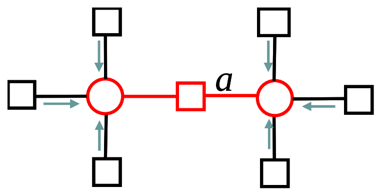
School of Computer Science
Carnegie Mellon

Probabilistic Graphical Models

Variational Inference

Eric Xing

Lecture 13, February 24, 2014



Reading: See class website

Inference Problems

- Compute the likelihood of observed data
- Compute the marginal distribution $p(x_A)$ over a particular subset of nodes $A \subset V$
- Compute the conditional distribution $p(x_A | x_B)$ for disjoint subsets *A* and *B*
- Compute a mode of the density $\hat{x} = \arg\max_{x \in \mathcal{X}^m} p(x)$

- Methods we have

Brute force **Elimination** $\Rightarrow$ **Message Passing**

(Forward-backward , Max-product /BP, Junction Tree)

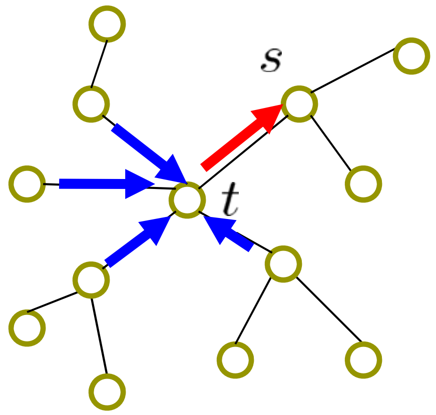
Individual computations independent

Sharing intermediate terms

Sum-Product Revisited

- Tree-structured GMs

$$p(x_1, \cdots, x_m) = \frac{1}{Z} \prod_{s \in V} \psi_s(x_s) \prod_{(s,t) \in E} \psi_{st}(x_s, x_t)$$



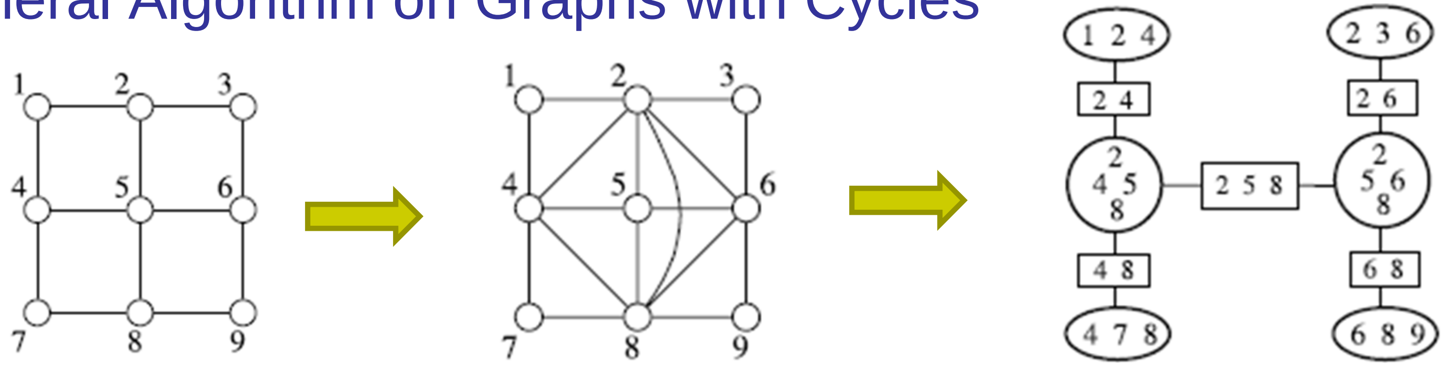
- Message Passing on Trees:

$$M_{t \to s}(x_s) \leftarrow \kappa \sum_{x'_t} \left\{ \psi_{st}(x_s, x'_t) \psi_t(x'_t) \prod_{u \in N(t) \backslash s} M_{u \to t}(x'_t) \right\}$$

