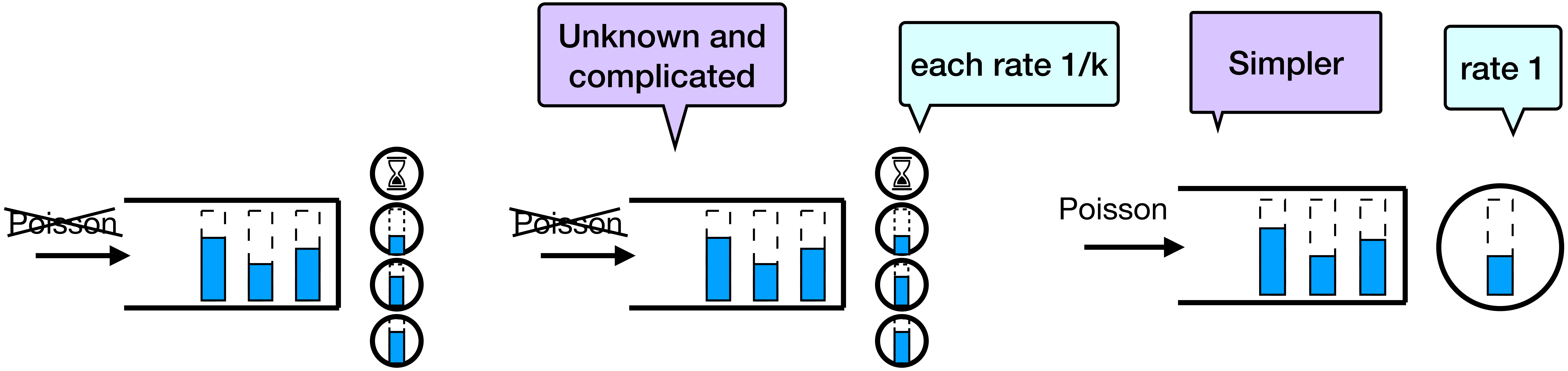


number in **G/G/k/setup-Gittins** ≥ **G/G/k/setup-OPT** ≥ **G/G/1-OPT**

Gap

Analysis roadmap of G/G/k/setup

based on [SGH20]

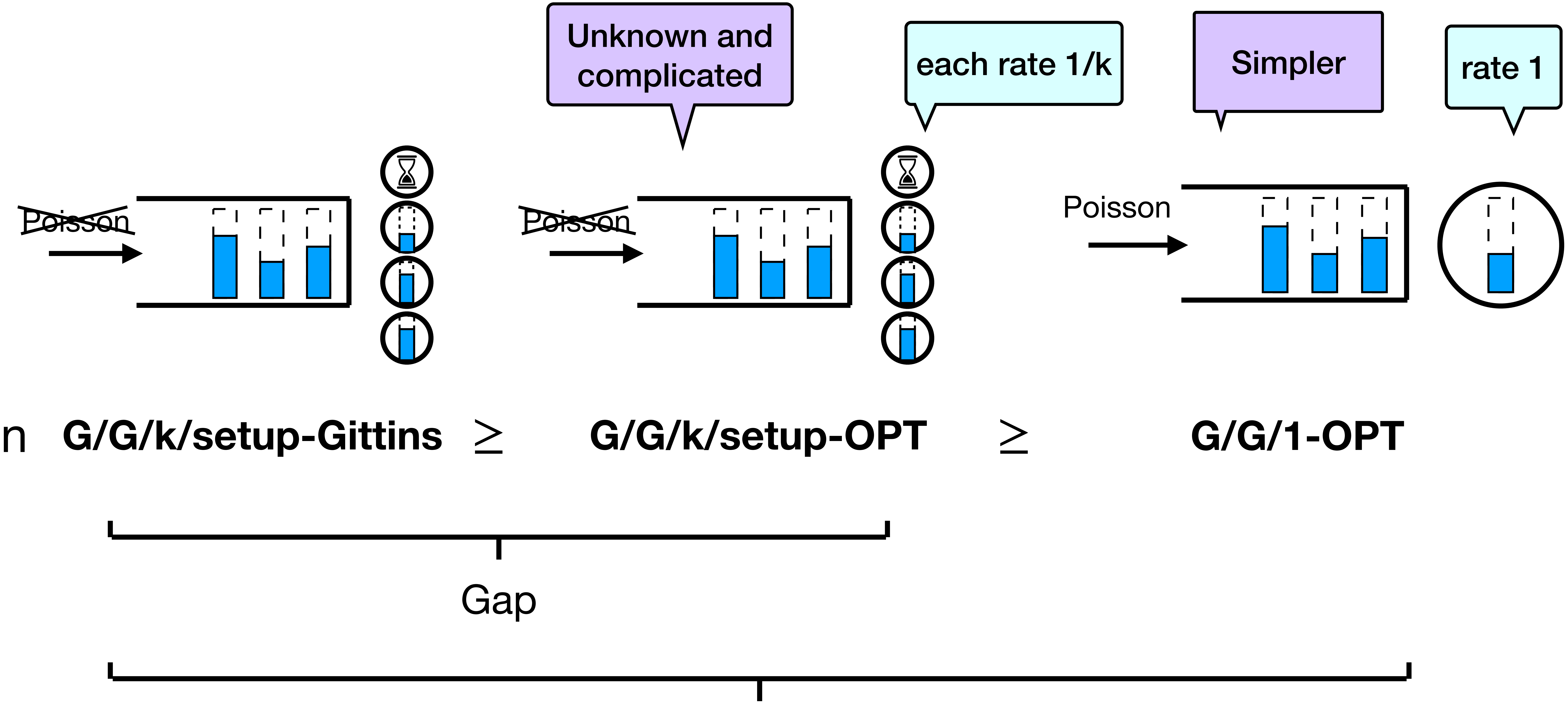


number in **G/G/k/setup-Gittins** ≥ **G/G/k/setup-OPT** ≥ **G/G/1-OPT**

Gap

Analysis roadmap of G/G/k/setup

based on [SGH20]



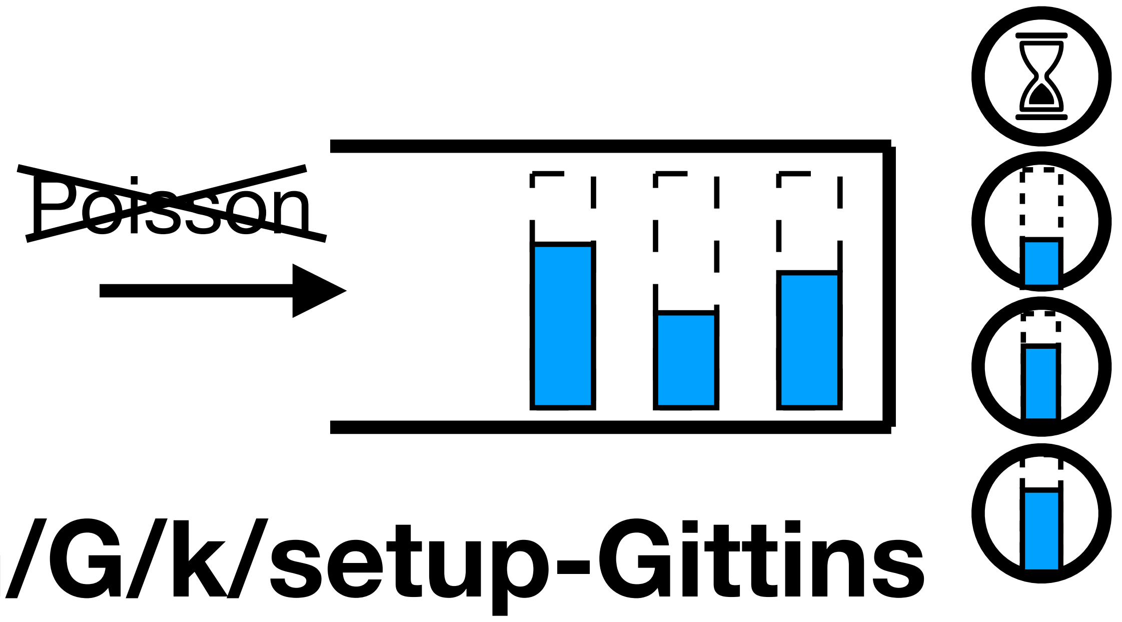
number in **G/G/k/setup-Gittins** $\geq$ **G/G/k/setup-OPT** $\geq$ **G/G/1-OPT**

Gap

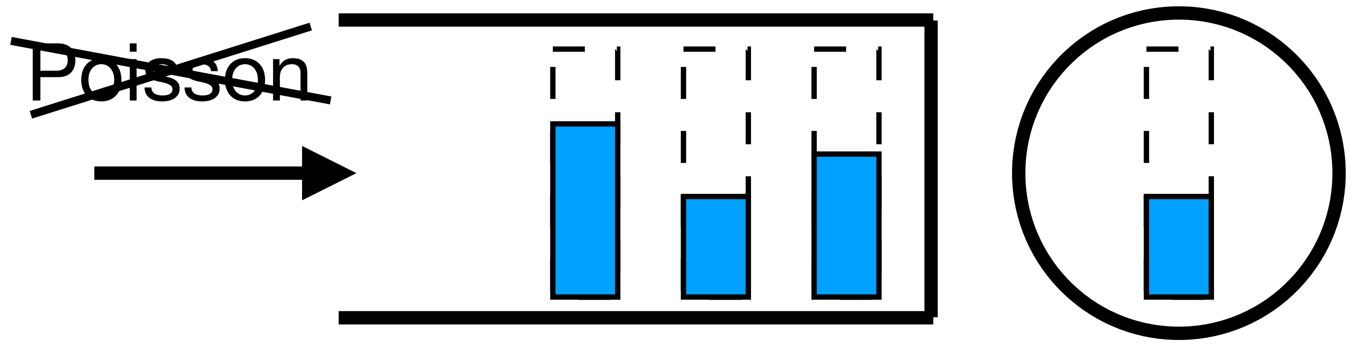
Enough to bound the difference

Analysis roadmap of G/G/k/setup

based on [SGH20]



G/G/k/setup-Gittins

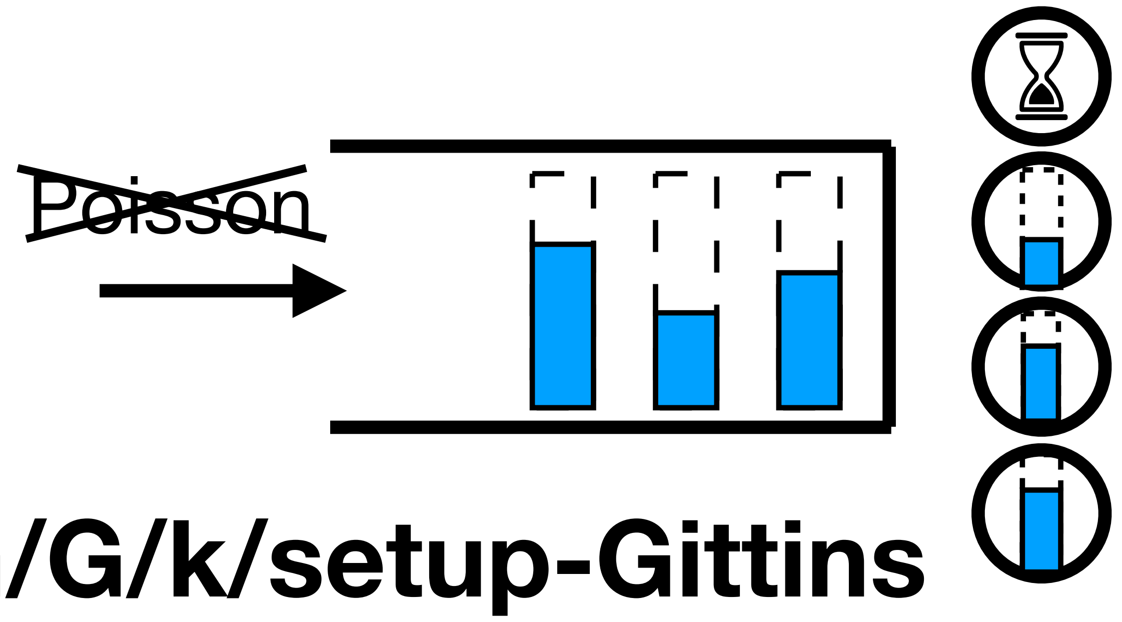


G/G/1-OPT

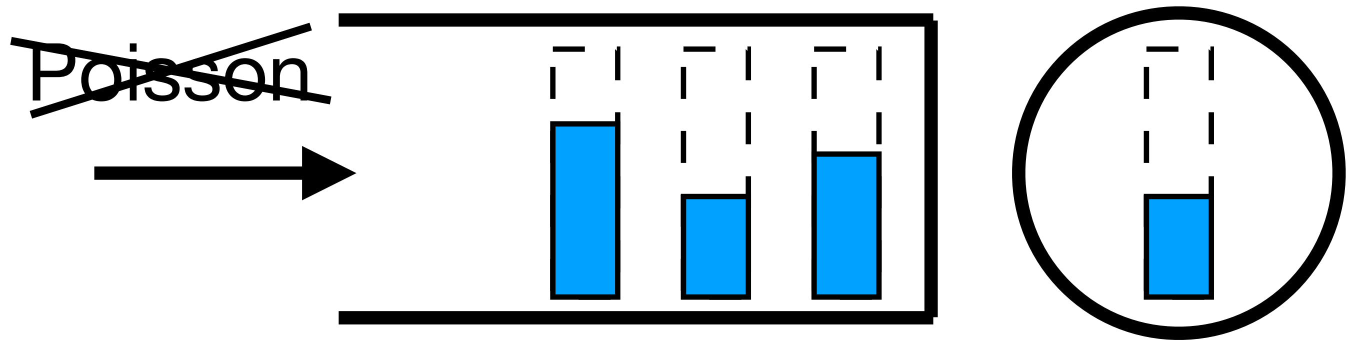
number in G/G/k-setup ←- - - gap - - -→ number in G/G/1

Analysis roadmap of G/G/k/setup

based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT

number in G/G/k-setup ↔ gap ↔ number in G/G/1

↕

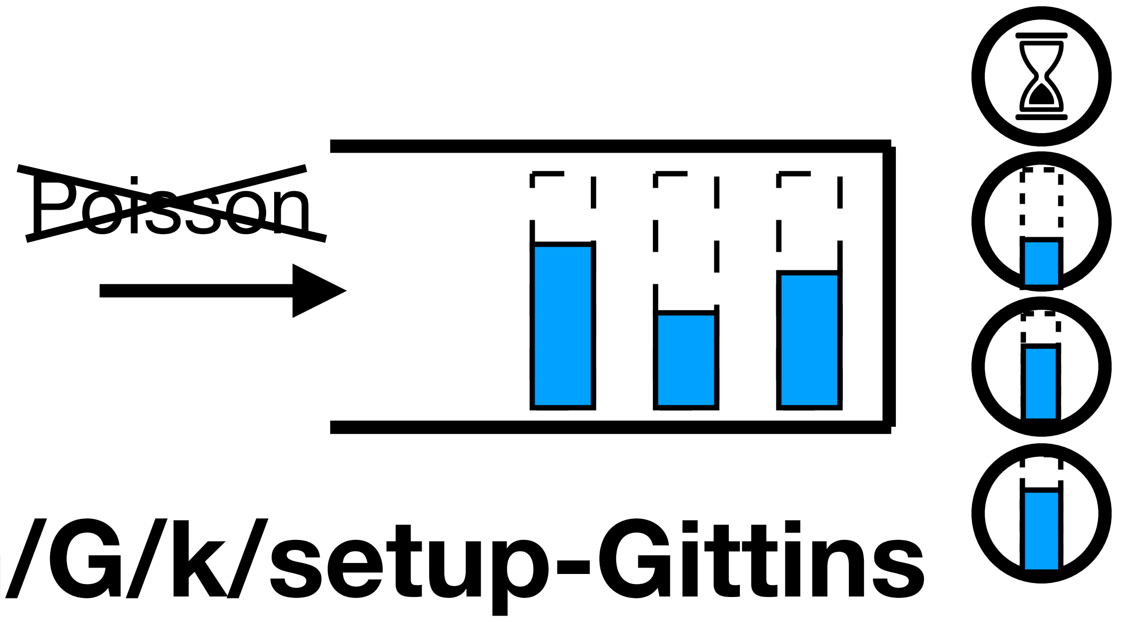
work in G/G/k-setup

↕

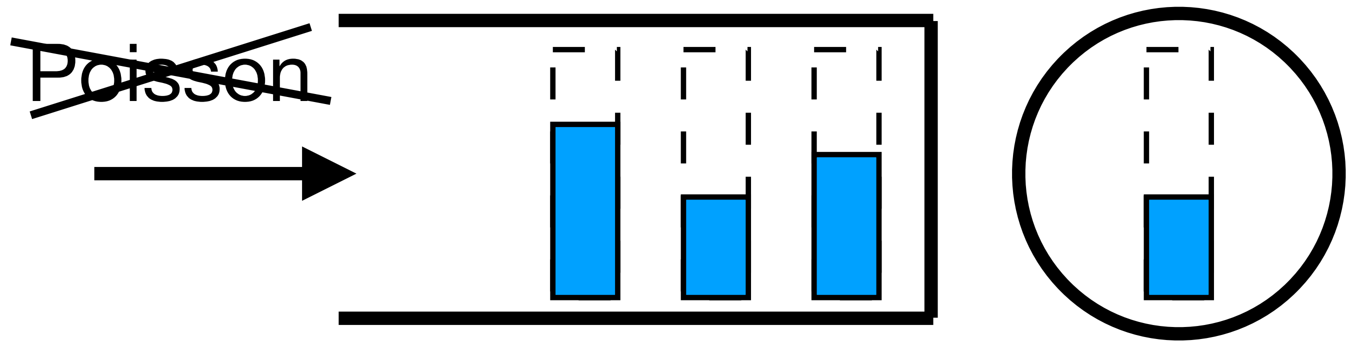
work in G/G/1

Analysis roadmap of G/G/k/setup

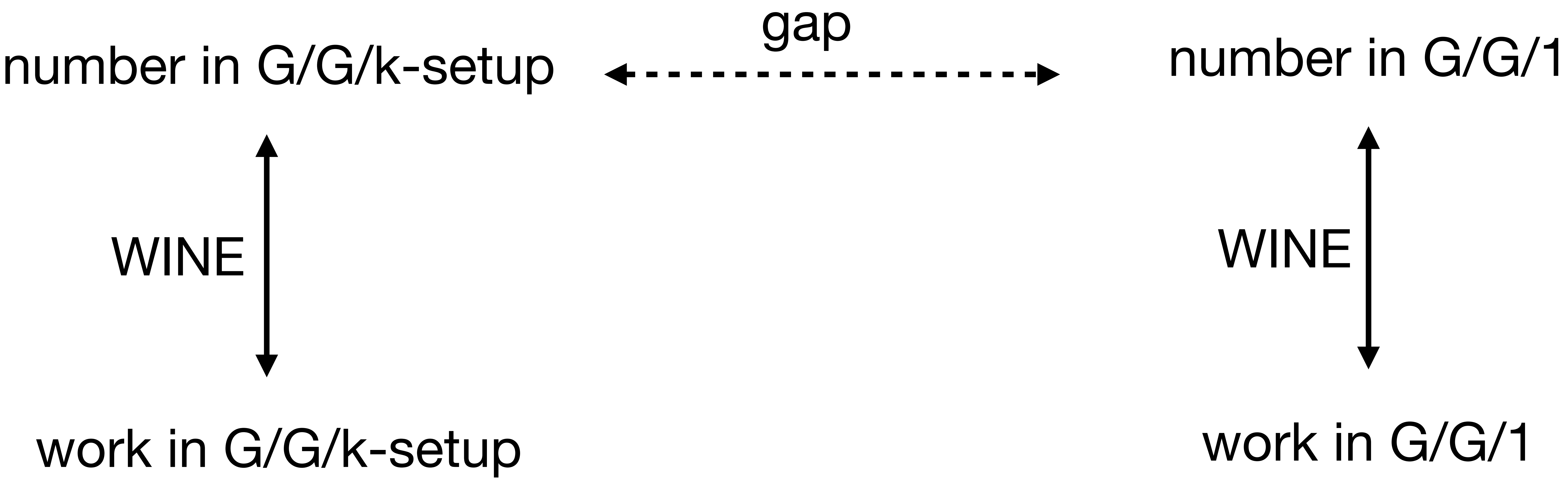
based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT



number in G/G/k-setup

gap

number in G/G/1

WINE

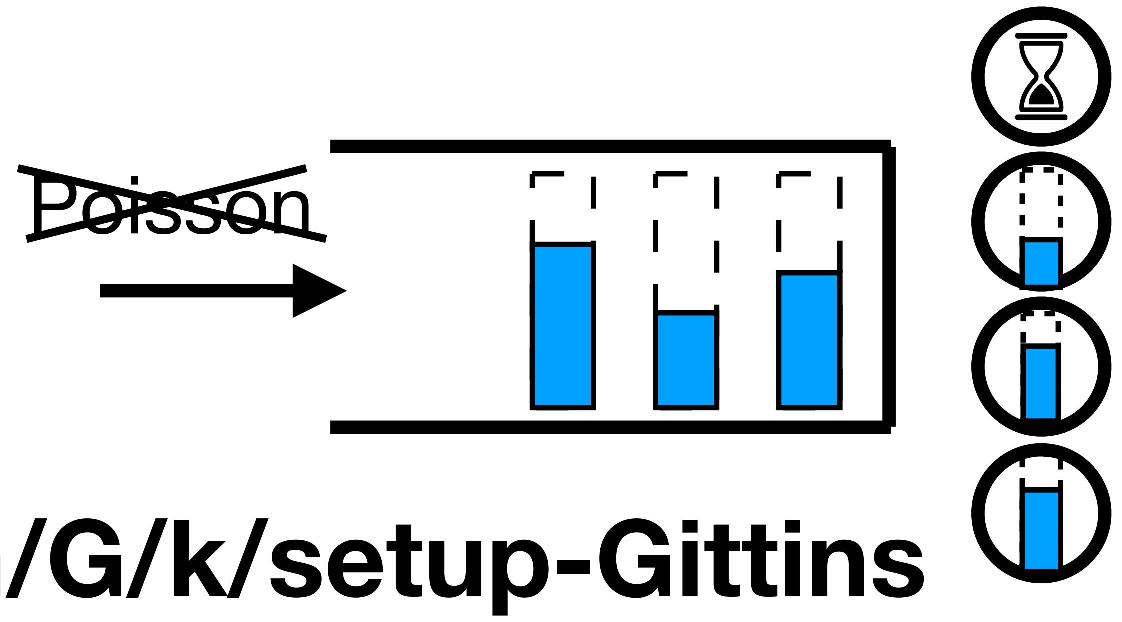
WINE

work in G/G/k-setup

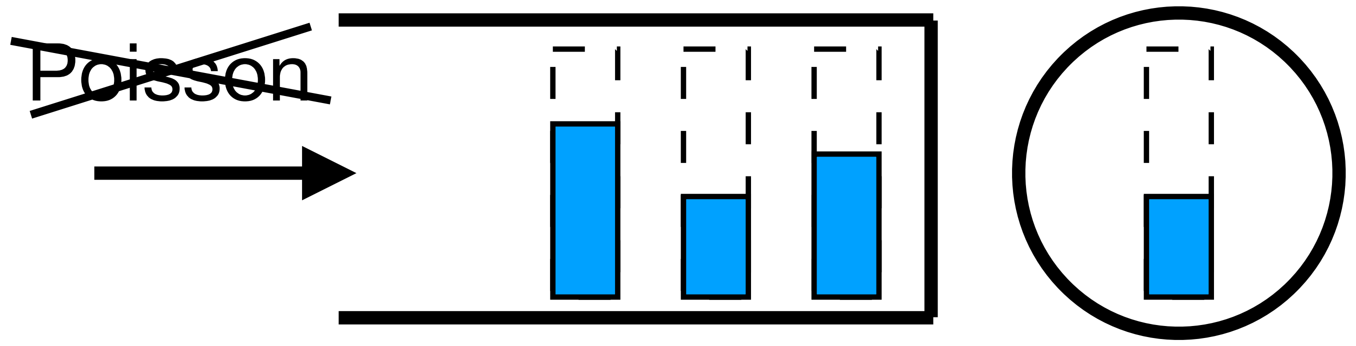
work in G/G/1

Analysis roadmap of G/G/k/setup

based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT

number in G/G/k-setup ← gap → number in G/G/1

SGH20, general

WINE ↕

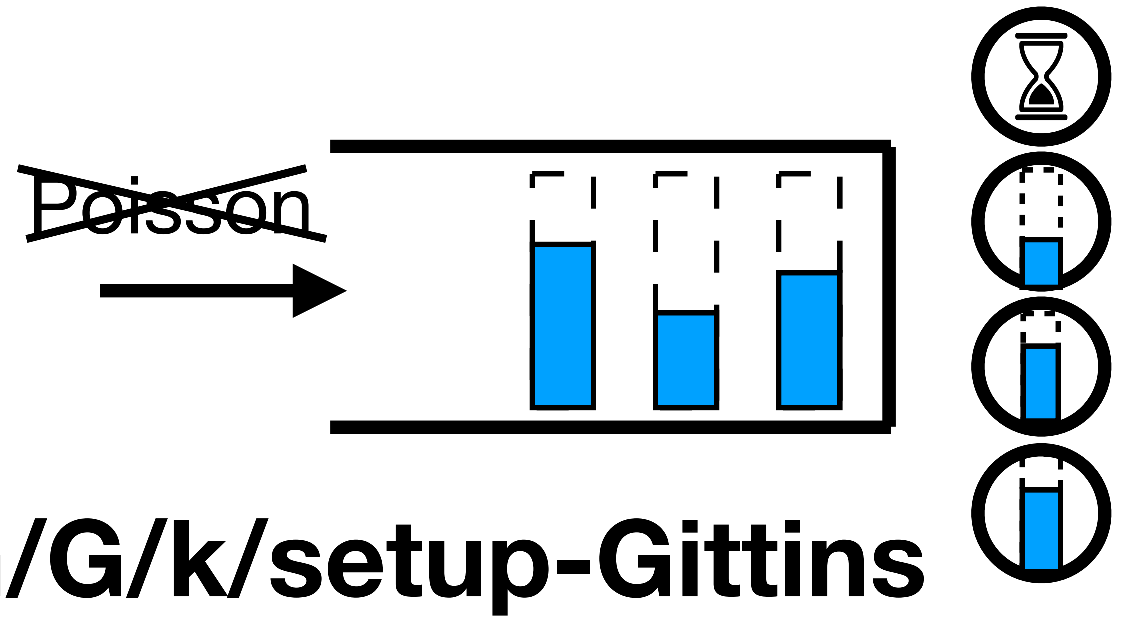
WINE ↕

work in G/G/k-setup

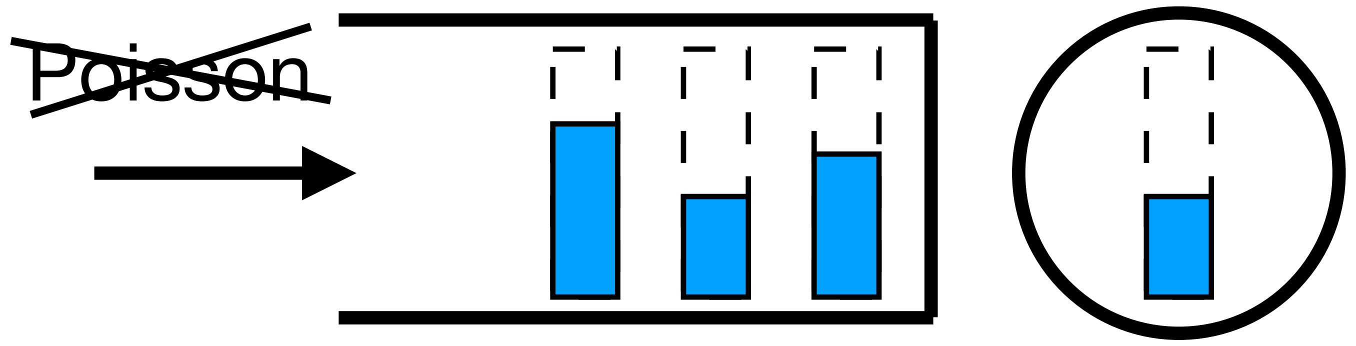
work in G/G/1

Analysis roadmap of G/G/k/setup

based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT

number in G/G/k-setup ←gap→ number in G/G/1

SGH20, general

WINE ↕

work in G/G/k-setup

r-work

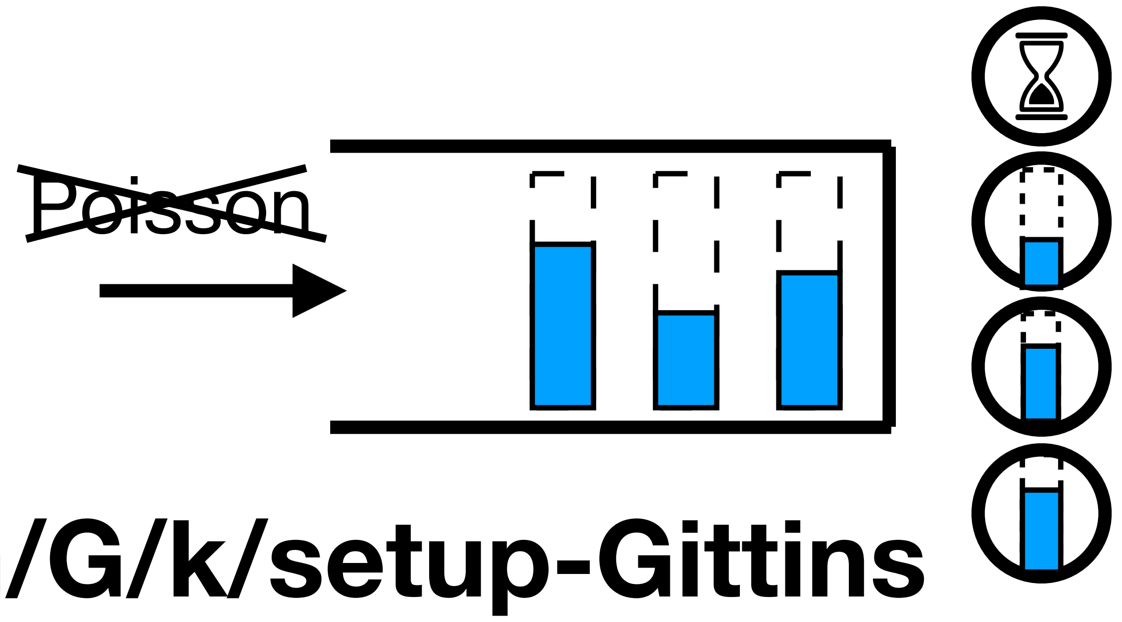
WINE ↕

work in G/G/1

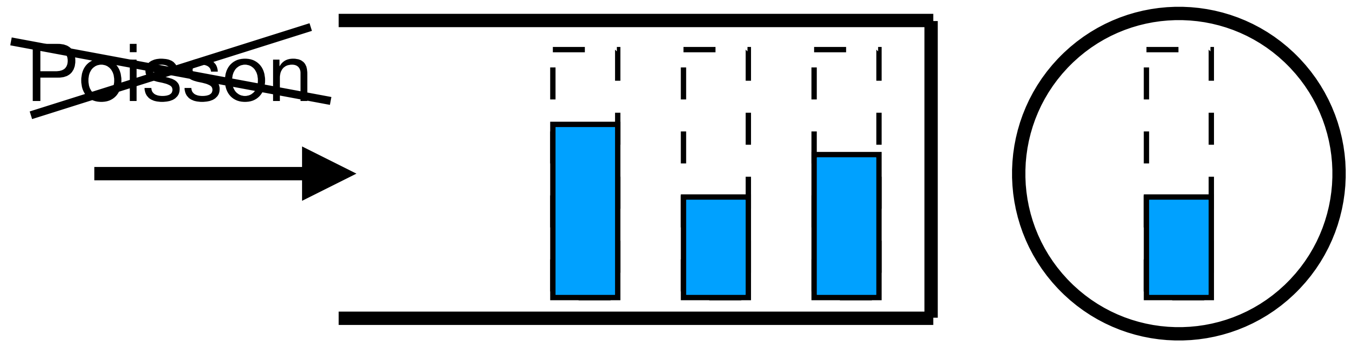
r-work

Analysis roadmap of G/G/k/setup

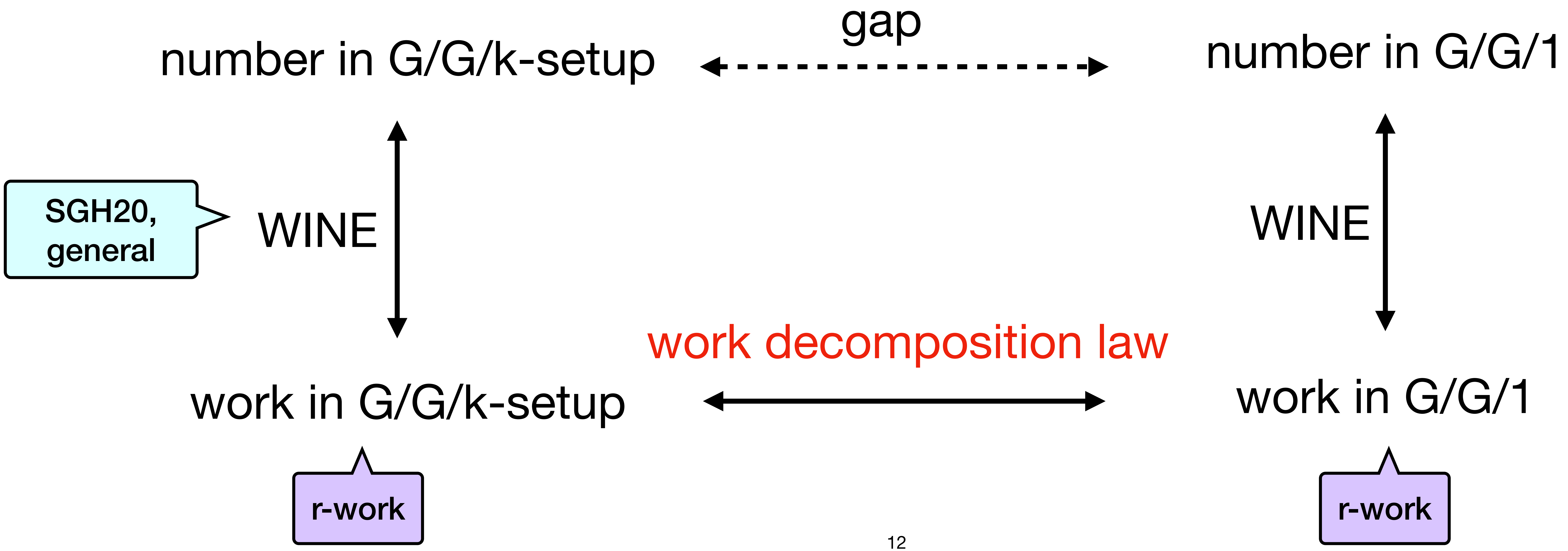
based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT



number in G/G/k-setup

gap

number in G/G/1

SGH20, general

WINE

WINE

work decomposition law

work in G/G/k-setup

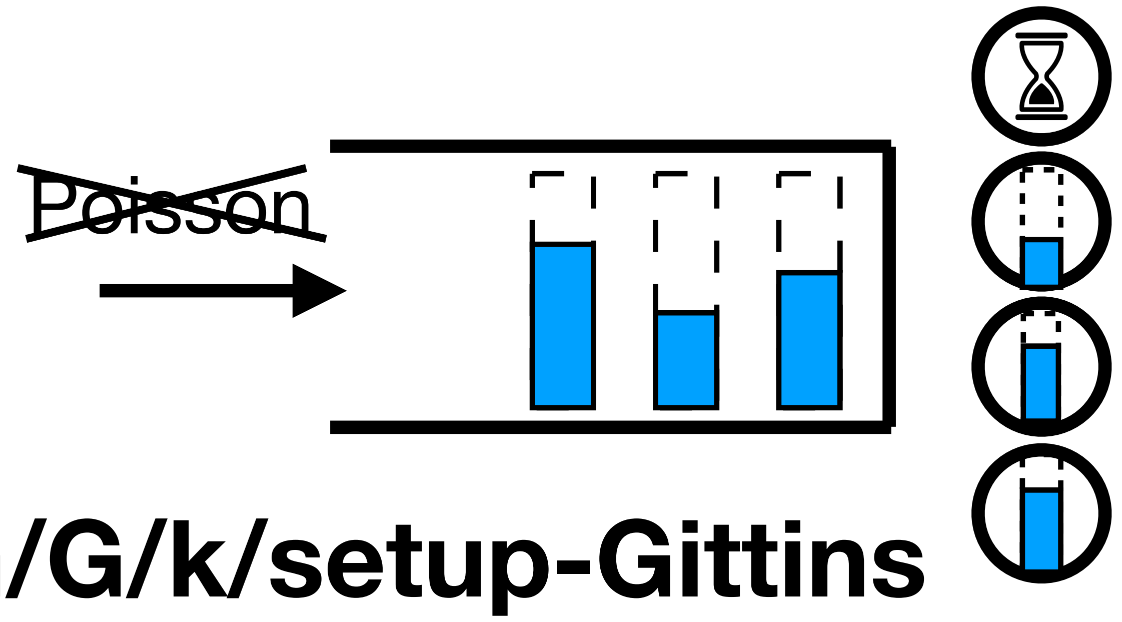
work in G/G/1

r-work

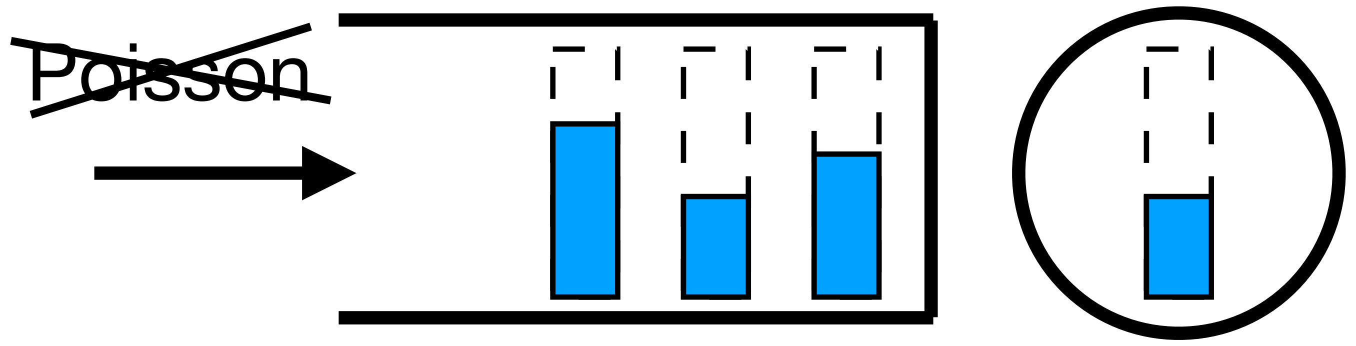
r-work

Analysis roadmap of G/G/k/setup

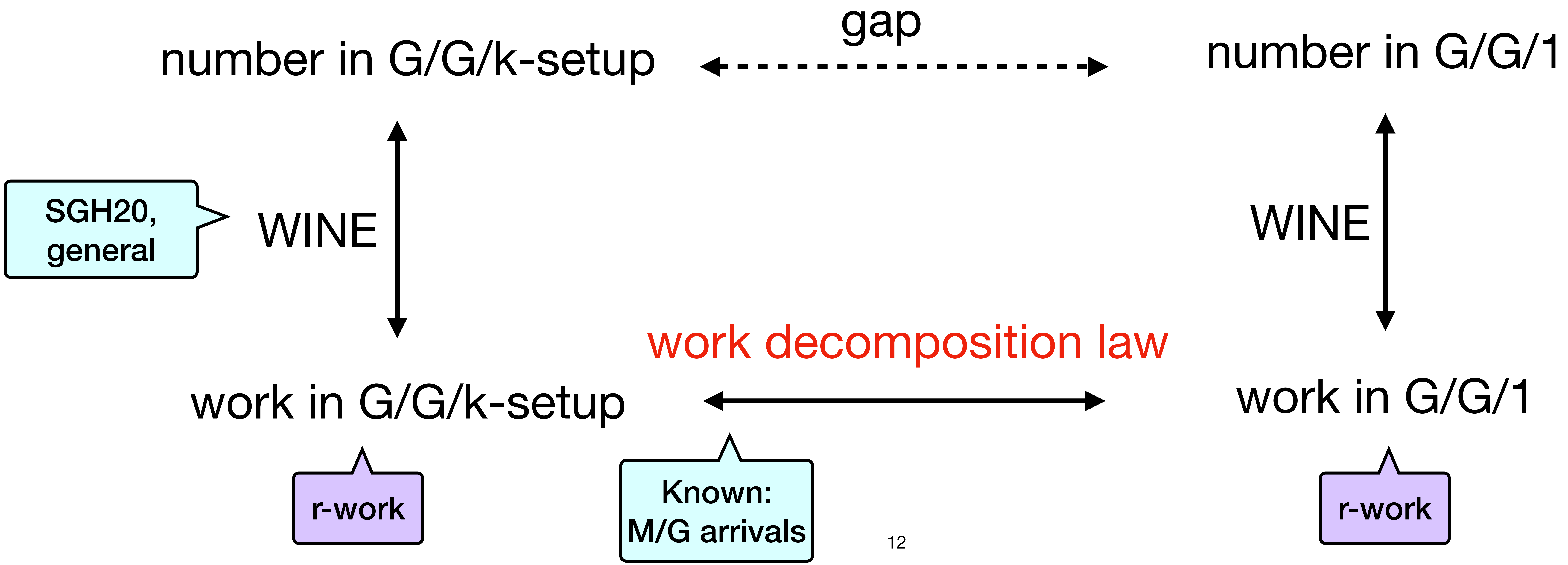
based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT



number in G/G/k-setup

gap

number in G/G/1

SGH20, general

WINE

WINE

work decomposition law

work in G/G/k-setup

work in G/G/1

r-work

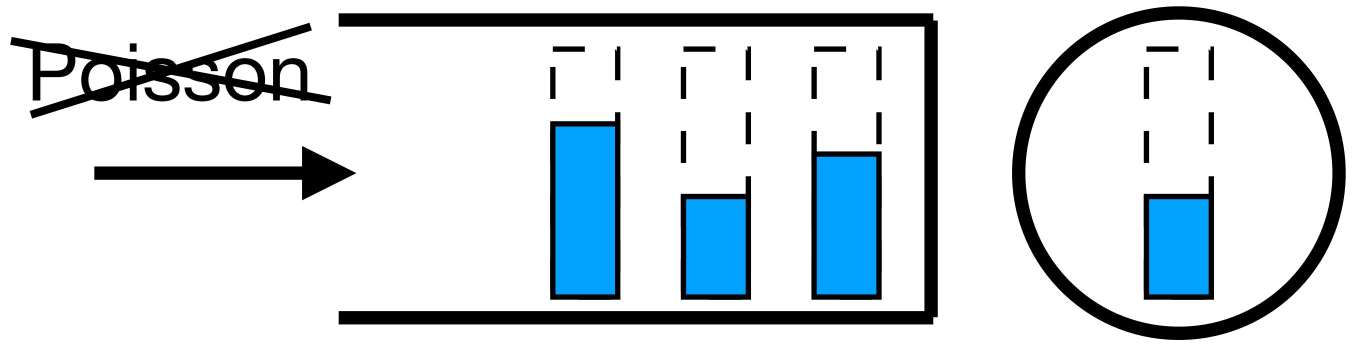
Known: M/G arrivals

r-work

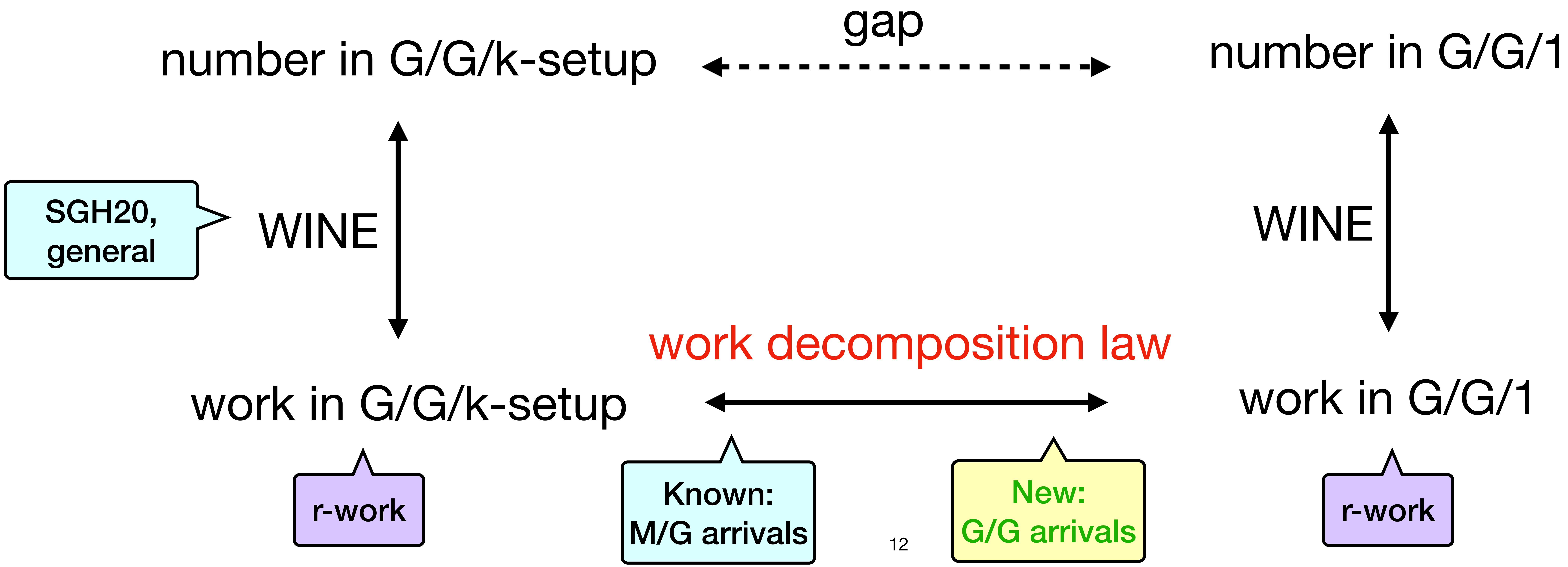
Analysis roadmap of G/G/k/setup

based on [SGH20]