  - On trees, converge to a unique fixed point after a finite number of iterations

Junction Tree Revisited

- General Algorithm on Graphs with Cycles

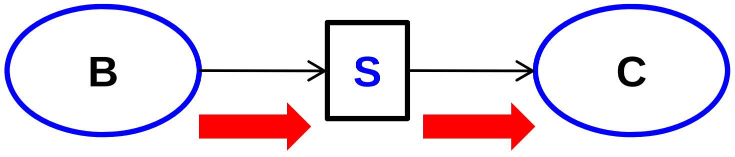


- Steps: => Triangularization => Construct JTs

=> Message Passing on Clique Trees

$$\tilde{\phi}_S(x_S) \leftarrow \sum_{x_{B\setminus S}} \phi_B(x_B)$$

$$\phi_C(x_C) \leftarrow \frac{\tilde{\phi}_S(x_S)}{\phi_S(x_S)} \phi_C(x_C)$$



Local Consistency

- Given a set of functions $\{\tau_C,\ C \in \mathcal{C}\}$ and $\{\tau_S,\ S \in \mathcal{S}\}$ associated with the cliques and separator sets

- They are locally consistent if:

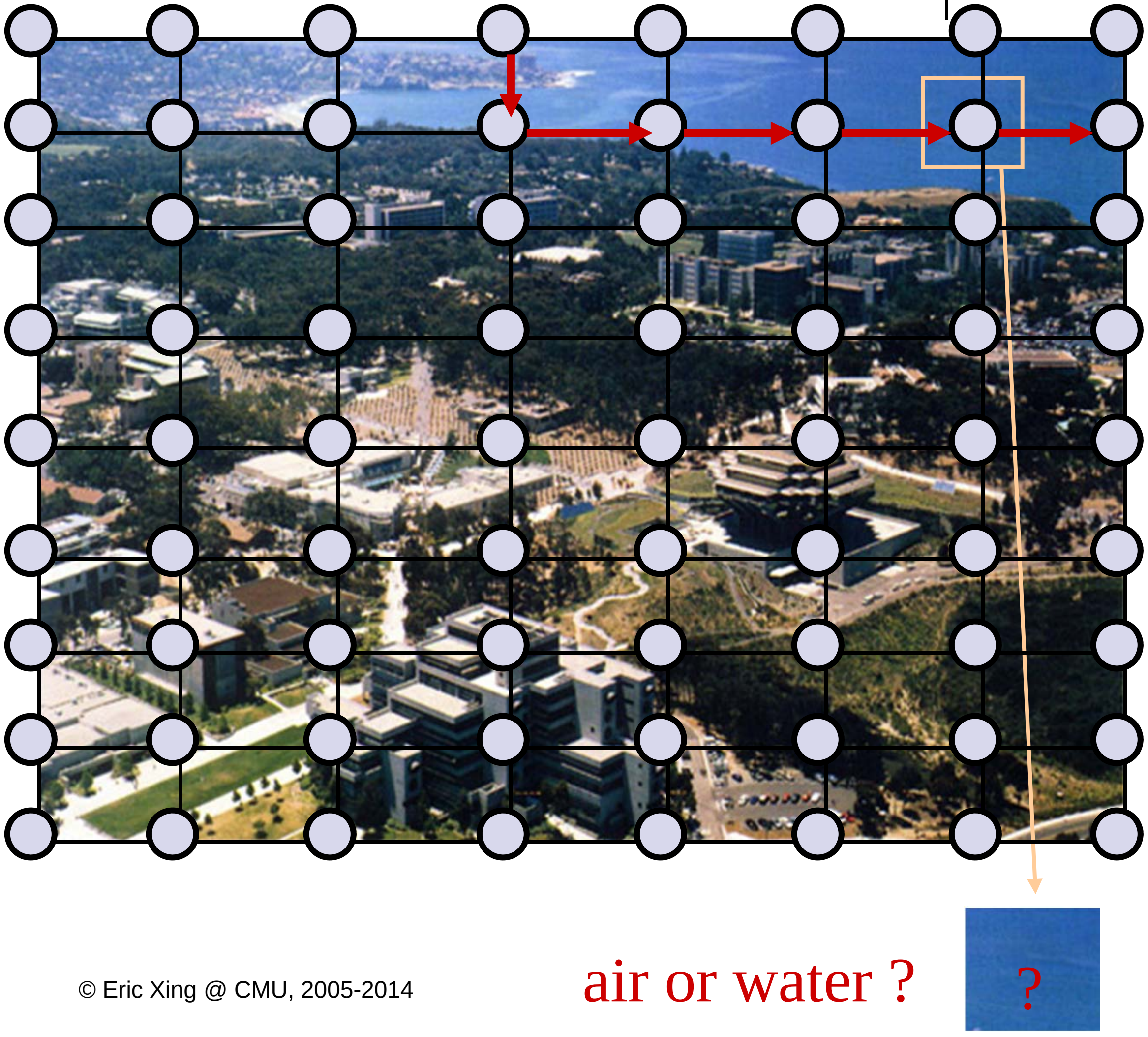
$$\sum_{x'_S} \tau_S(x'_S) = 1,\ \forall S \in \mathcal{S}$$

$$\sum_{x'_C | x'_S = x_S} \tau_C(x'_C) = \tau_S(x_S),\ \forall C \in \mathcal{C},\ S \subset C$$

- For junction trees, local consistency is equivalent to global consistency!

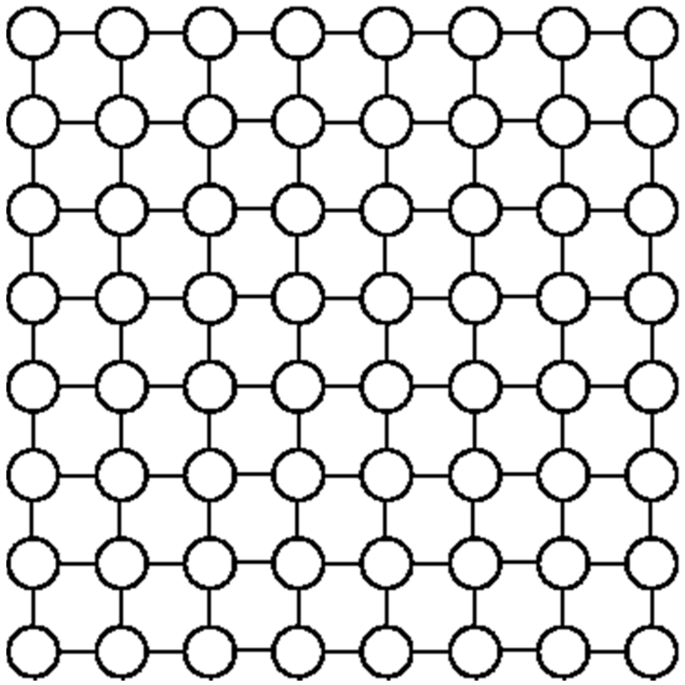
An Ising model on 2-D image

- Nodes encode hidden information (patch-identity).
- They receive local information from the image (brightness, color).
- Information is propagated though the graph over its edges.
- Edges encode 'compatibility' between nodes.



Why Approximate Inference?

- Why can't we just run junction tree on this graph?



$$p(X) = \frac{1}{Z}\exp\left\{\sum_{i<j}\theta_{ij}X_iX_j + \sum_i \theta_{i0}X_i\right\}$$

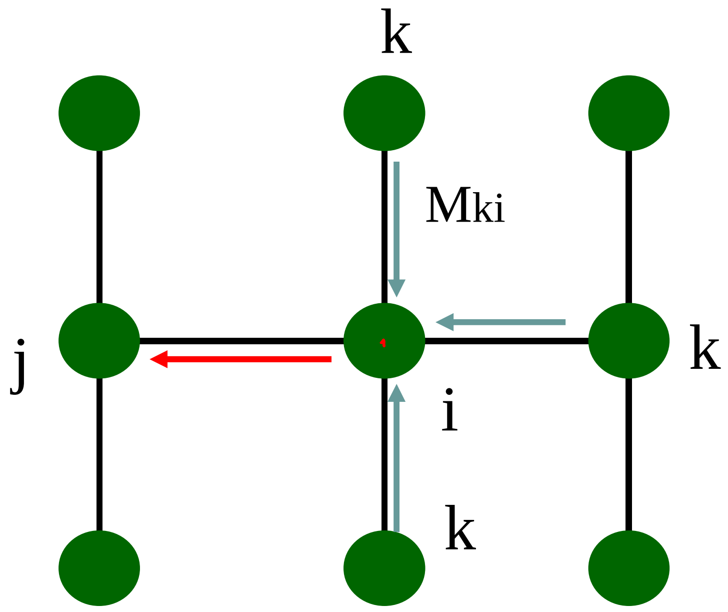
- If NxN grid, tree width at least N
- N can be a huge number(~1000s of pixels)
    - If N~O(1000), we have a clique with 2^{100} entries