G/G/k/setup-Gittins



G/G/1-OPT



gap

number in G/G/k-setup

number in G/G/1

SGH20, general

WINE

WINE

work decomposition law

work in G/G/k-setup

work in G/G/1

r-work

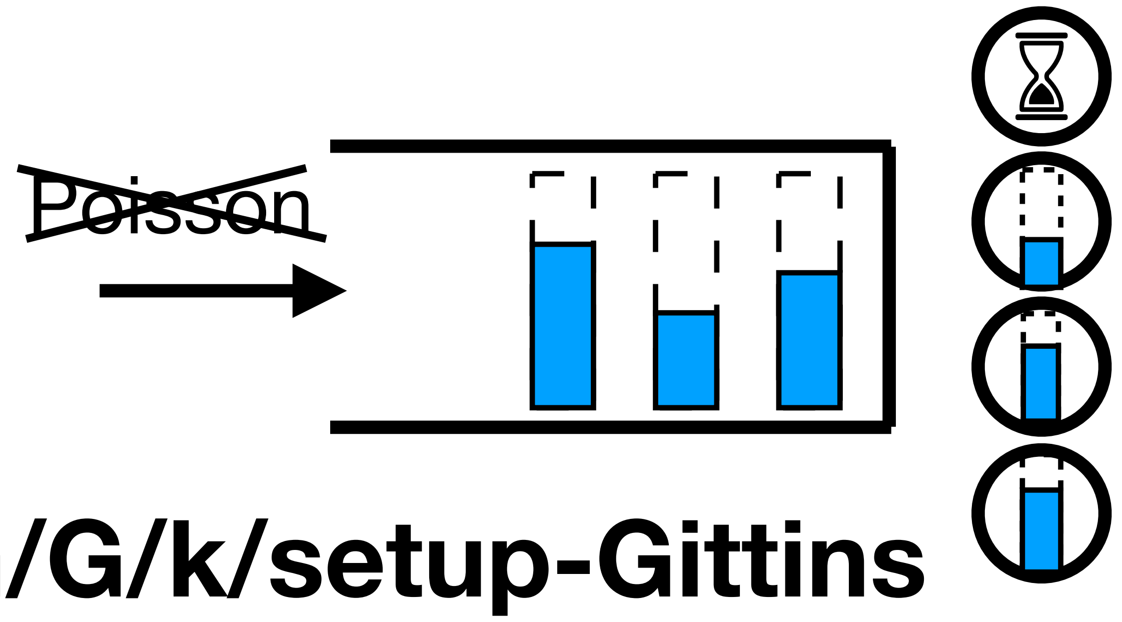
Known:
M/G arrivals

New:
G/G arrivals

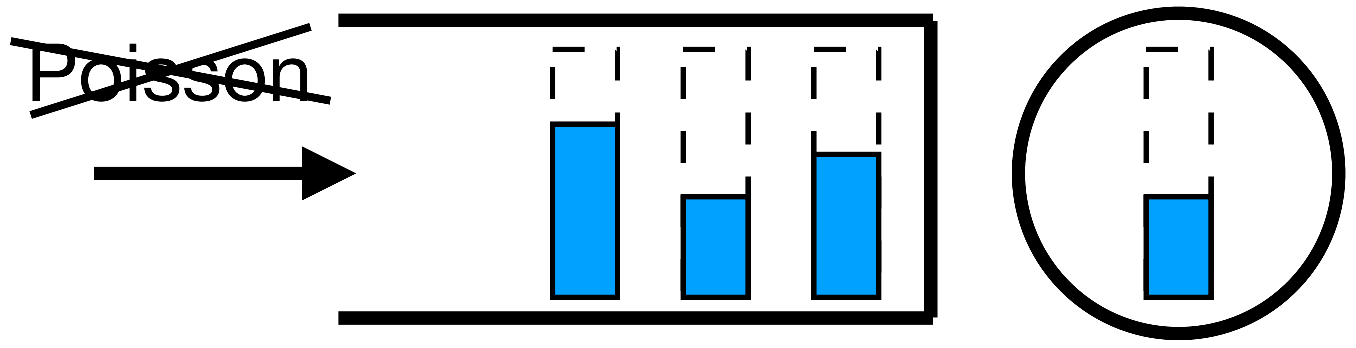
r-work

Analysis roadmap of G/G/k/setup

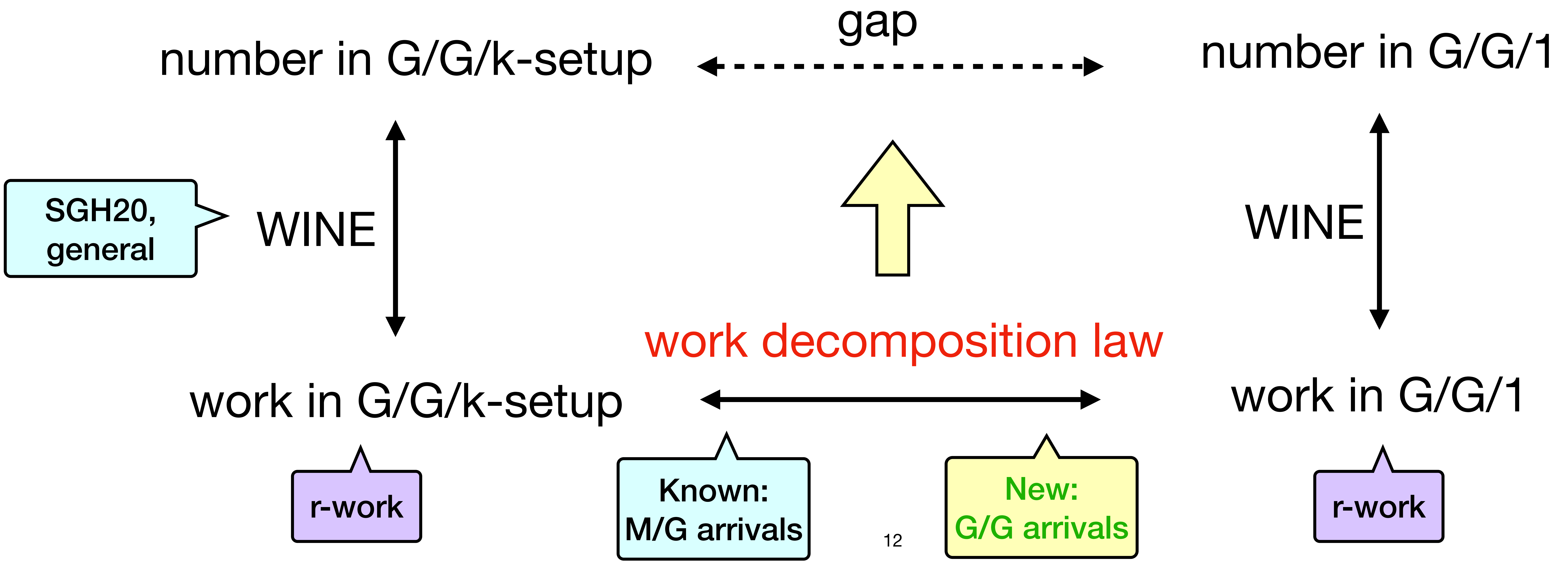
based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT



number in G/G/k-setup

gap

number in G/G/1

SGH20, general

WINE

WINE

work decomposition law

work in G/G/k-setup

work in G/G/1

r-work

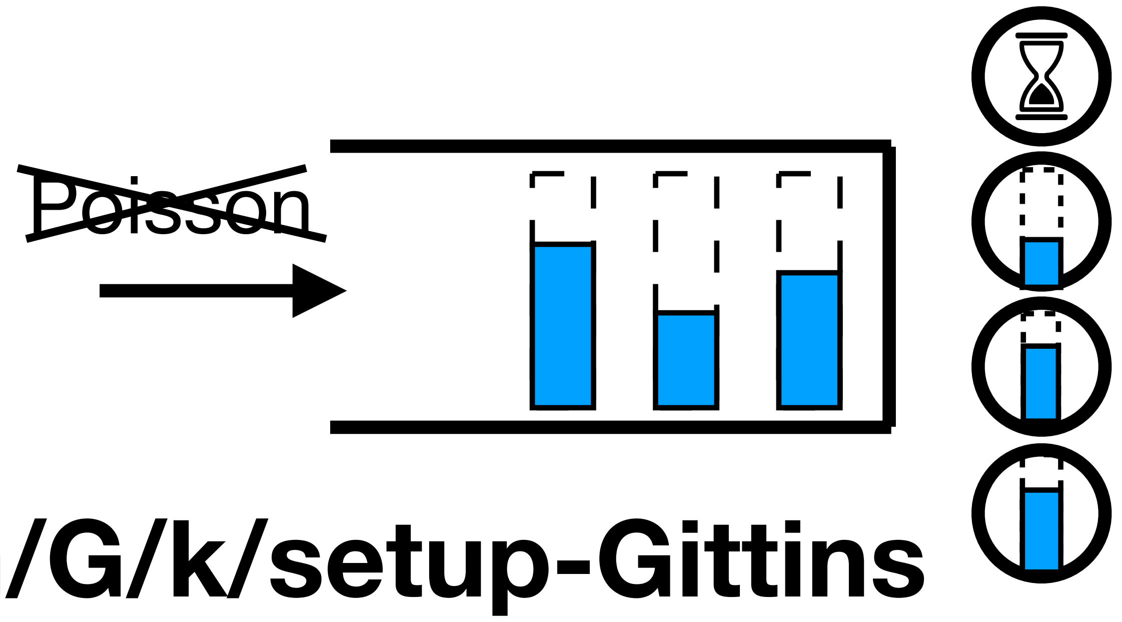
Known: M/G arrivals

New: G/G arrivals

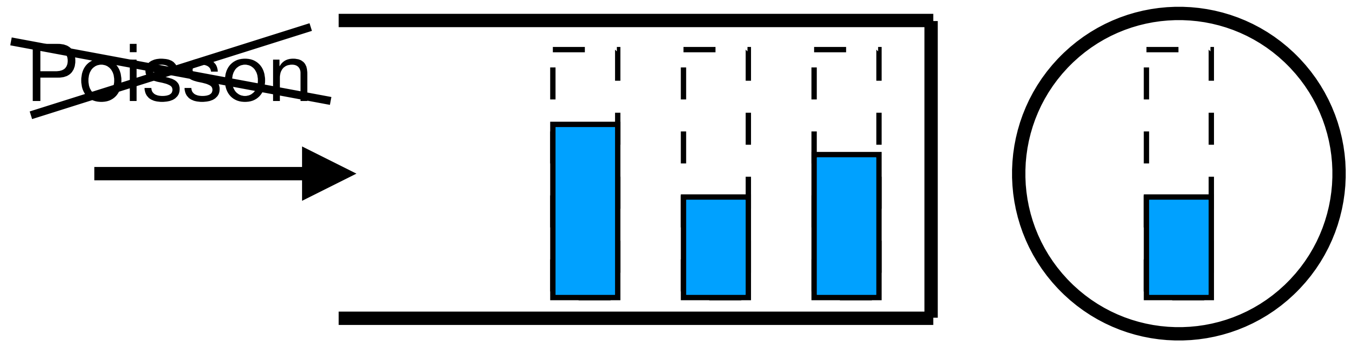
r-work

Analysis roadmap of G/G/k/setup

based on [SGH20]



G/G/k/setup-Gittins



G/G/1-OPT

number in G/G/k-setup ↔ gap ↔ number in G/G/1

$\leq \ell_{(a)} + \ell_{(b)} + \ell_{(c)}$

SGH20, general

WINE

WINE

work decomposition law

work in G/G/k-setup ↔ work in G/G/1

r-work

Known: M/G arrivals

New: G/G arrivals

r-work

G/G work-decomposition law

$$
\begin{aligned}
\mathbb{E}[W] = {} & \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2} \\
& + \frac{\mathbb{E}[I_{off}\,W]}{1-\rho} \\
& + \frac{\mathbb{E}[I_{setup}\,W]}{1-\rho} \\
& - \frac{\rho}{1-\rho}\mathbb{E}[I\,R]
\end{aligned}
$$

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

same for all G/G systems & policies

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

depend on system & policy

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho} \longrightarrow$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

depend on system & policy

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

→ Captures the effect of multiple servers

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

depend on system & policy

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

Captures the effect of multiple servers

- I_{off}: fraction of servers that are off

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

depend on system & policy

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

depend on system & policy

Captures the effect of multiple servers

- I_{off}: fraction of servers that are off
- $k = 1$:

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

Captures the effect of multiple servers

- I_{off}: fraction of servers that are off
- $k = 1$:
  - whenever $W > 0$, $I_{off} = 0$

depend on system & policy

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

depend on system & policy

→ Captures the effect of multiple servers

- I_{off}: fraction of servers that are off
- $k = 1$:
  - whenever $W > 0,\ I_{off} = 0$
- k > 1:

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

depend on system & policy

Captures the effect of multiple servers

- I_{off} : fraction of servers that are off
- $k = 1$:
  - whenever $W > 0$, $I_{off} = 0$
- k > 1:
  - $I_{off}W$ can be positive

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

depend on system & policy

Captures the effect of multiple servers

- I_{off}: fraction of servers that are off
- $k = 1$:
  - whenever $W > 0$, $I_{off} = 0$
- k > 1:
  - $I_{off}W$ can be positive
  - whenever $\geq k$ jobs, $I_{off} = 0$, so $\mathbb{E}[I_{off}W]$ small

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

depend on system & policy

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

same for all G/G systems & policies

depend on system & policy

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off} W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup} W]}{1-\rho}$$

Captures the effect of setup times

$$-\frac{\rho}{1-\rho}\mathbb{E}[I R]$$

depend on system & policy

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}\,W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}\,W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[I\,R]$$

depend on system & policy

Captures the effect of setup times

- I_{setup} : fraction of servers setting up

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

depend on system & policy

Captures the effect of setup times

- I_{setup} : fraction of servers setting up
- Without setup times,

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

same for all G/G systems & policies

depend on system & policy

Captures the effect of setup times

- I_{setup} : fraction of servers setting up
- Without setup times,
  - $I_{setup} = 0$

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho}$$

$$+\frac{\mathbb{E}[I_{setup}W]}{1-\rho}$$

$$-\frac{\rho}{1-\rho}\mathbb{E}[IR]$$

depend on system & policy

Captures the effect of setup times

- I_{setup} : fraction of servers setting up
- Without setup times,
  - $I_{setup} = 0$
- With setup times,

G/G work-decomposition law

$$\mathbb{E}[W] = \frac{\left(c_A^2 + \rho c_S^2\right)\mathbb{E}[S]}{2(1-\rho)} + \frac{\mathbb{E}[S]}{2}$$

same for all G/G systems & policies

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Captures the effect of setup times

- I_{setup} : fraction of servers setting up
- Without setup times,
  - $I_{setup} = 0$
- With setup times,
  - Long busy cycle, servers only set up at the beginning, $\mathbb{E}[I_{setup}W]$ small

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$$
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& - \frac{\rho}{1-\rho}\mathbb{E}[I\, R]
\end{aligned}
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same for all G/G systems & policies

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Captures the effect of G/G arrivals

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- In general,
  - assume $\exists R_{min}, R_{\max}$ $R_{\min} \leq \mathbb{E}[R \mid A_{age}] \leq R_{\max}$

G/G work-decomposition law + WINE in G/G/k/setup

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depend on system & policy

WINE

$\leq \ell_{(a)}$ under Gittins

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$$+\frac{\mathbb{E}[I_{off}W]}{1-\rho} \xrightarrow{\text{WINE}} \leq \ell_{(a)} \text{ under Gittins}$$

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$$\ell_{(b)} = \lambda(R_{\max} - R_{\min})$$

Outline for the rest of the talk

- What is our problem and result? ✓
- How does the G/G/k/setup anaysis work? ✓
- What is the main obstacle and how do we solve it?

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G/G work decomposition

Work analysis

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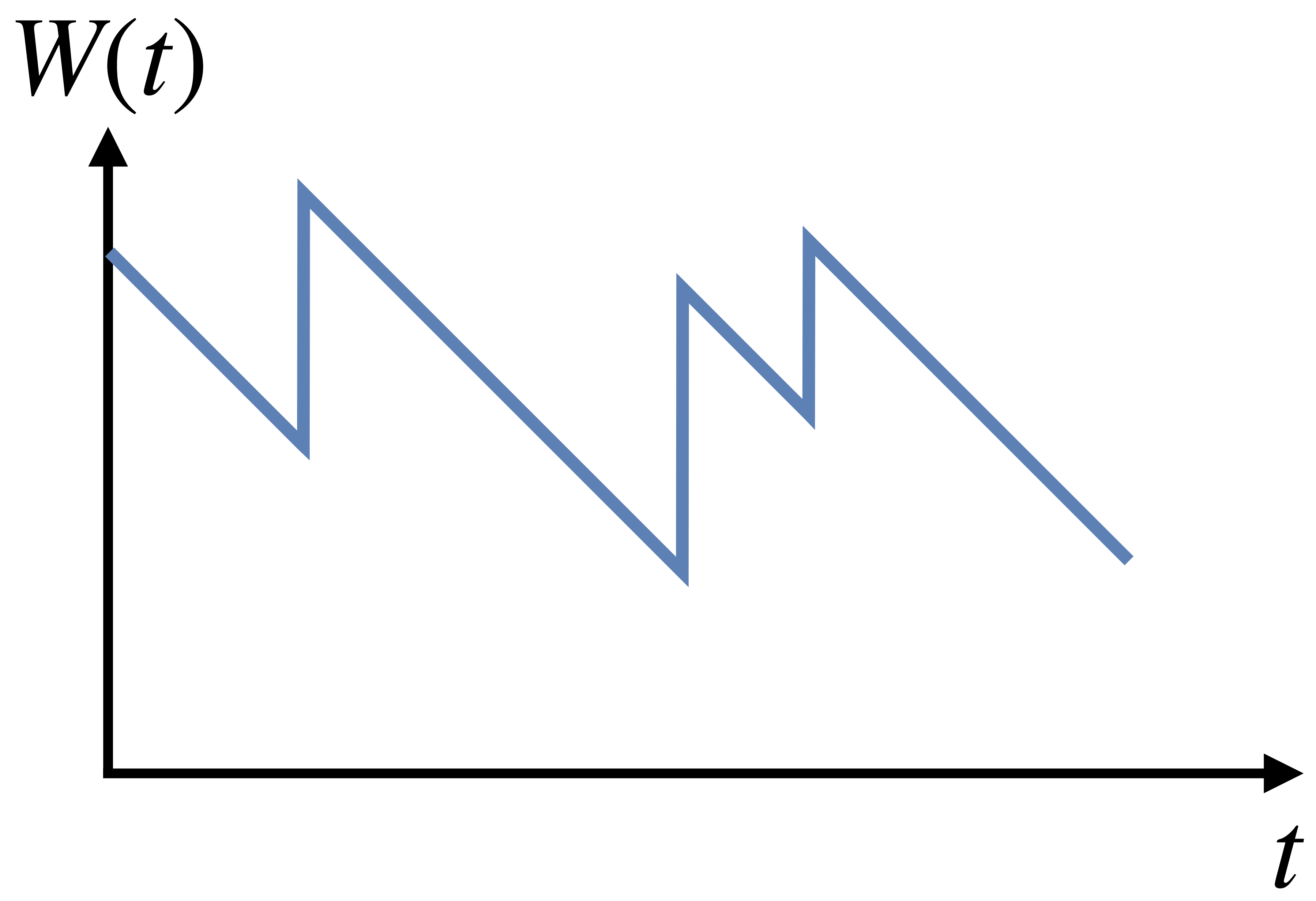

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- Why is G/G work analysis harder than M/G?
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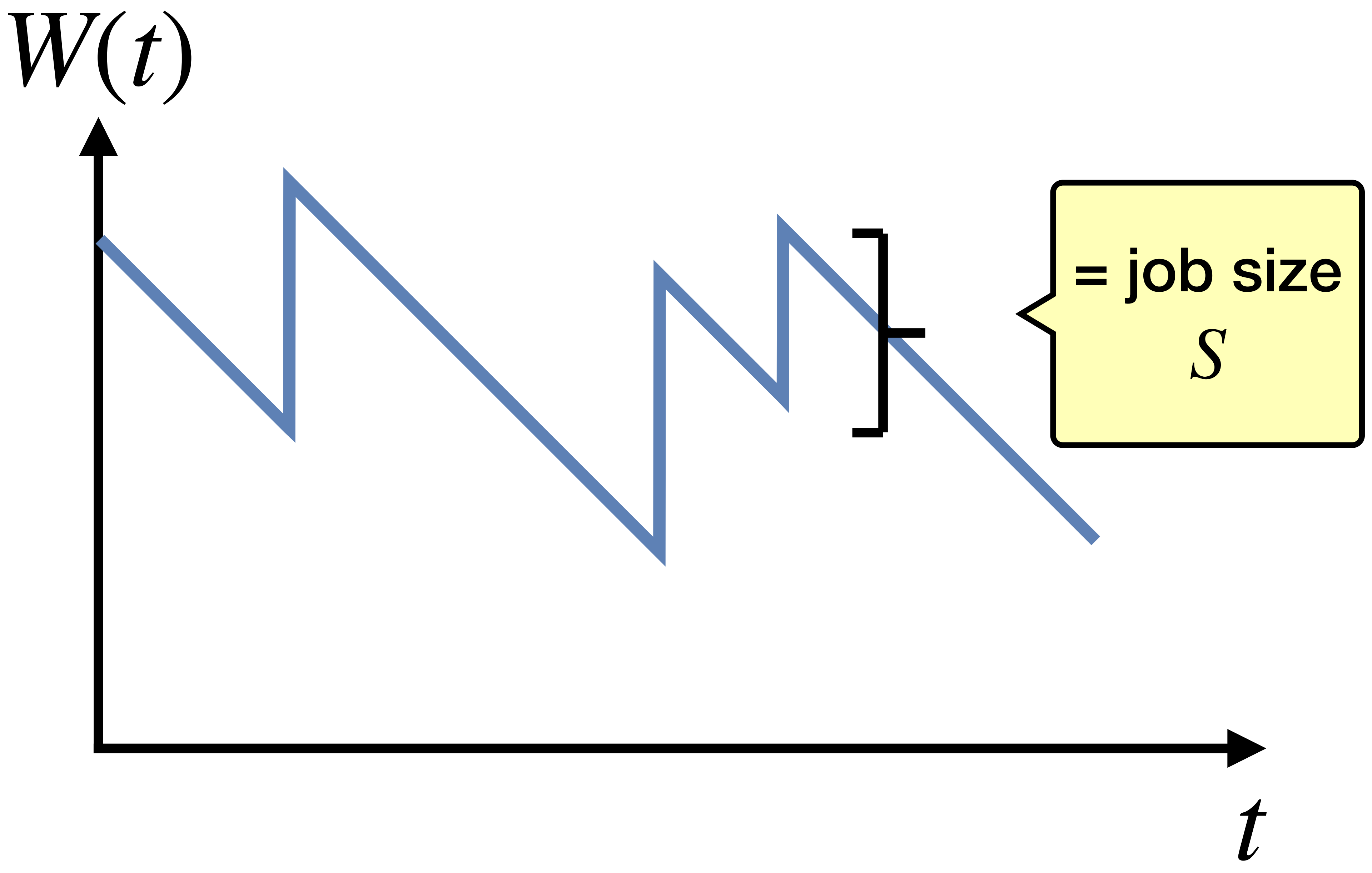

= job size S

Work analysis

- Why is G/G work analysis harder than M/G?
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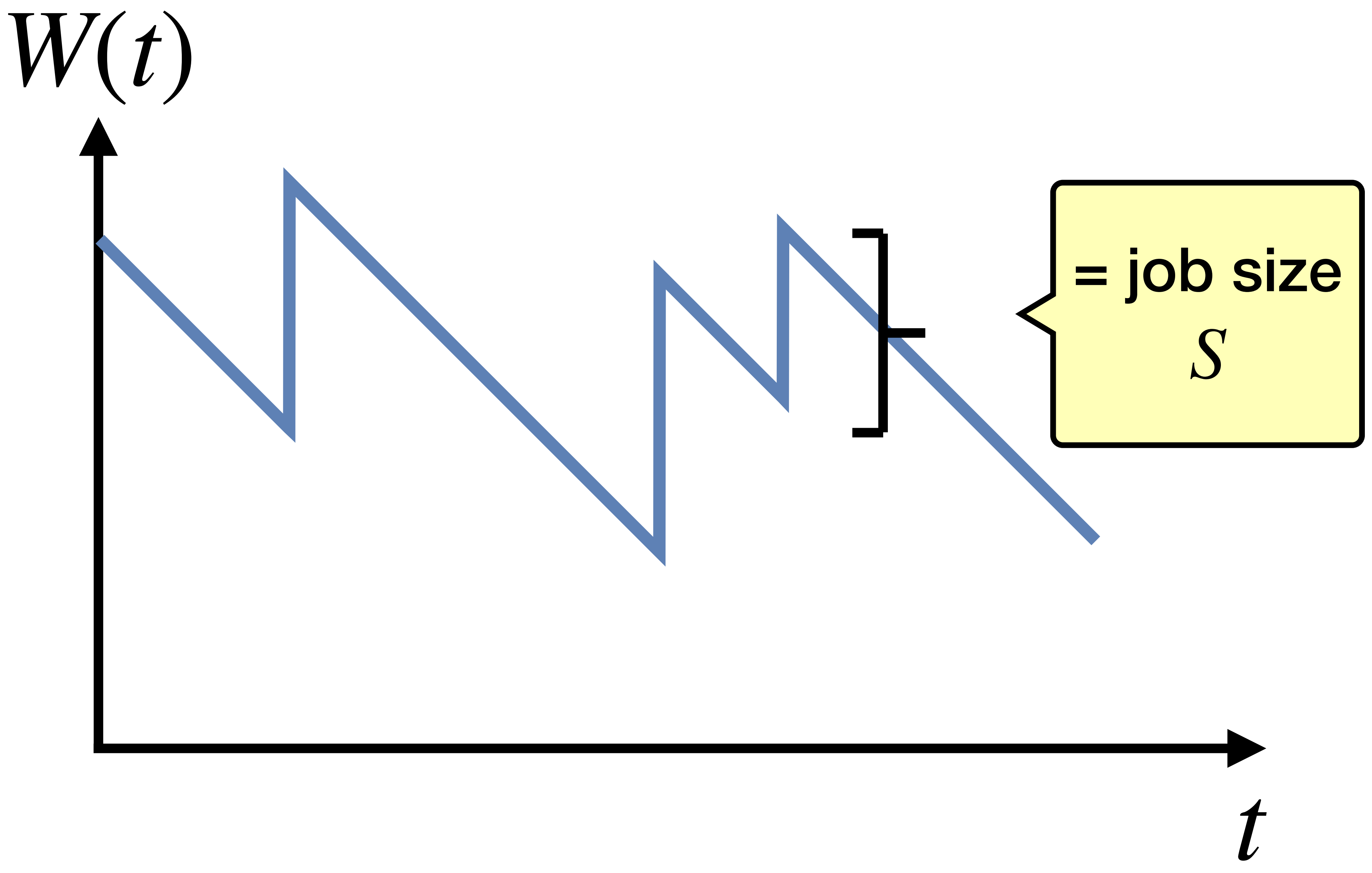

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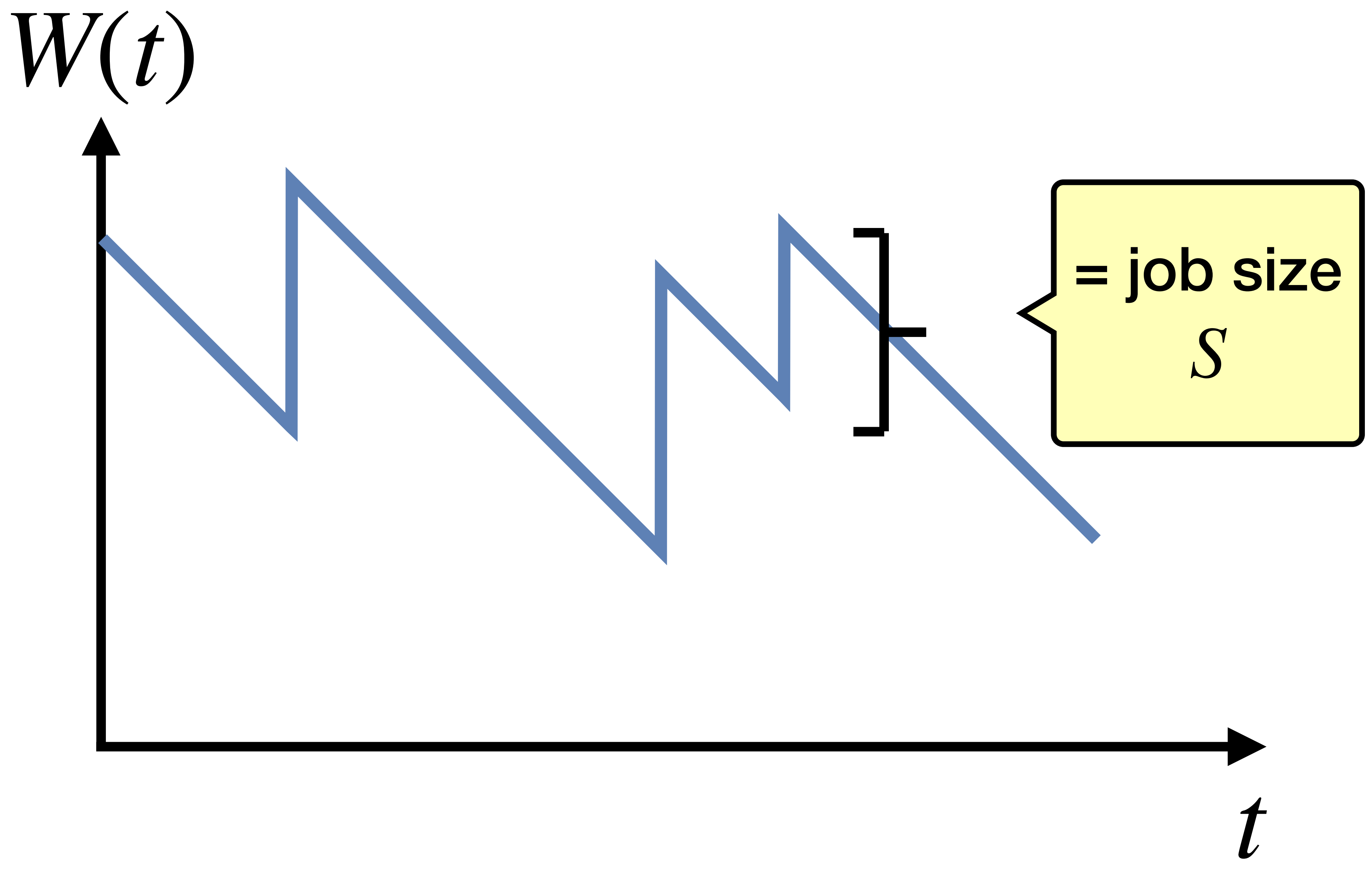

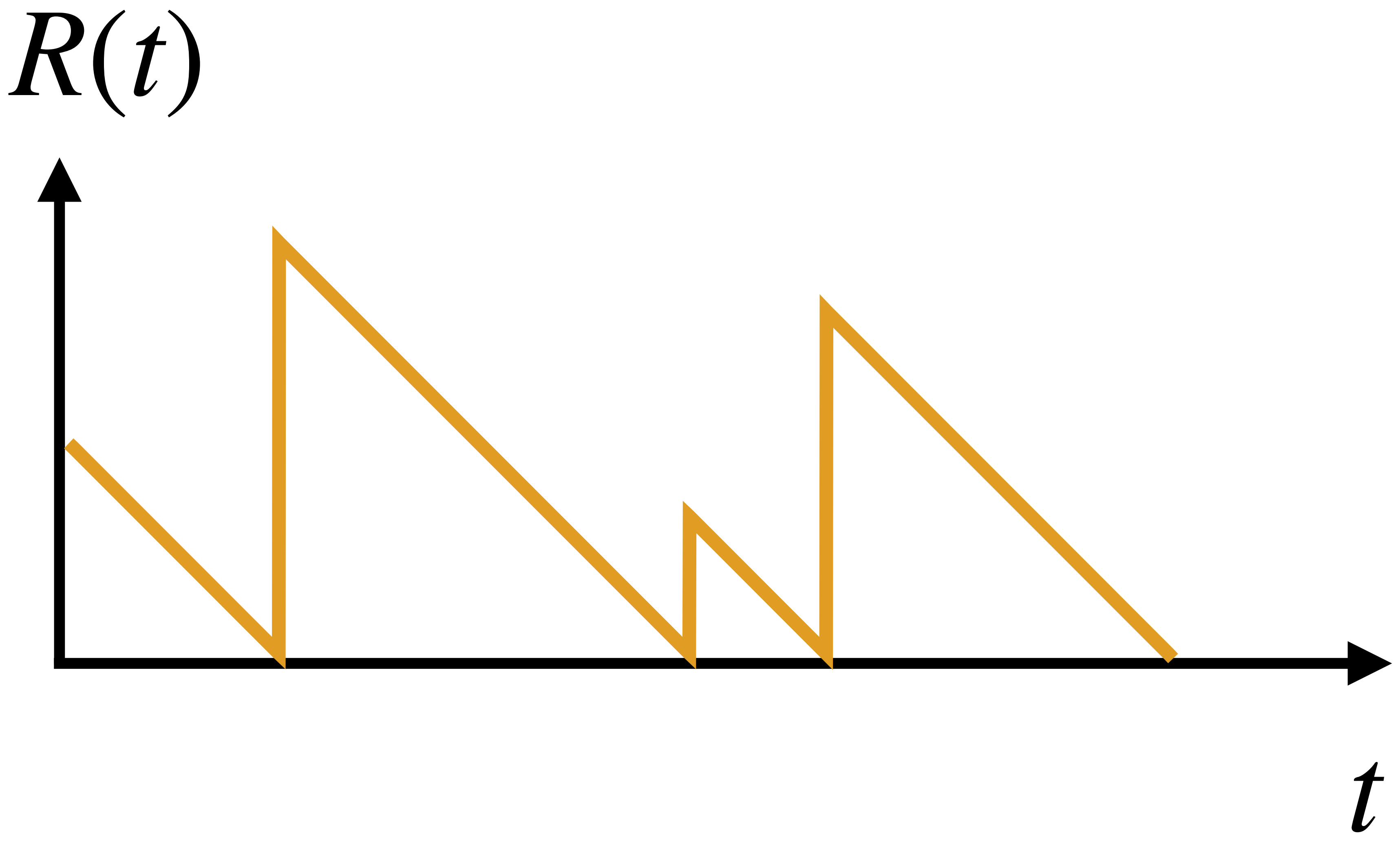

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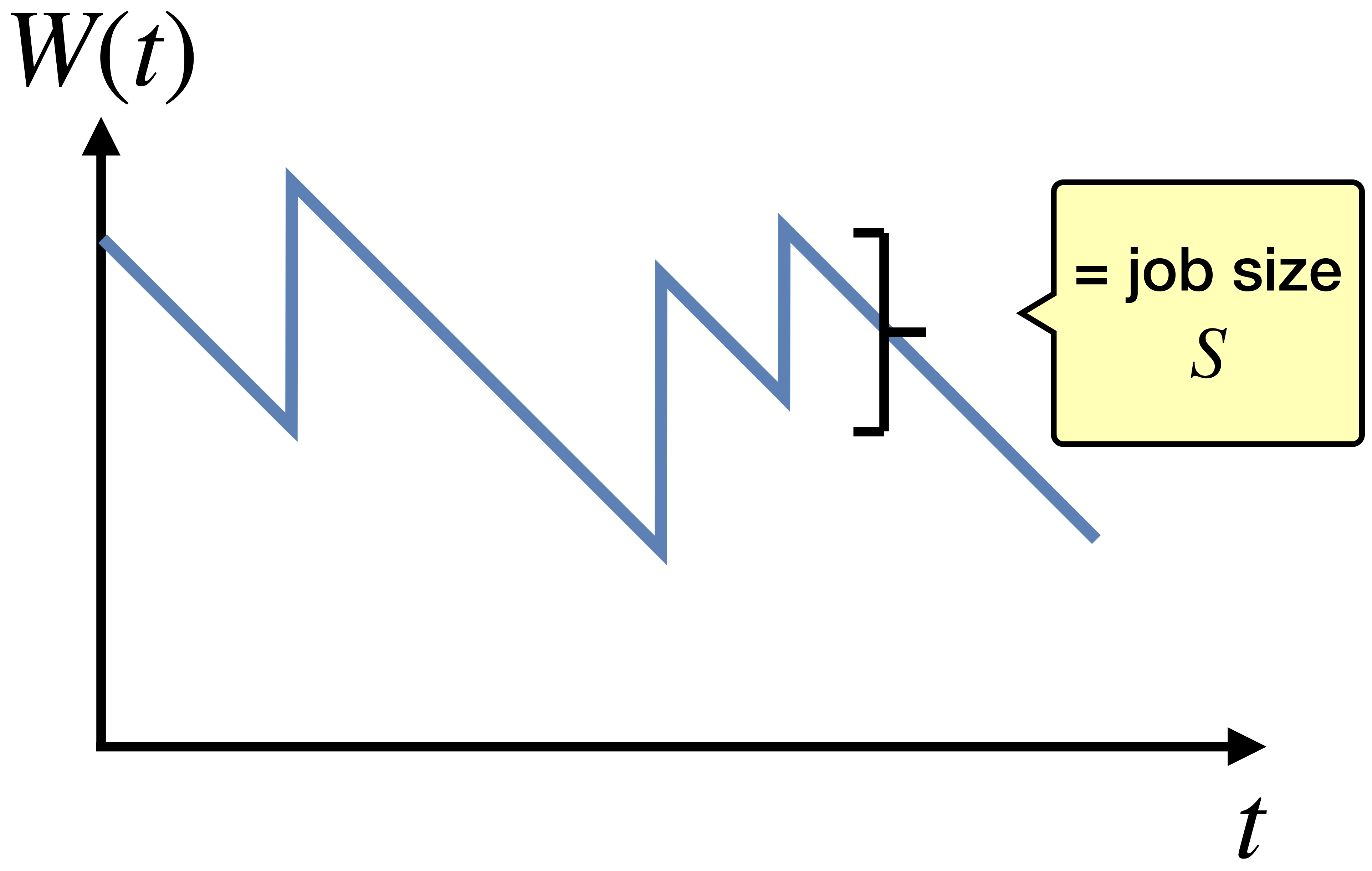

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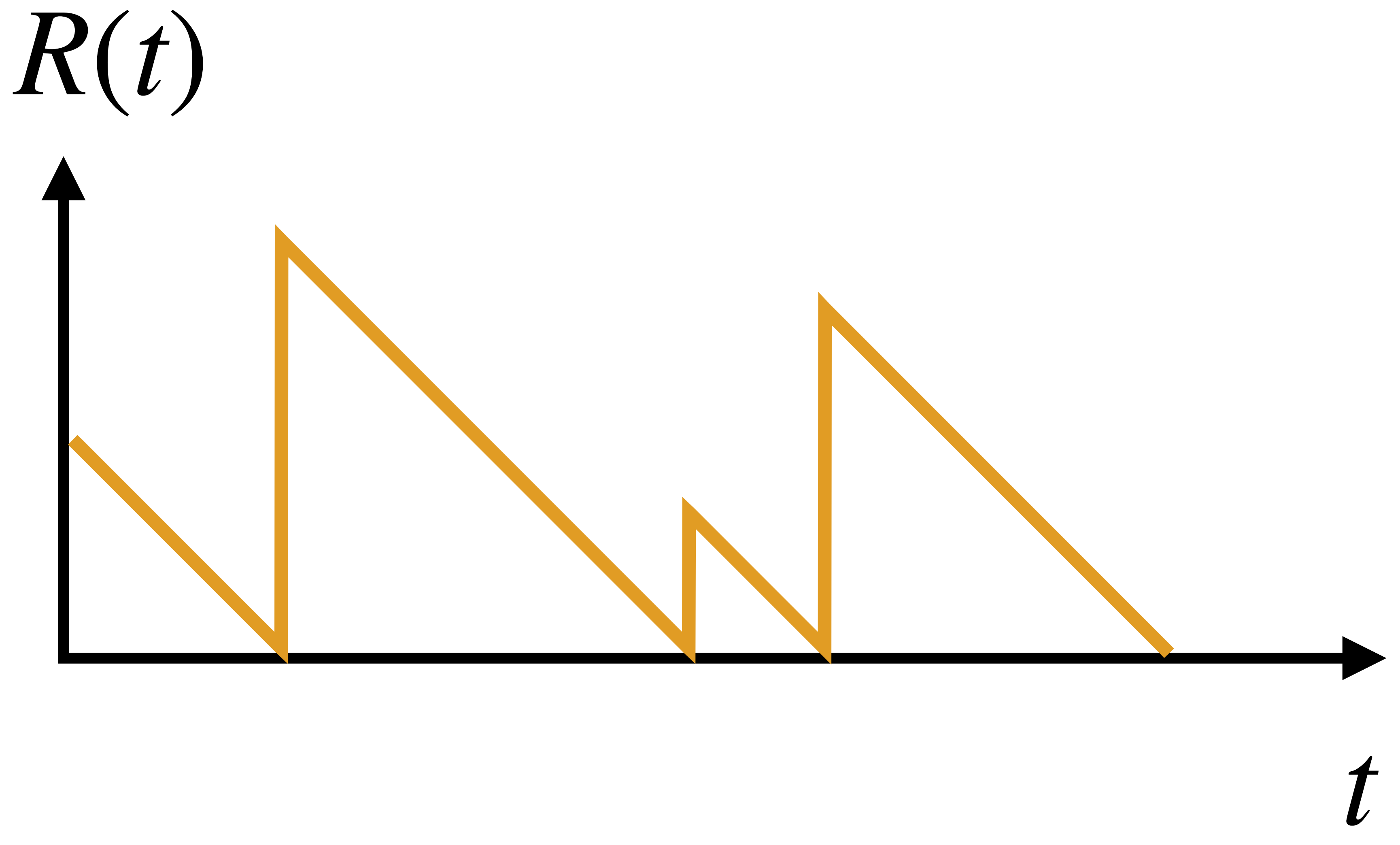
residual arrival time



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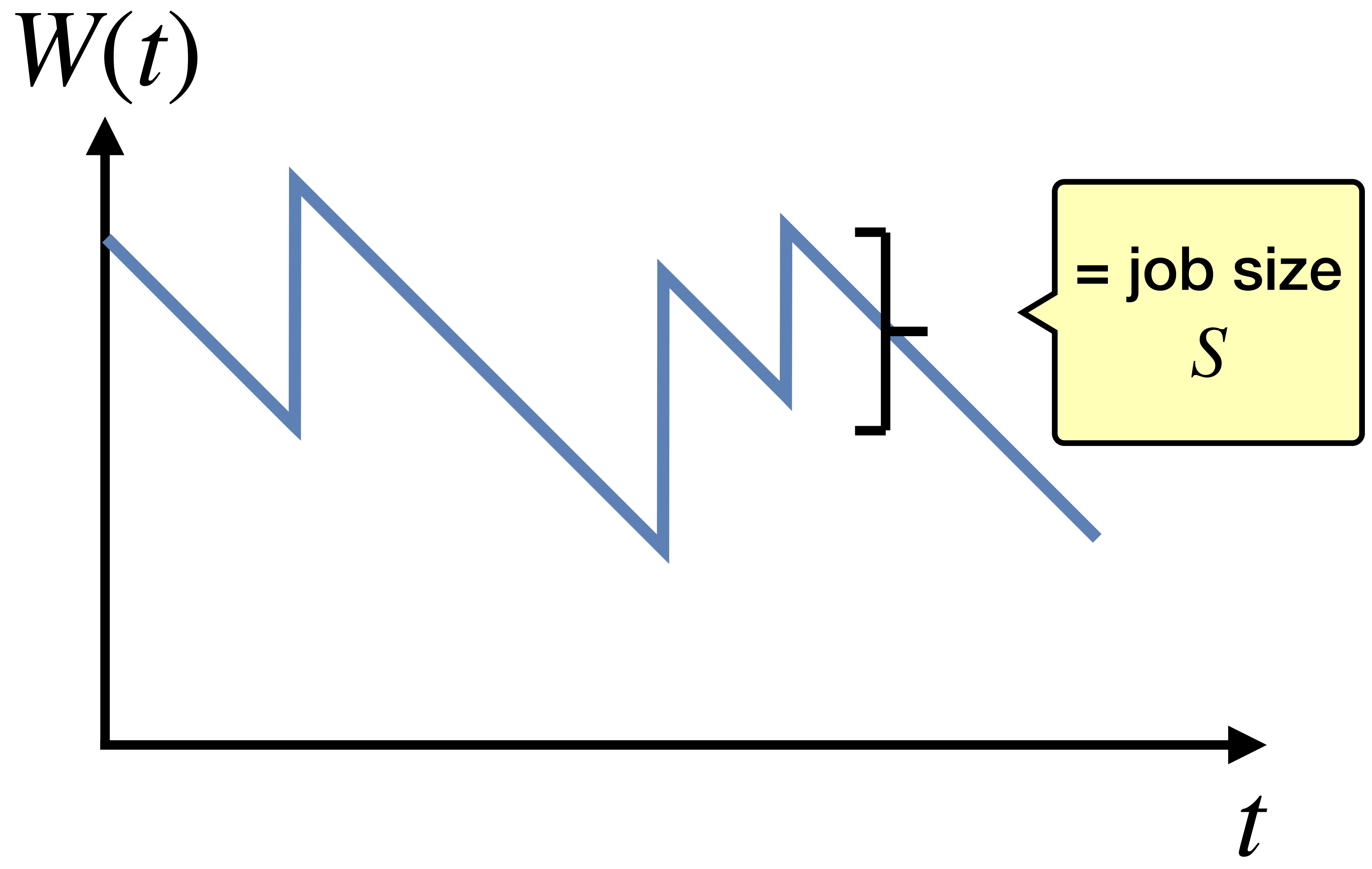

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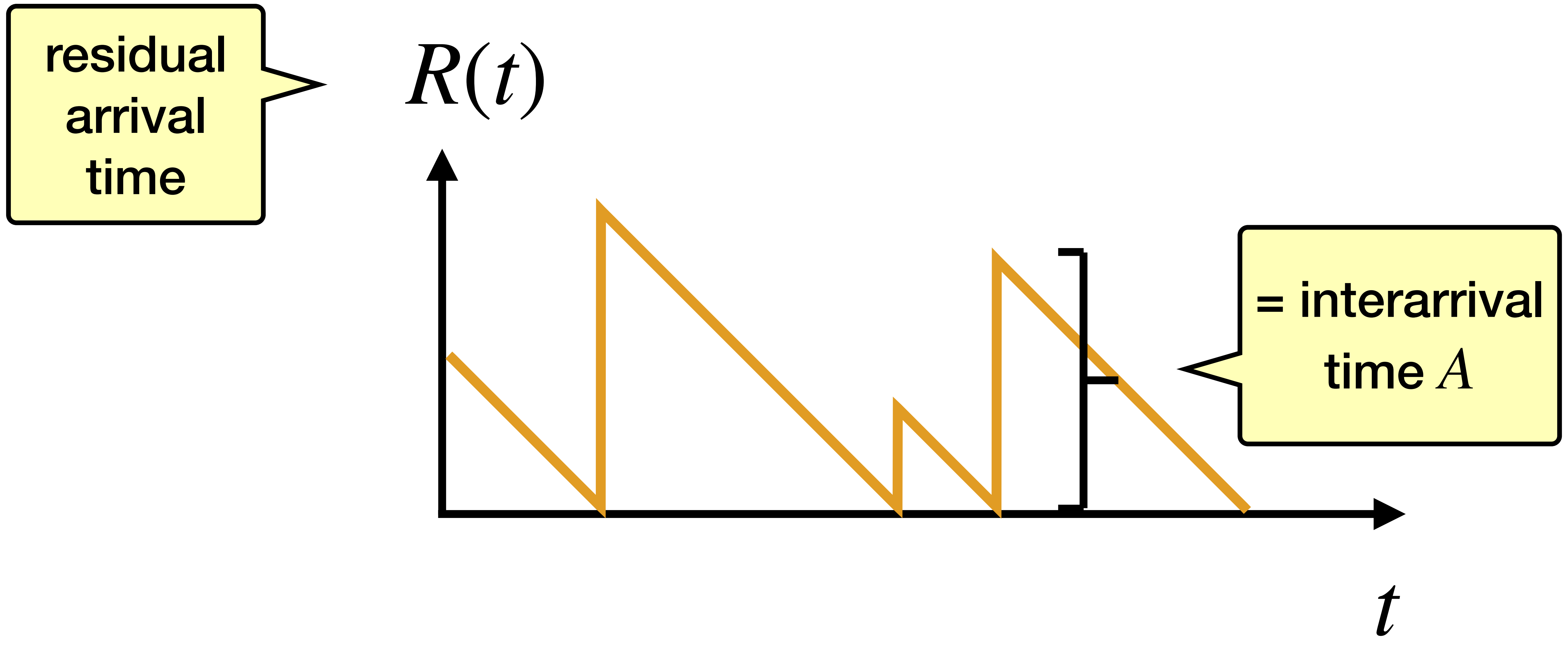
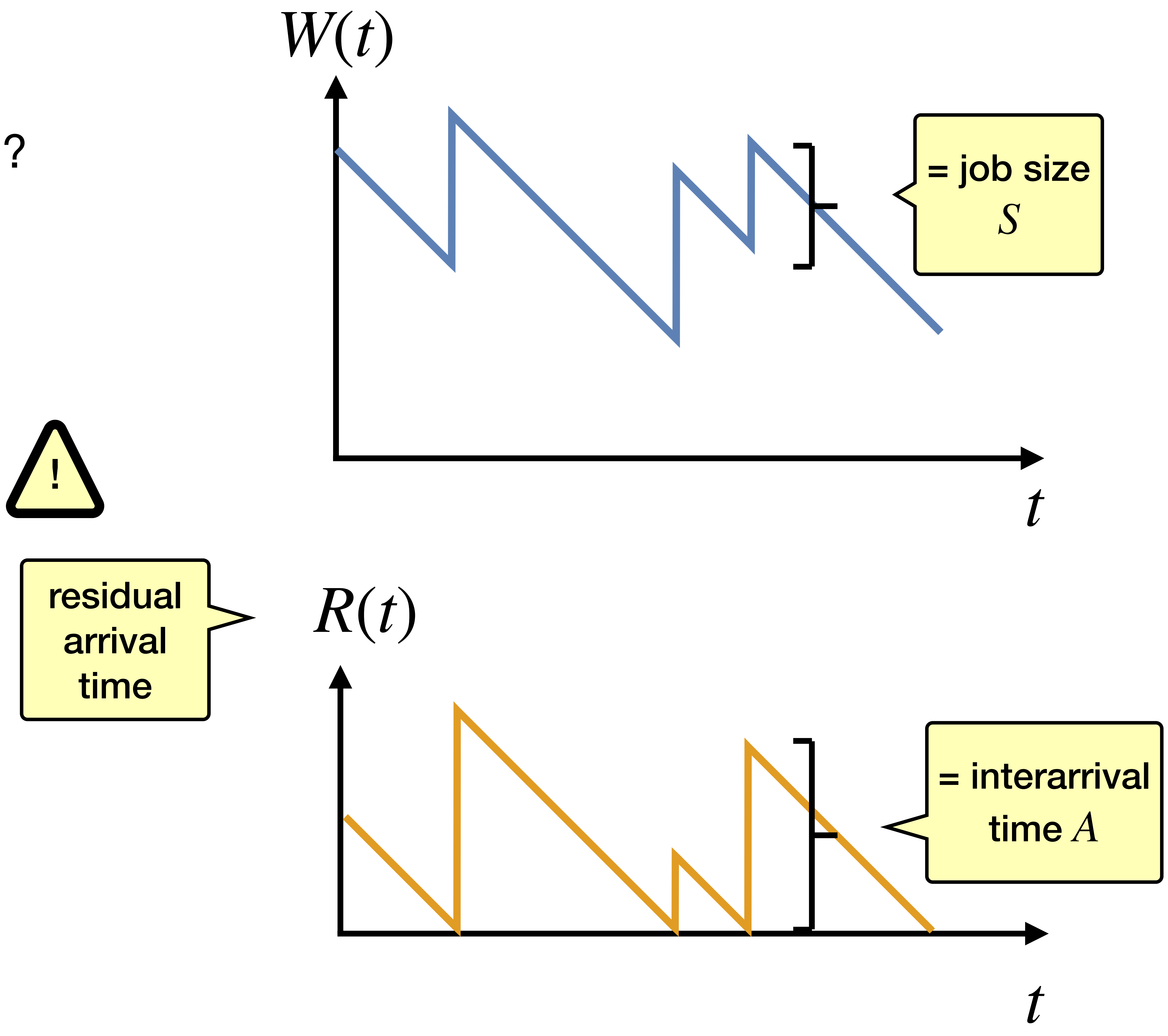


residual arrival time

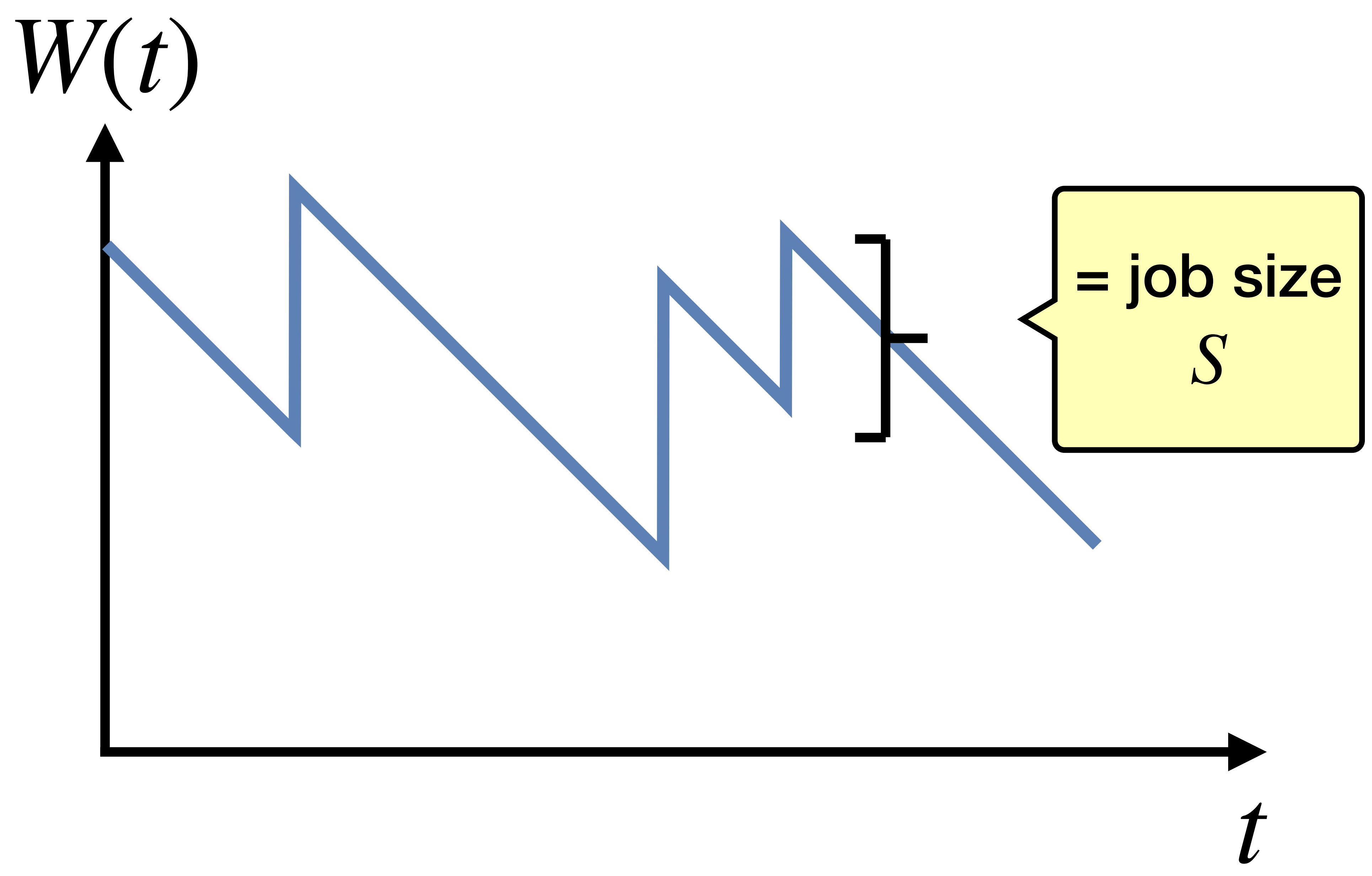
= interarrival time A

Work analysis

- Why is G/G work analysis harder than M/G?
- M/G/1: $W(t)$ Markov process ✓
- G/G/1: $W(t)$ not a Markov process ✗
- $(W(t), R(t))$ Markov, but two dimensional ⚠



Work analysis



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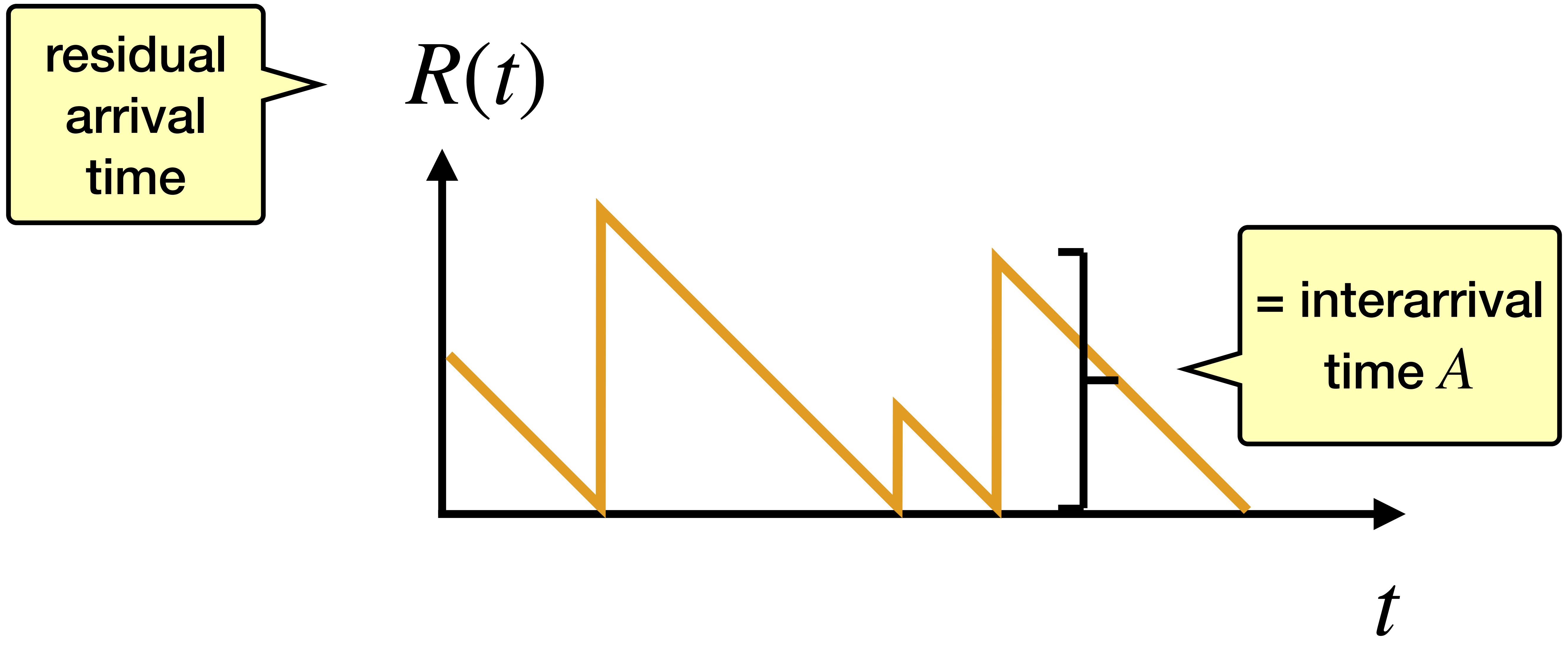

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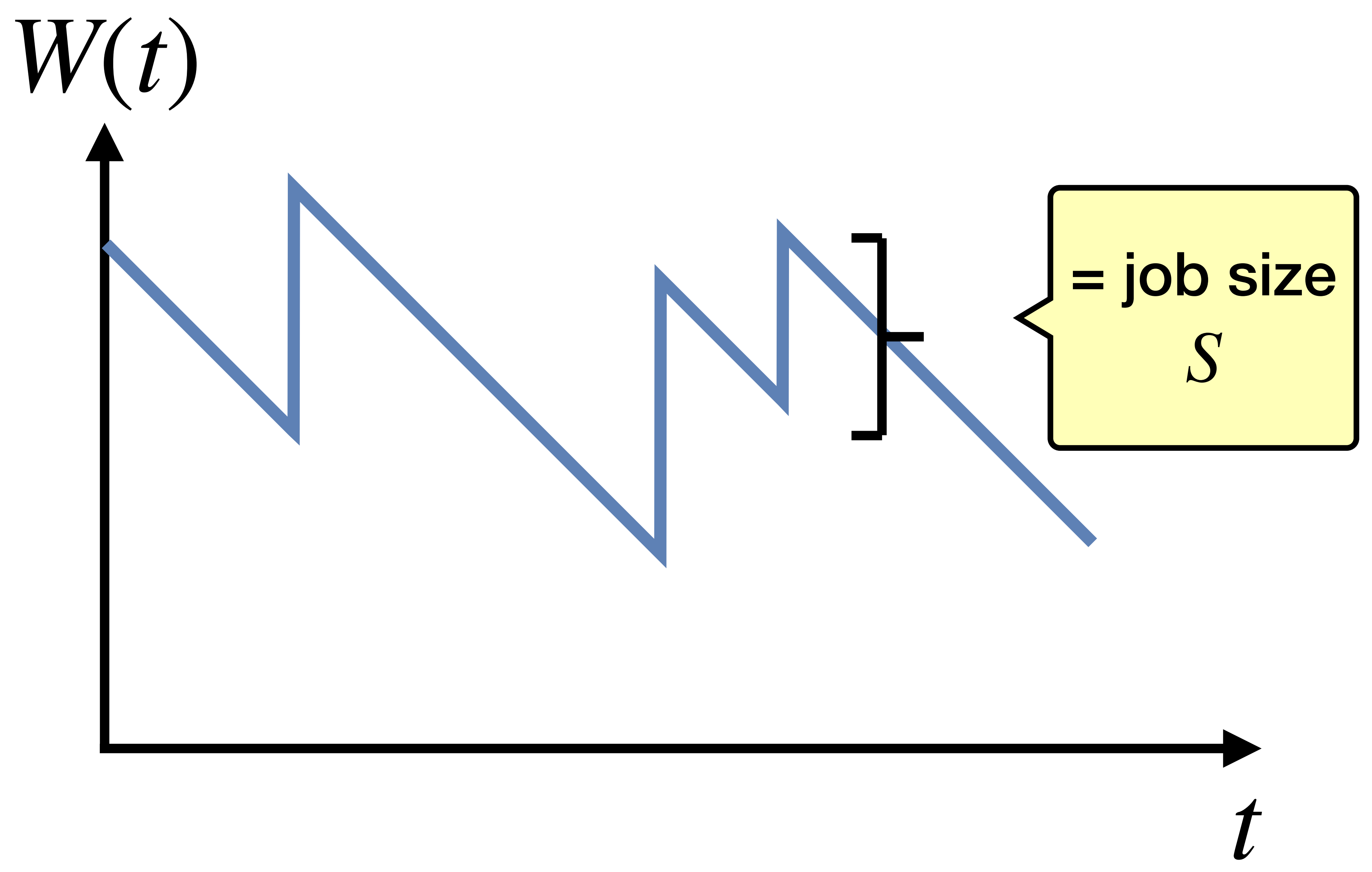
Work analysis

Idea: consider $W(t) - \rho R(t)$



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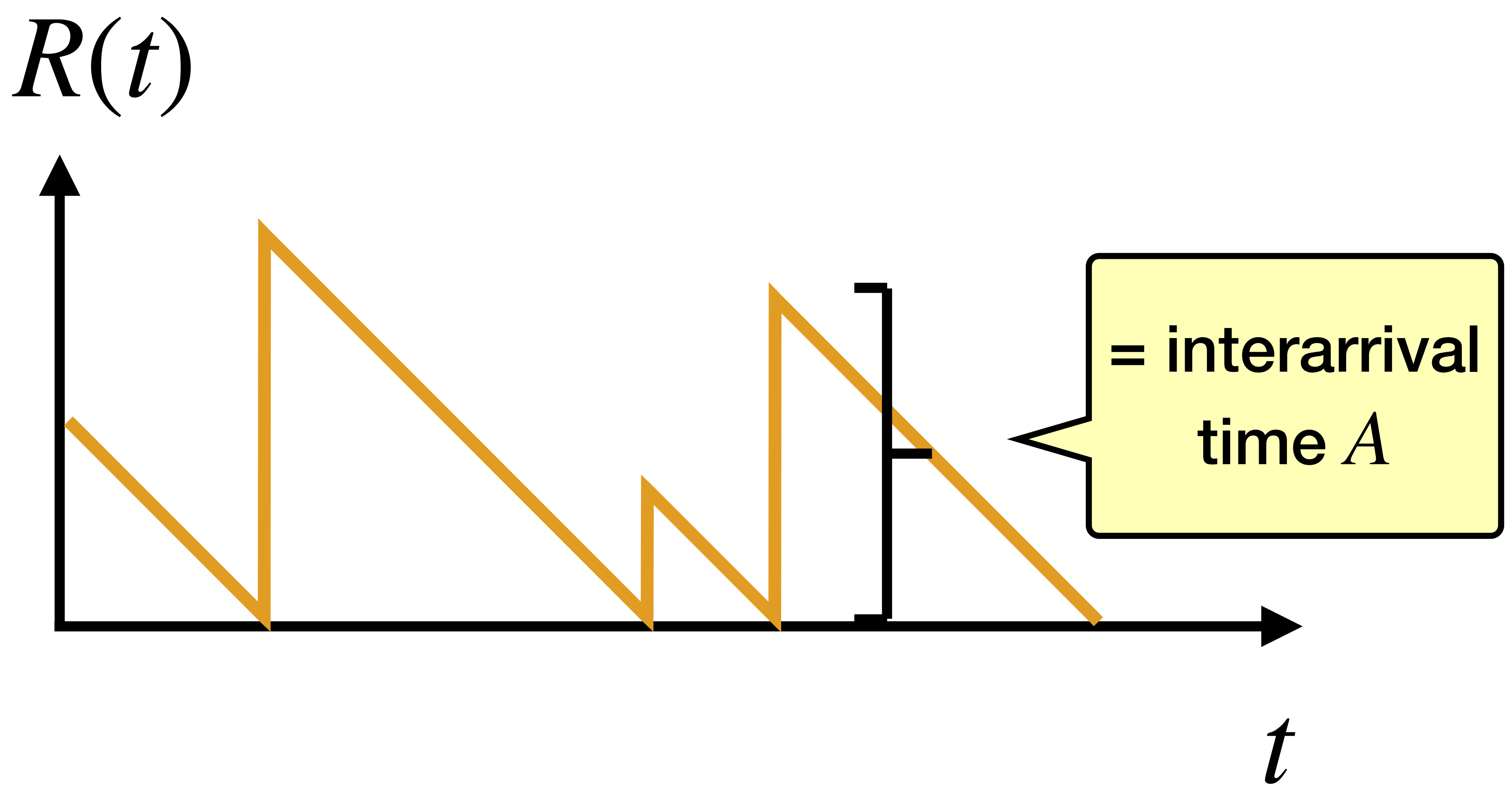

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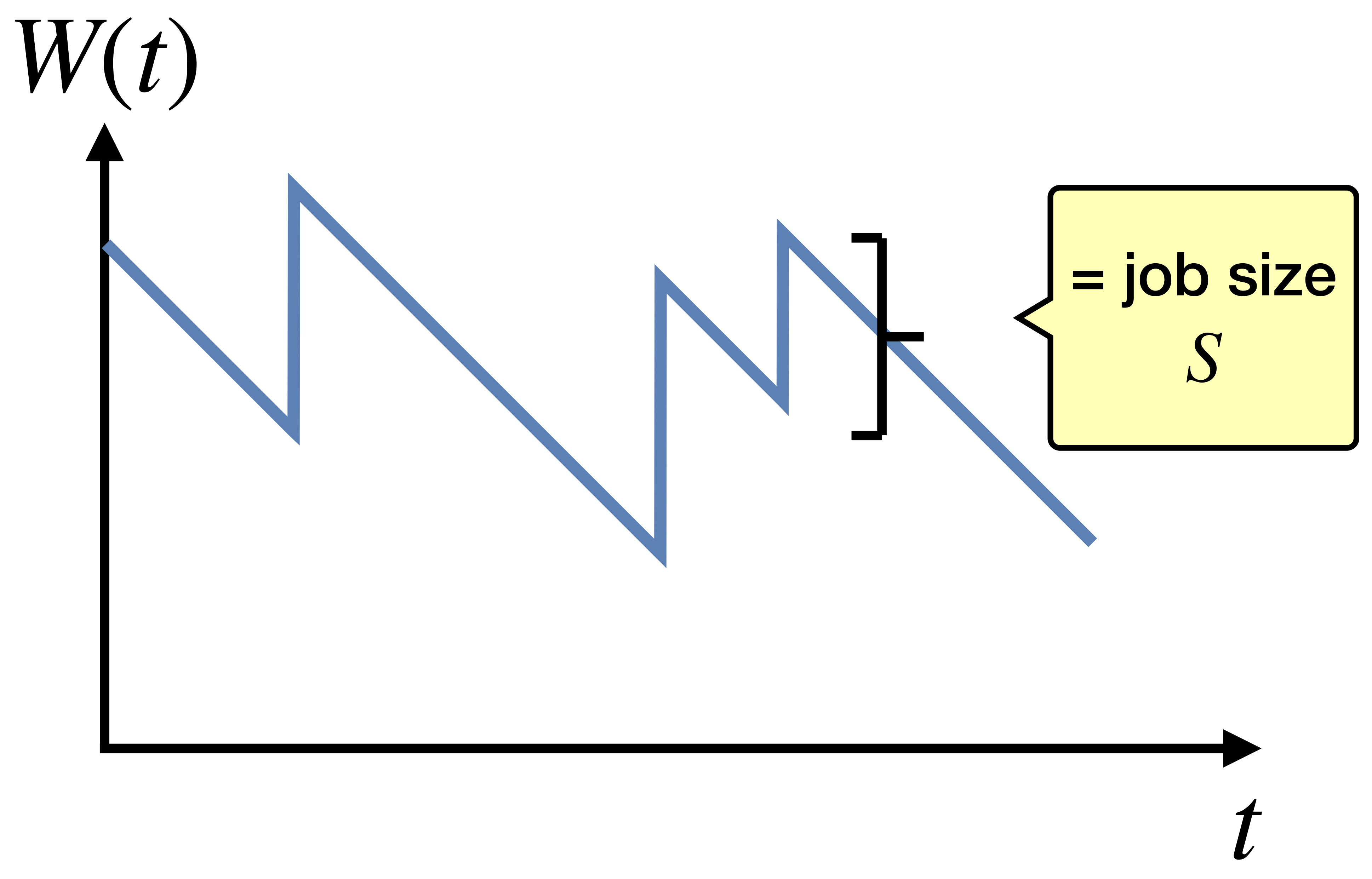

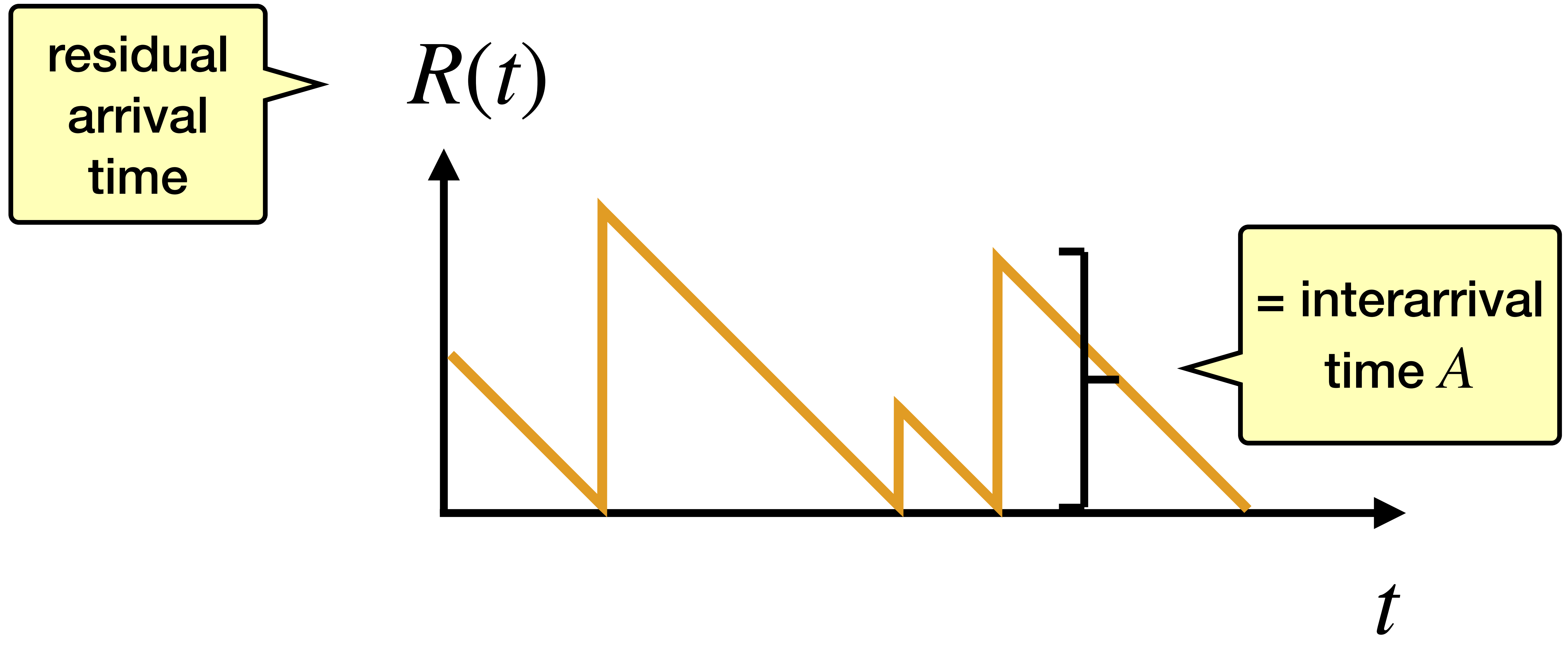

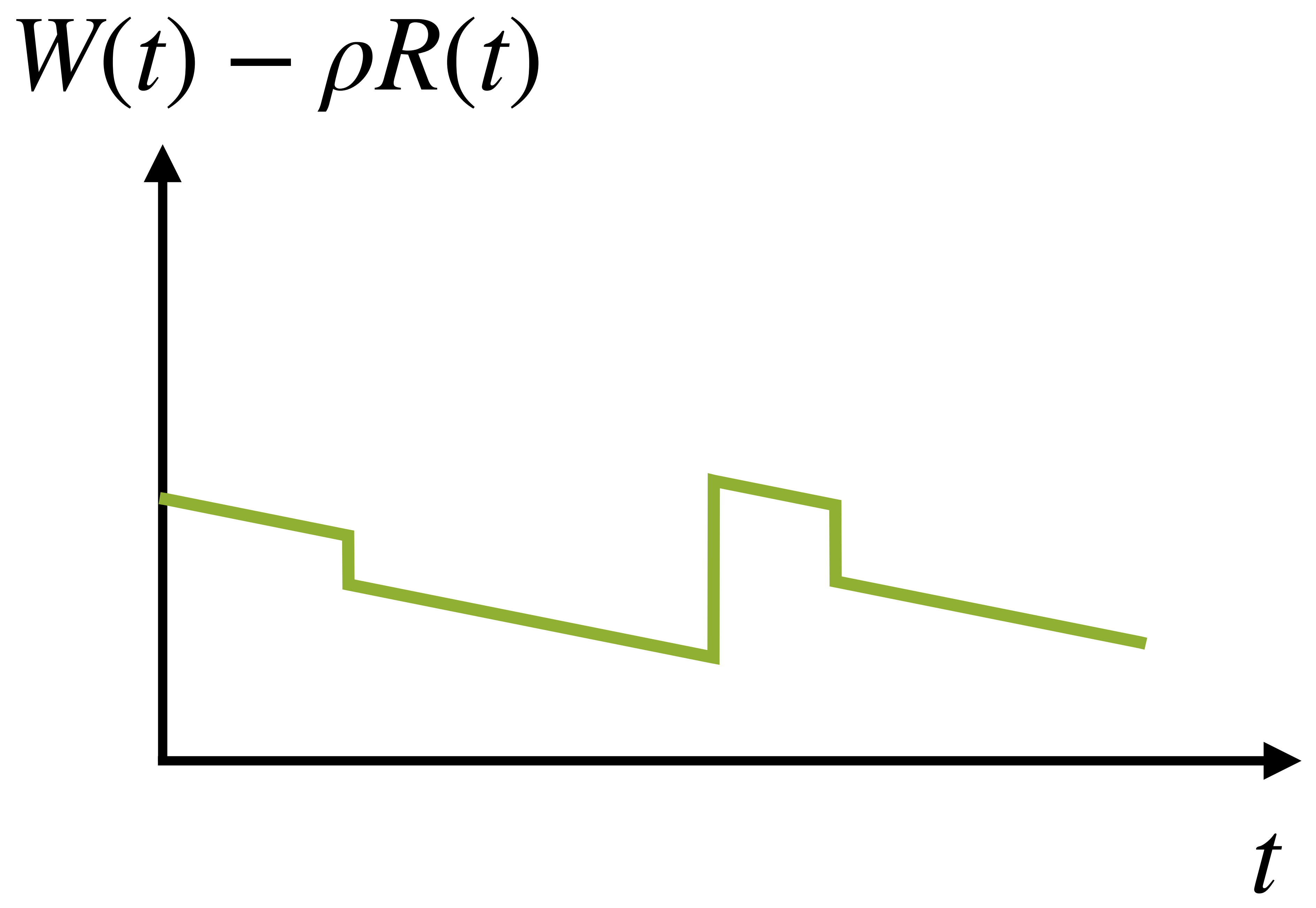

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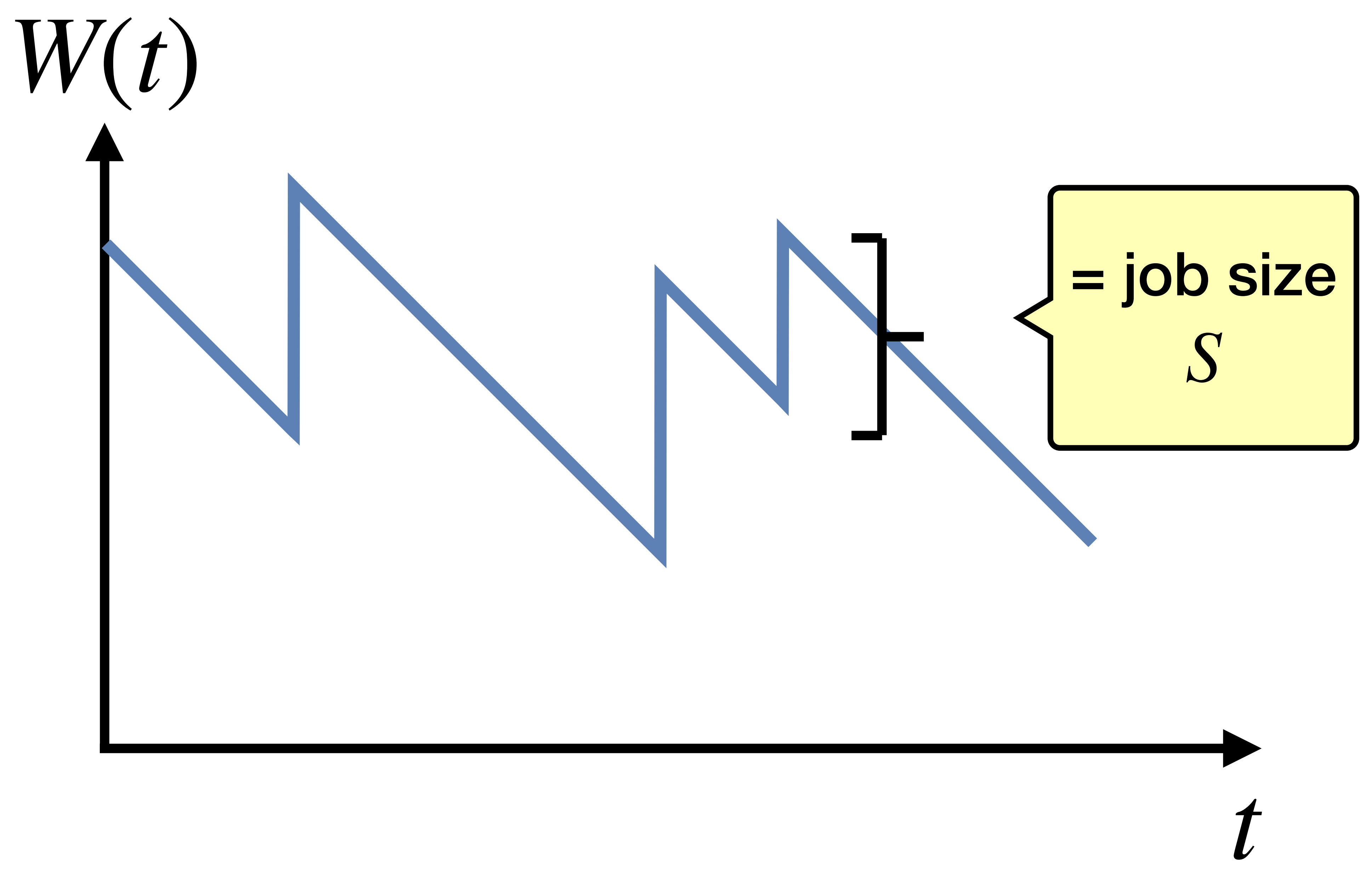

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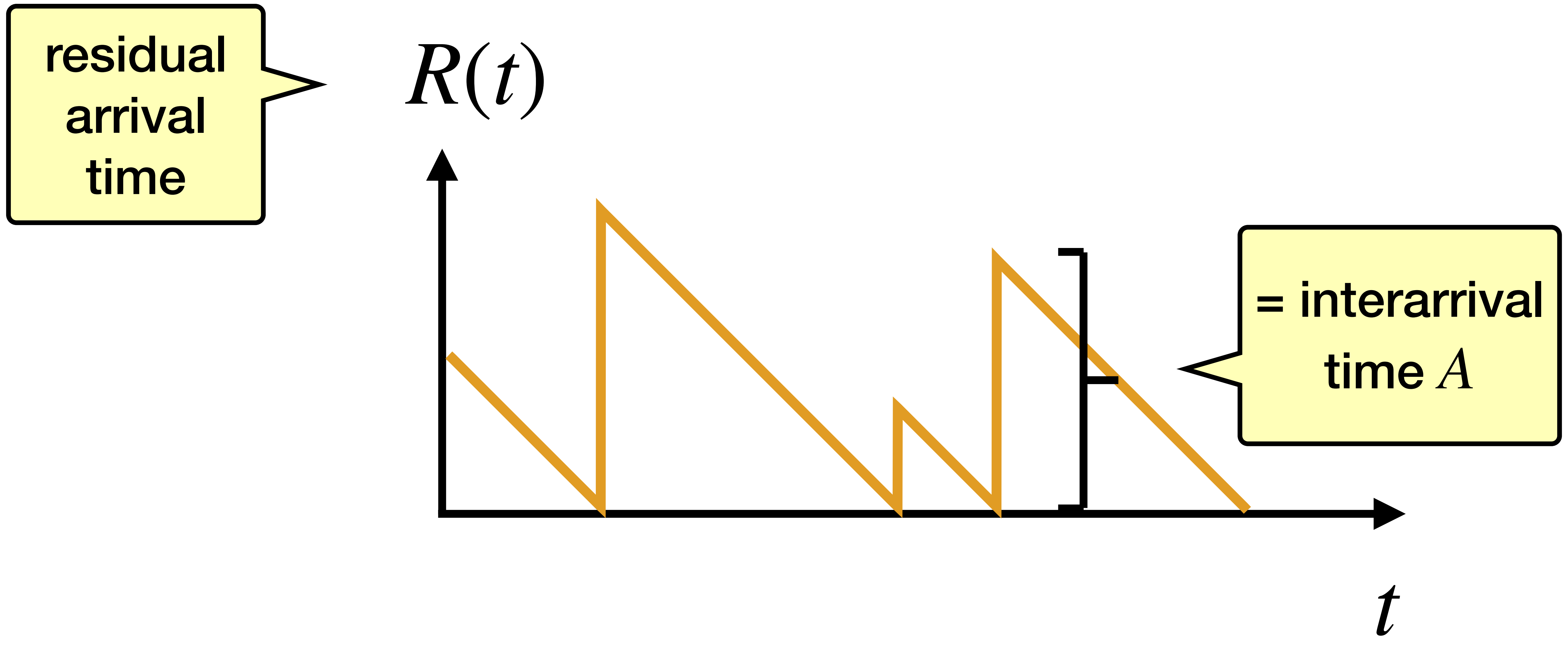

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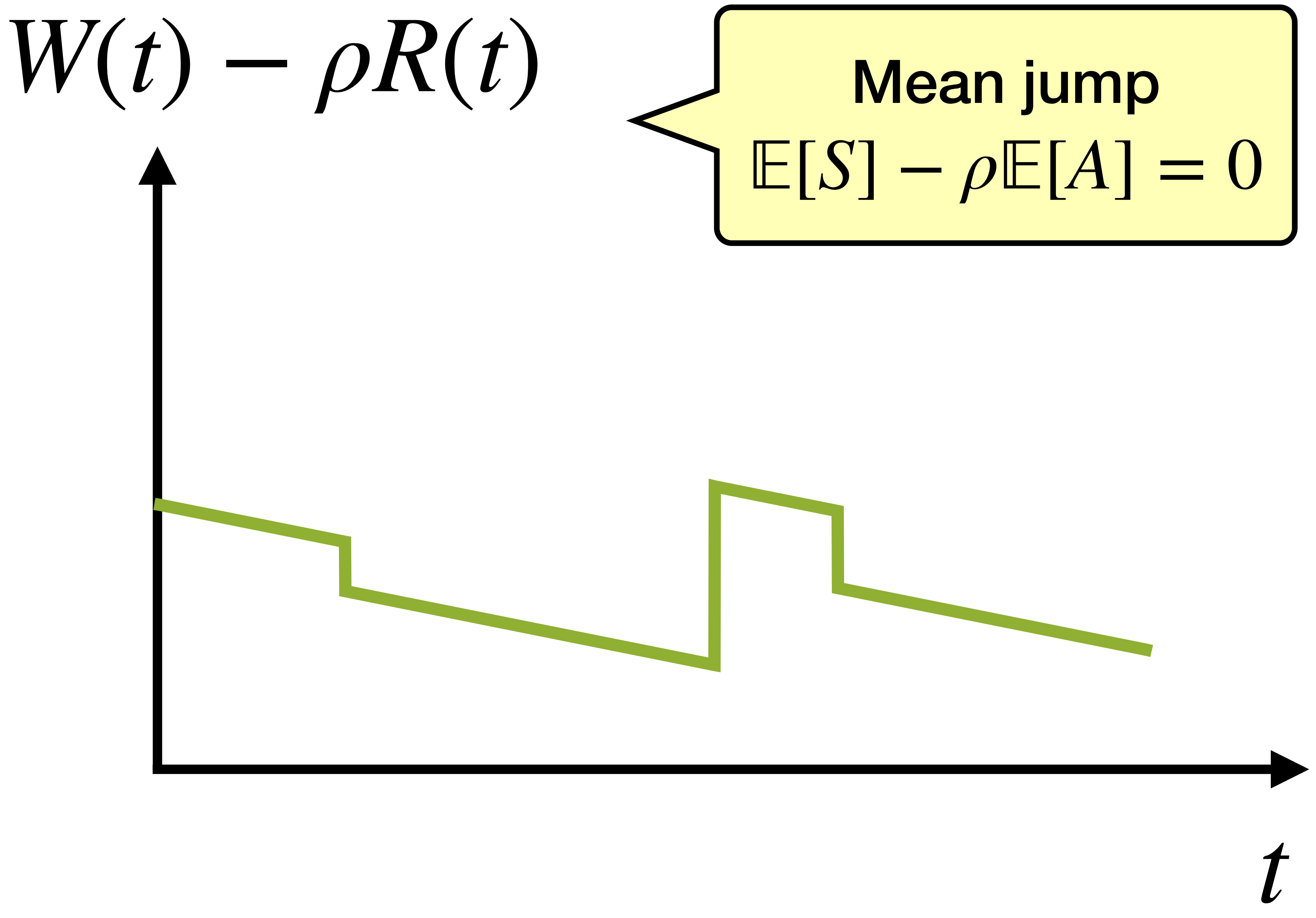
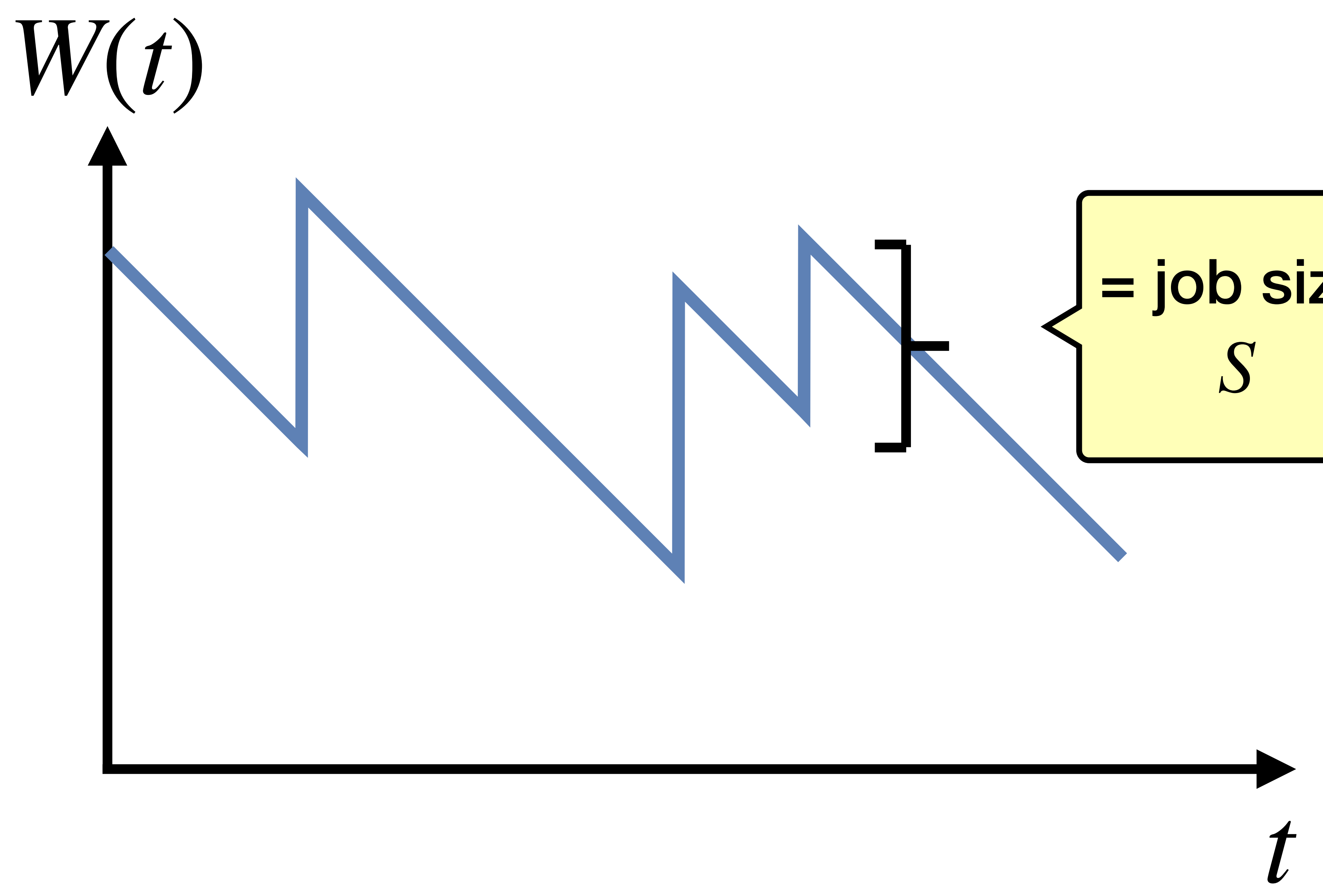


Mean jump $\mathbb{E}[S] - \rho\mathbb{E}[A] = 0$

Work analysis

Idea: consider $W(t) - \rho R(t)$

$W(t) - \rho R(t)$ decreases at const rate + noise



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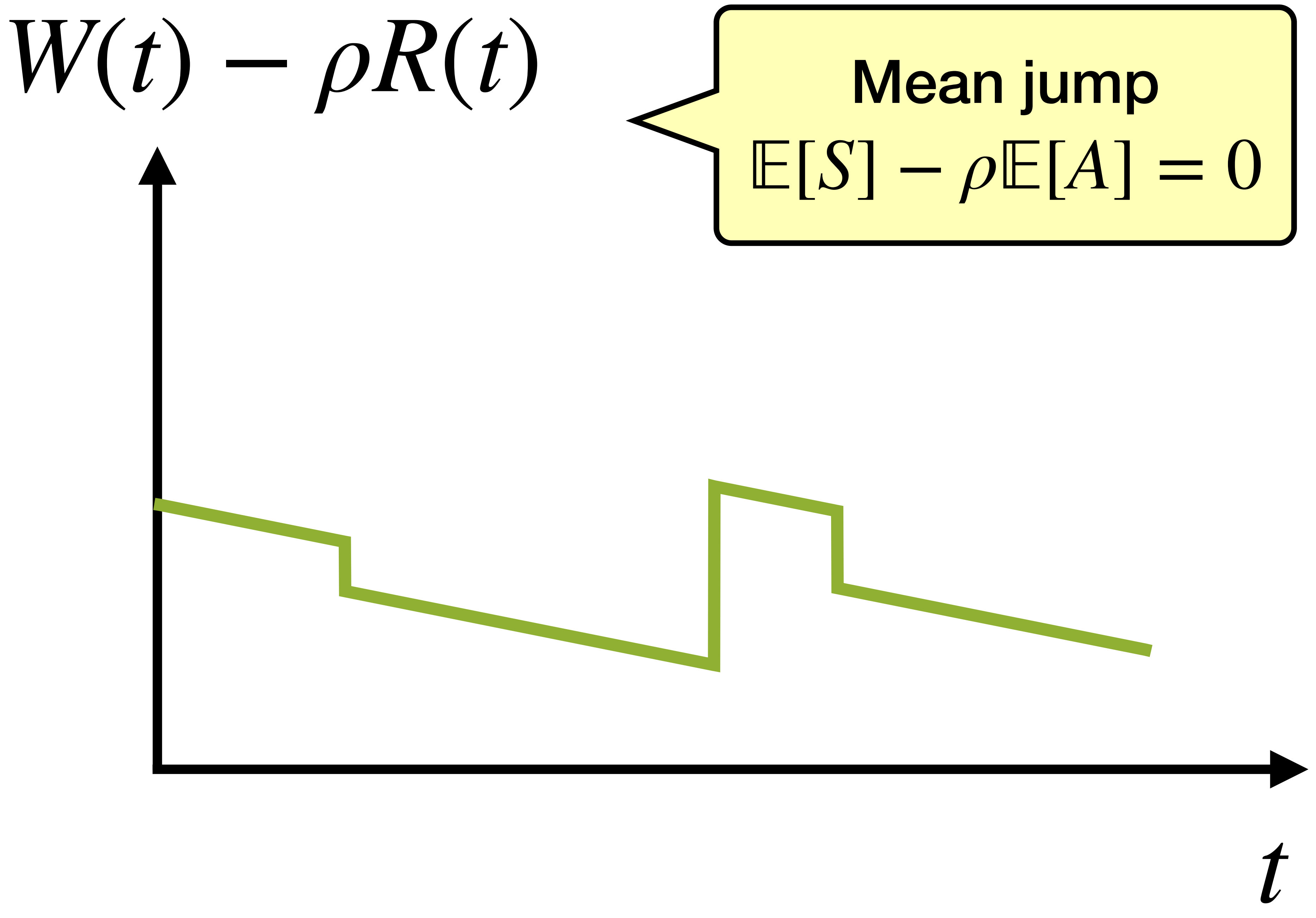

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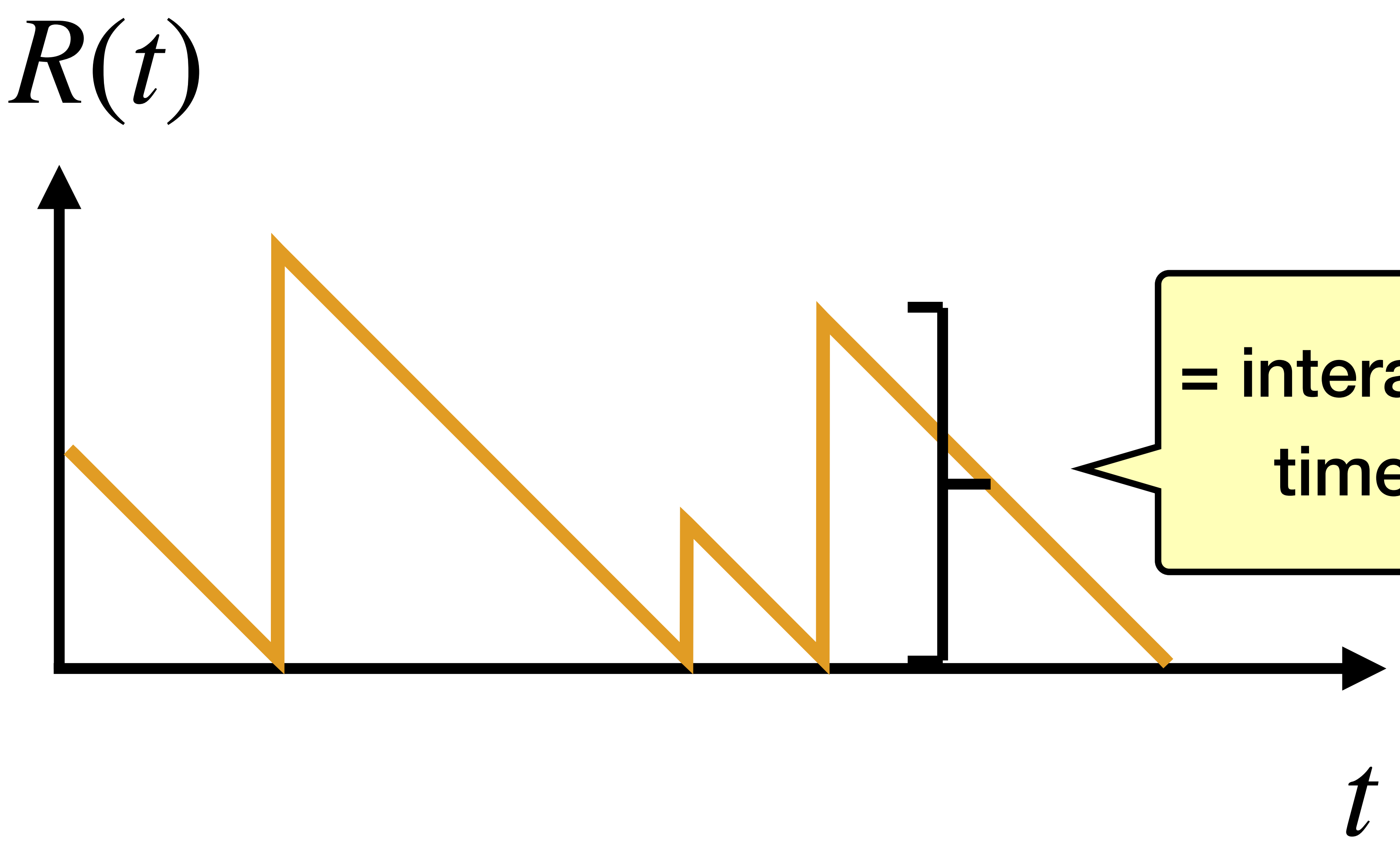

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Work analysis

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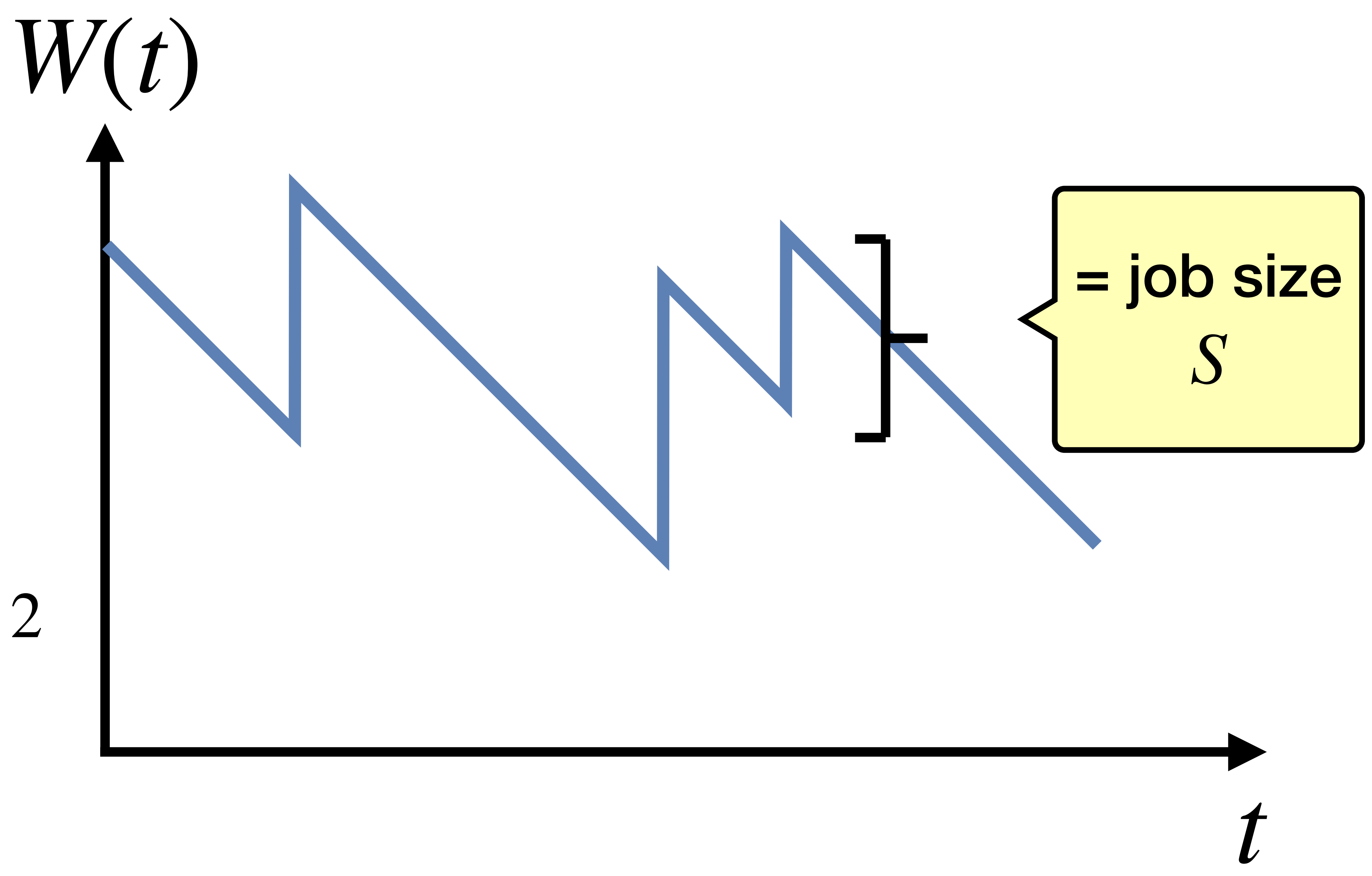
Formally, apply Rate Conservation Law to $\big(W(t) - \rho R(t)\big)^2$



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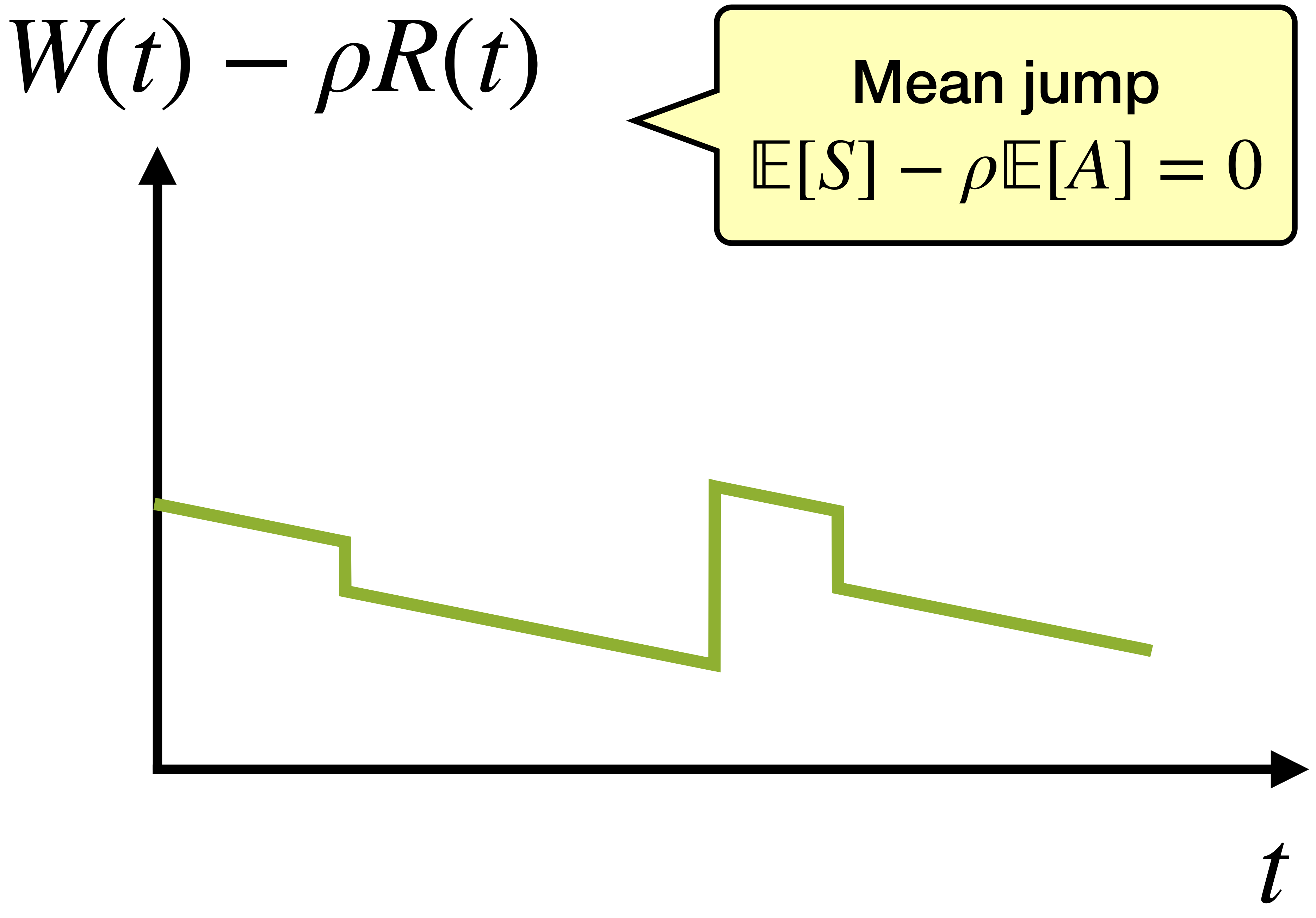

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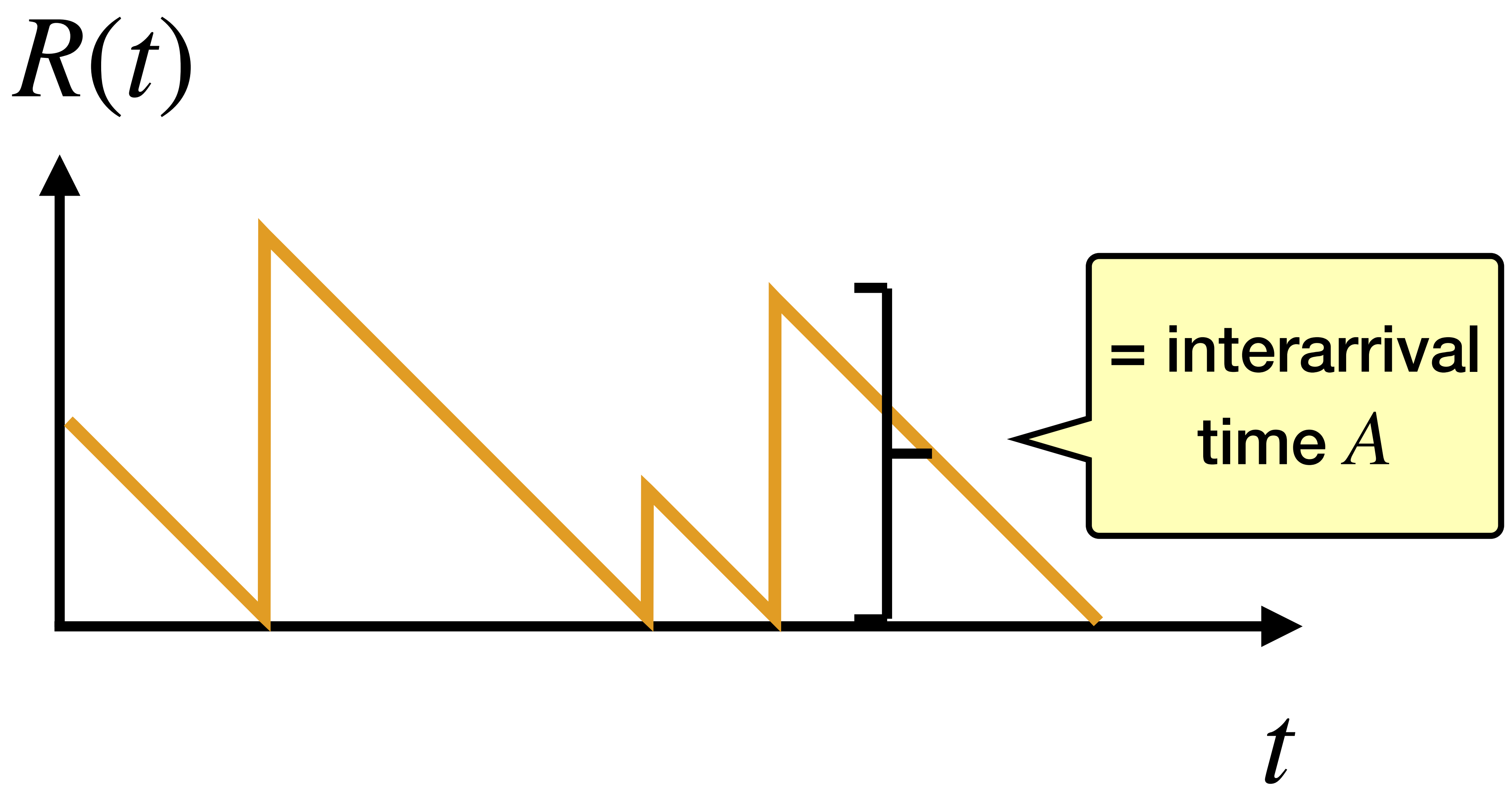
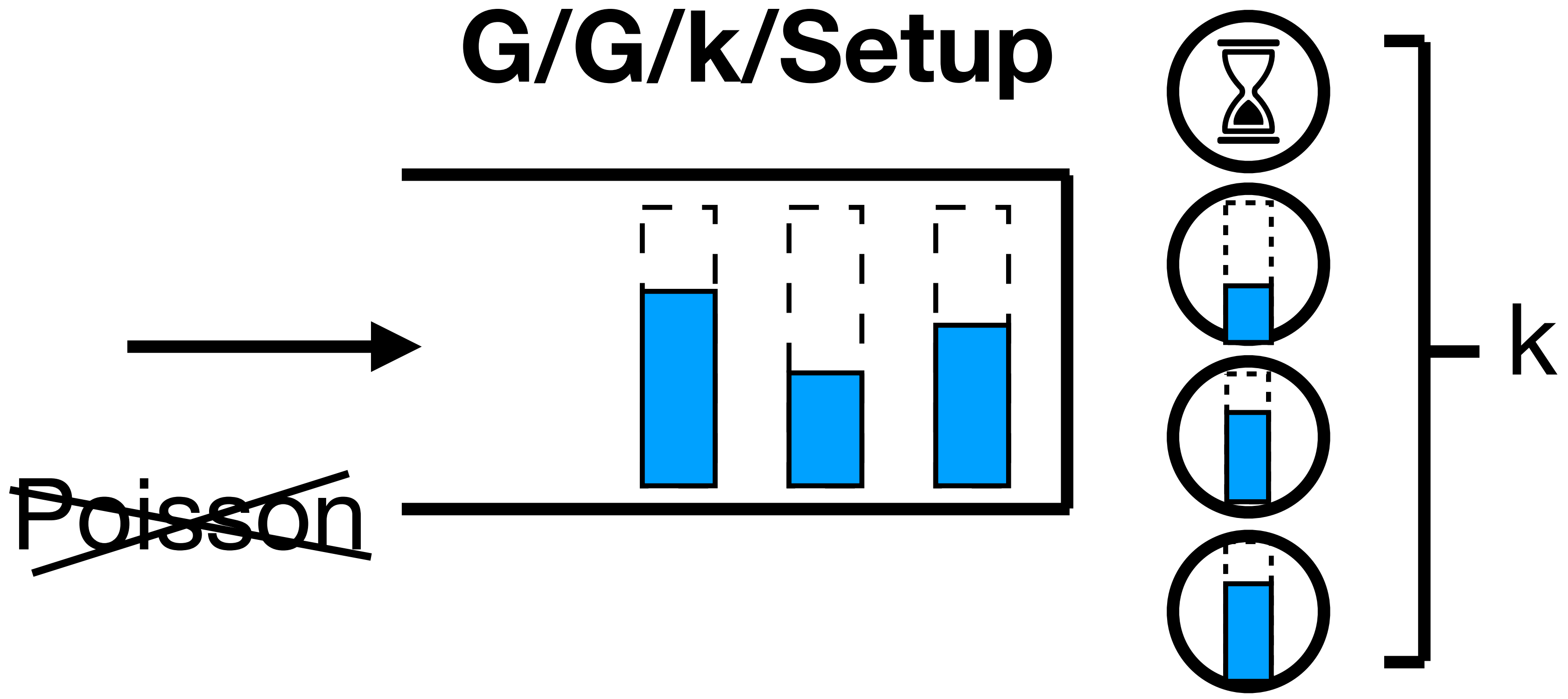


= interarrival time A

Conclusion



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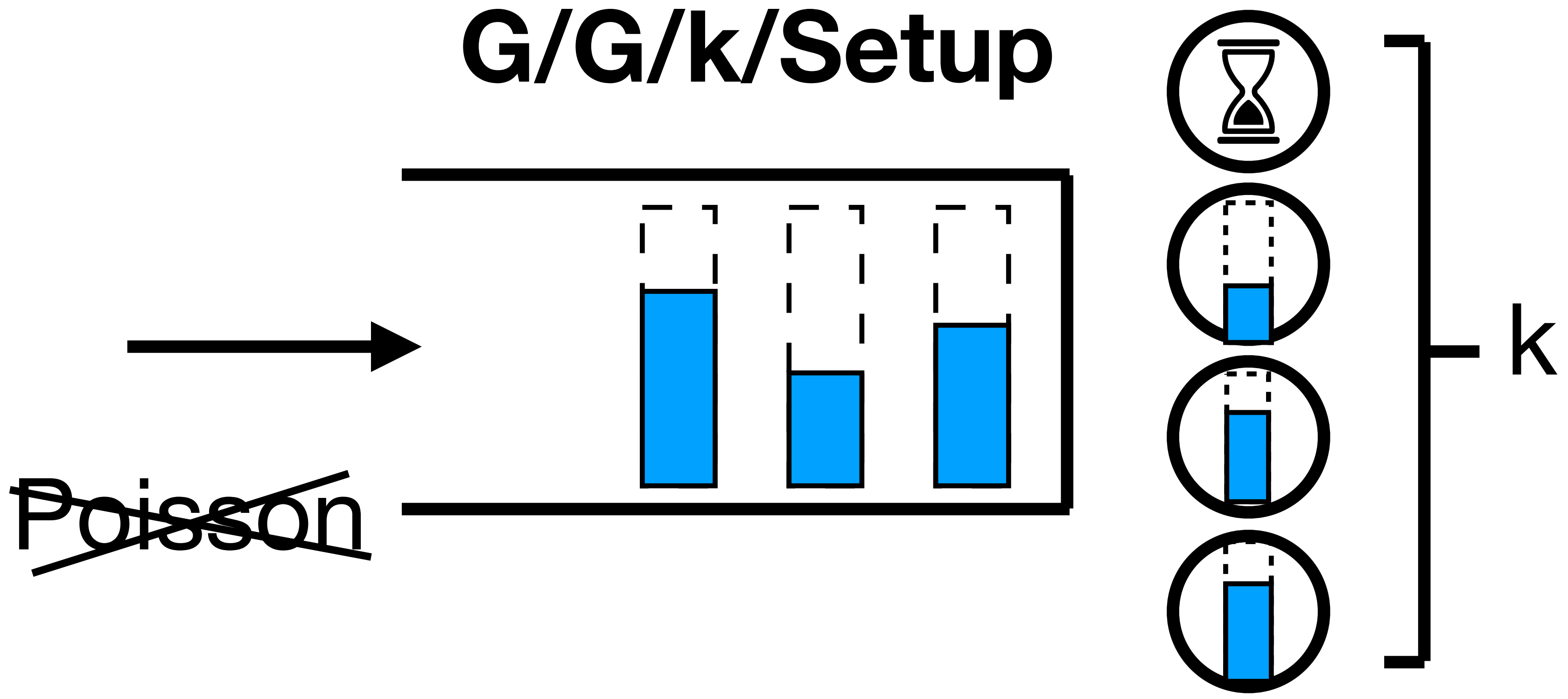
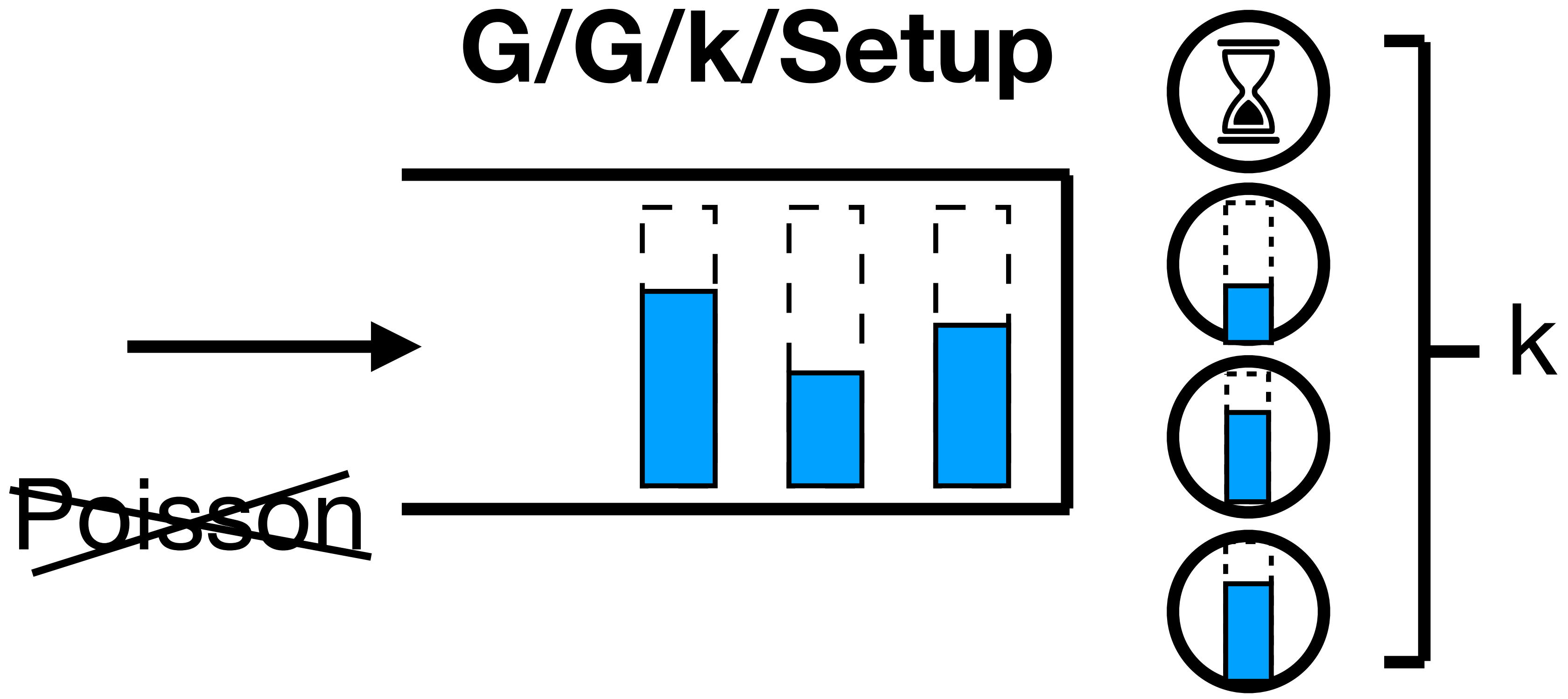


- **Problem:** scheduling in G/G/k/setup

Conclusion



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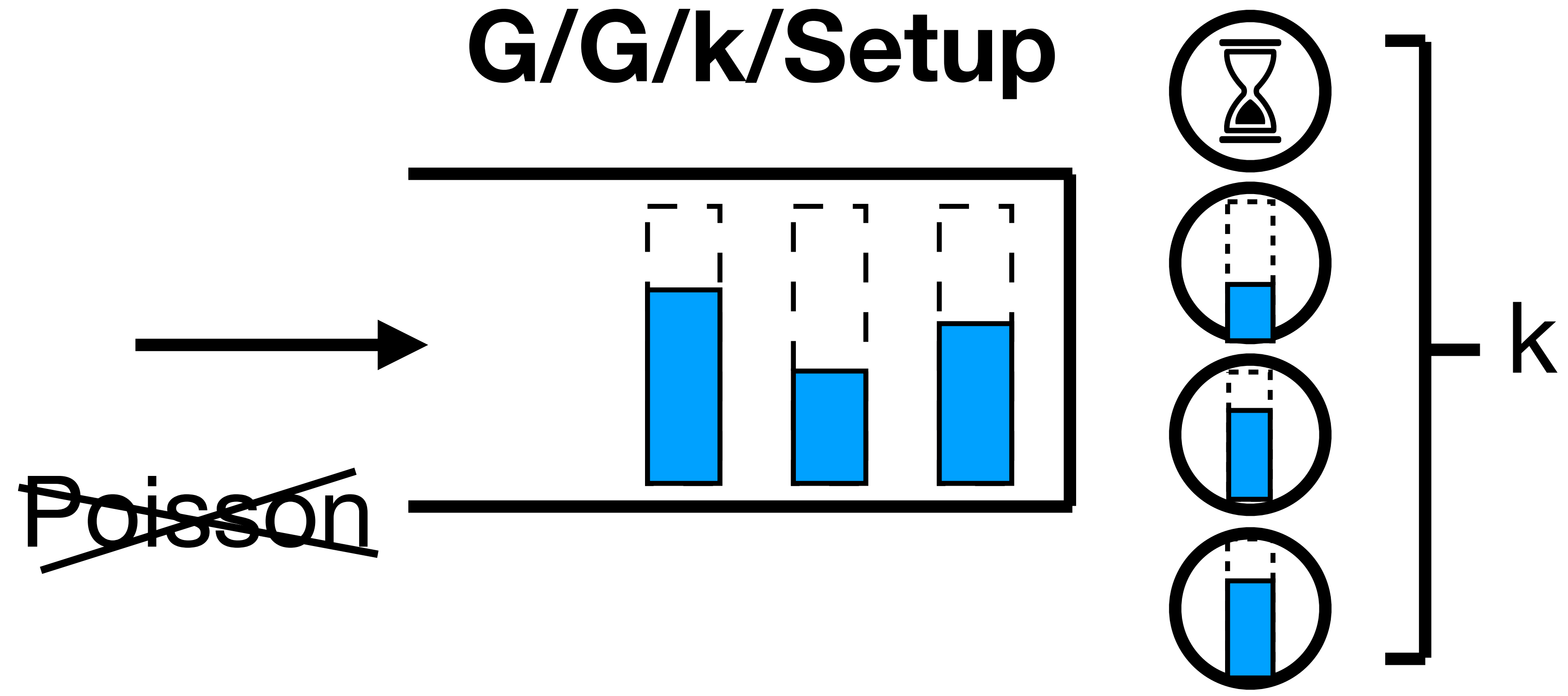
- **Question:** Is Gittins good?

Conclusion



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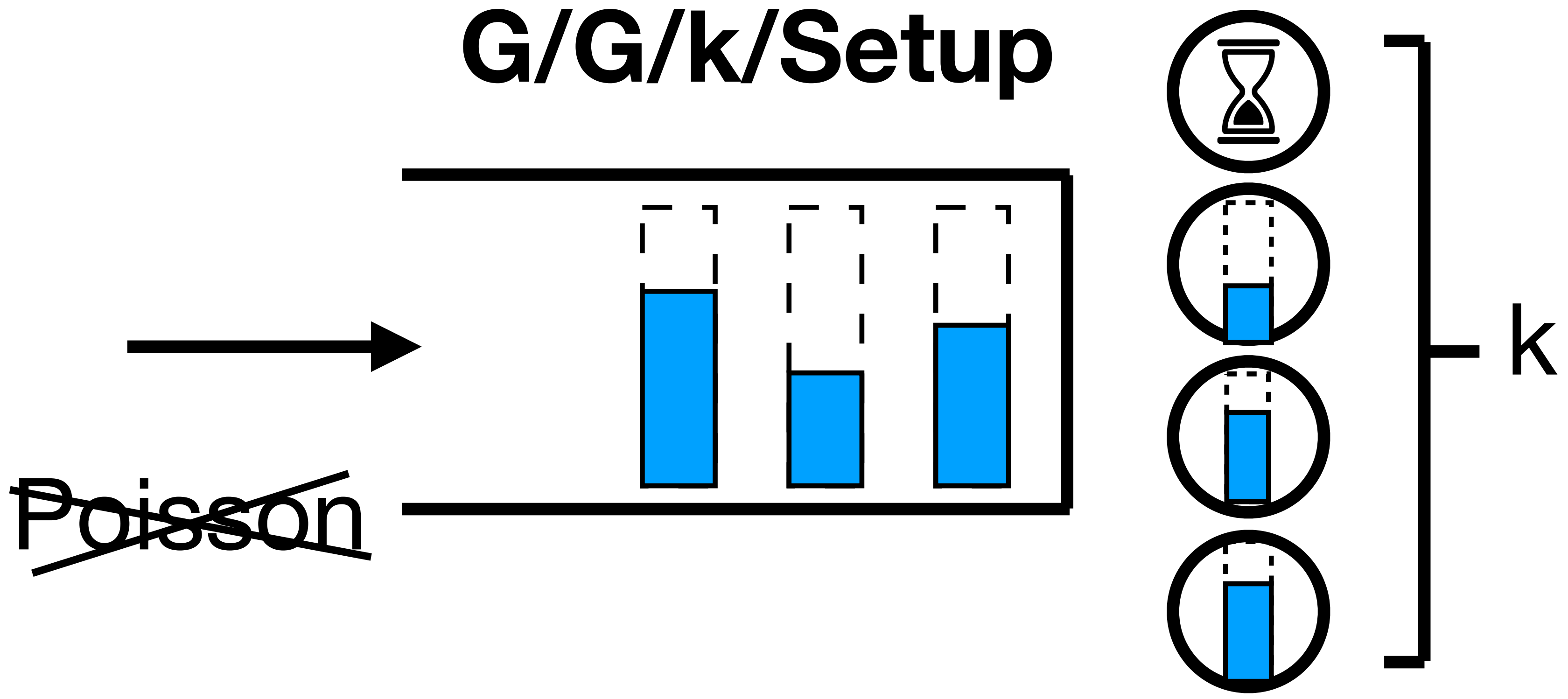

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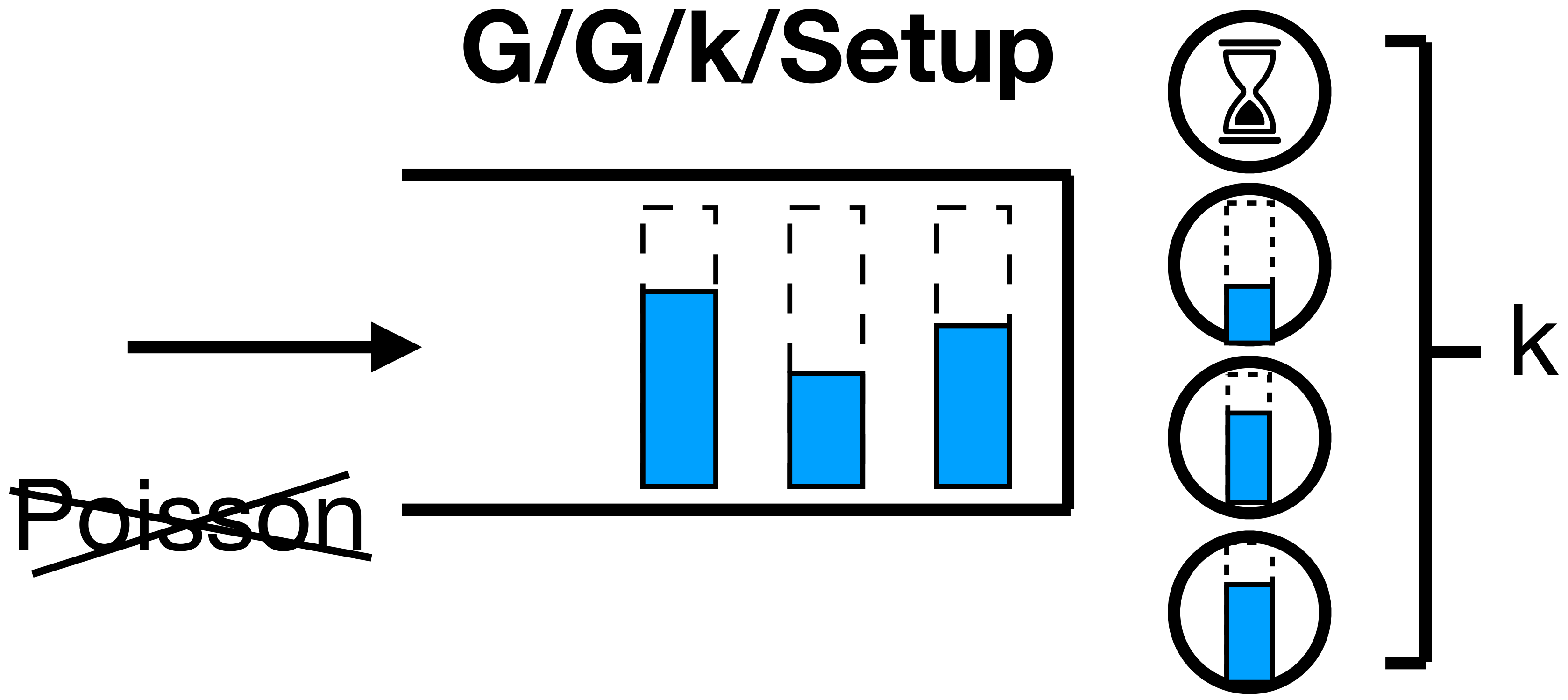
Conclusion



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Conclusion

G/G/k/Setup

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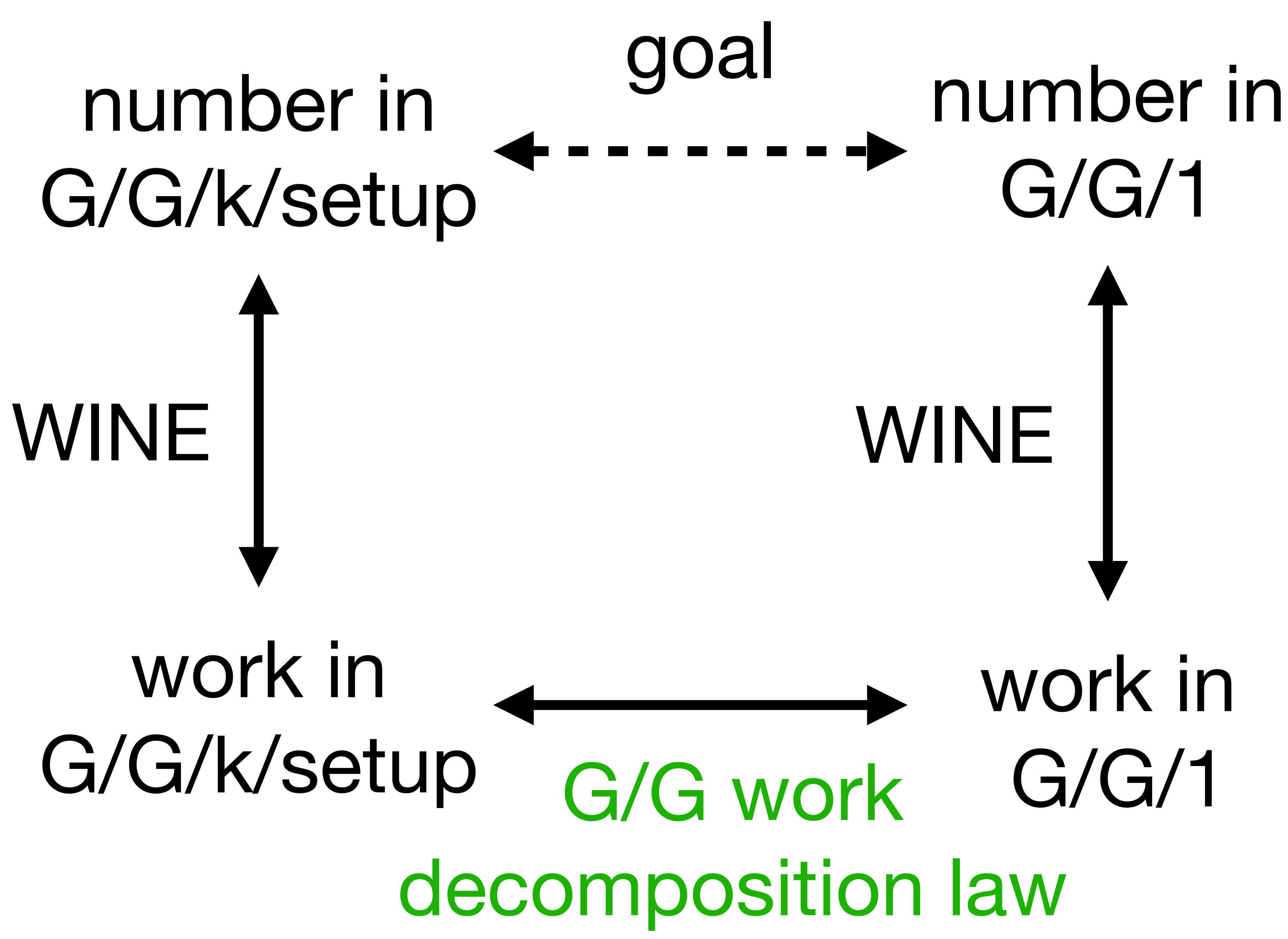


number in G/G/k/setup

goal

number in G/G/1

WINE

WINE

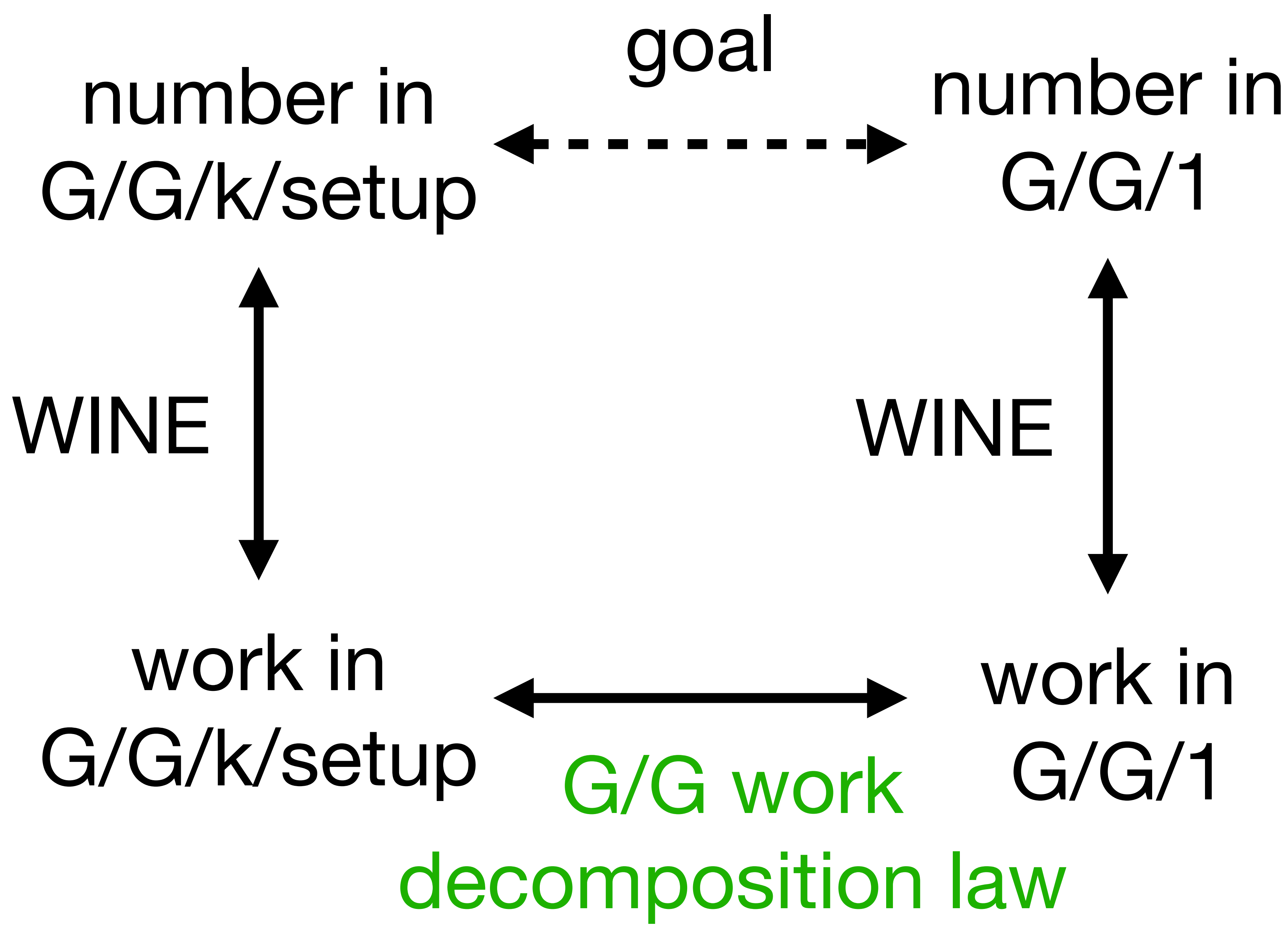
work in G/G/k/setup

work in G/G/1

G/G work decomposition law

Conclusion

- **Problem:** scheduling in G/G/k/setup
- **Question:** Is Gittins good?
- **Result:** gap $\leq \ell_{(a)} + \ell_{(b)} + \ell_{(c)}$
- **Corollary:** heavy-traffic optimal if $\mathbb{E}[S^2 \log S] < \infty$
- **Key tool:** new G/G work-decomposition law
- **Takeaway:** Gittins applicable far beyond M/G/1



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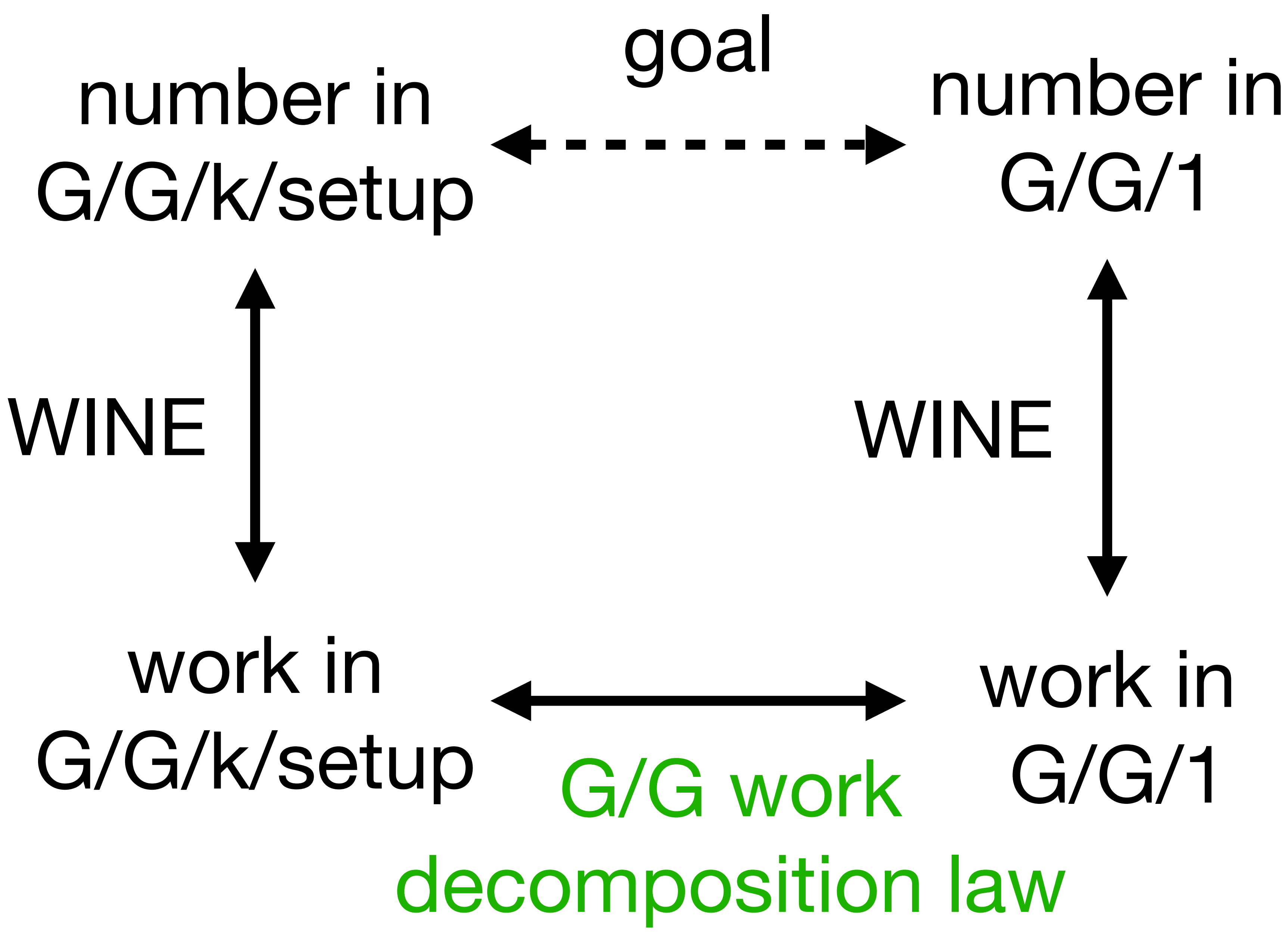

number in G/G/k/setup

goal

number in G/G/1

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work in G/G/k/setup

work in G/G/1

G/G work decomposition law

Thank you!