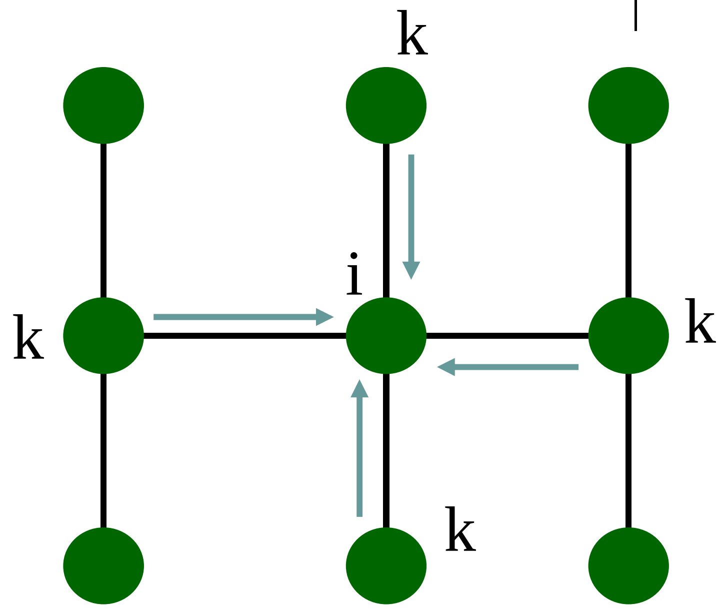
Approaches to inference

- Exact inference algorithms
  - The elimination algorithm
  - Message-passing algorithm (sum-product, belief propagation)
  - The junction tree algorithms
- Approximate inference techniques
  - Variational algorithms
    - Loopy belief propagation
    - Mean field approximation
  - Stochastic simulation / sampling methods
  - Markov chain Monte Carlo methods

Loopy Belief Propogation

Recap: Belief Propagation





- BP Message-update Rules

$$M_{i\to j}(x_j) \propto \sum_{x_i} \psi_{ij}(x_i, x_j)\psi_i(x_i)\prod_k M_{k\to i}(x_i)$$

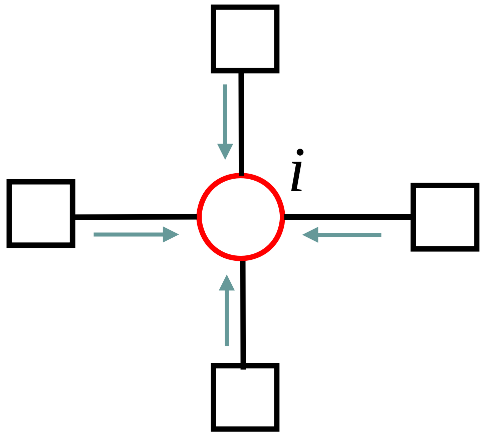
Compatibilities (interactions)

external evidence

$$b_i(x_i) \propto \psi_i(x_i)\prod_k M_k(x_k)$$

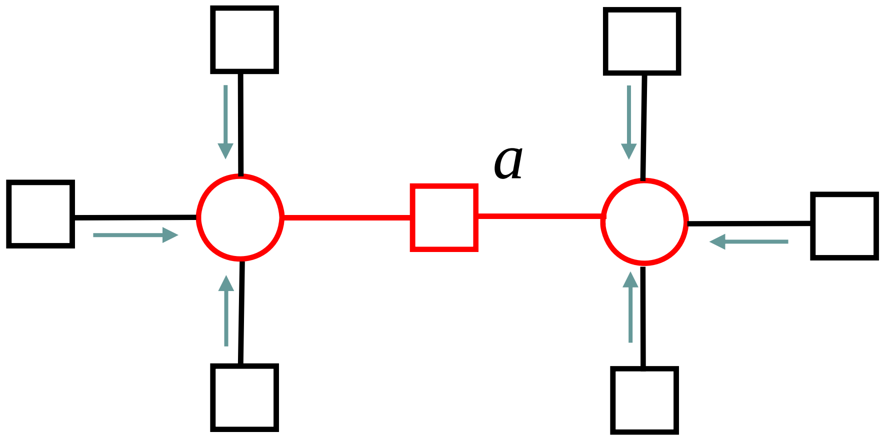
- BP on trees always converges to exact marginals (cf. Junction tree algorithm)

Beliefs and messages in FG



$$b_i(x_i) \propto f_i(x_i) \prod_{a \in N(i)} m_{a \to i}(x_i)$$

"beliefs" "messages"

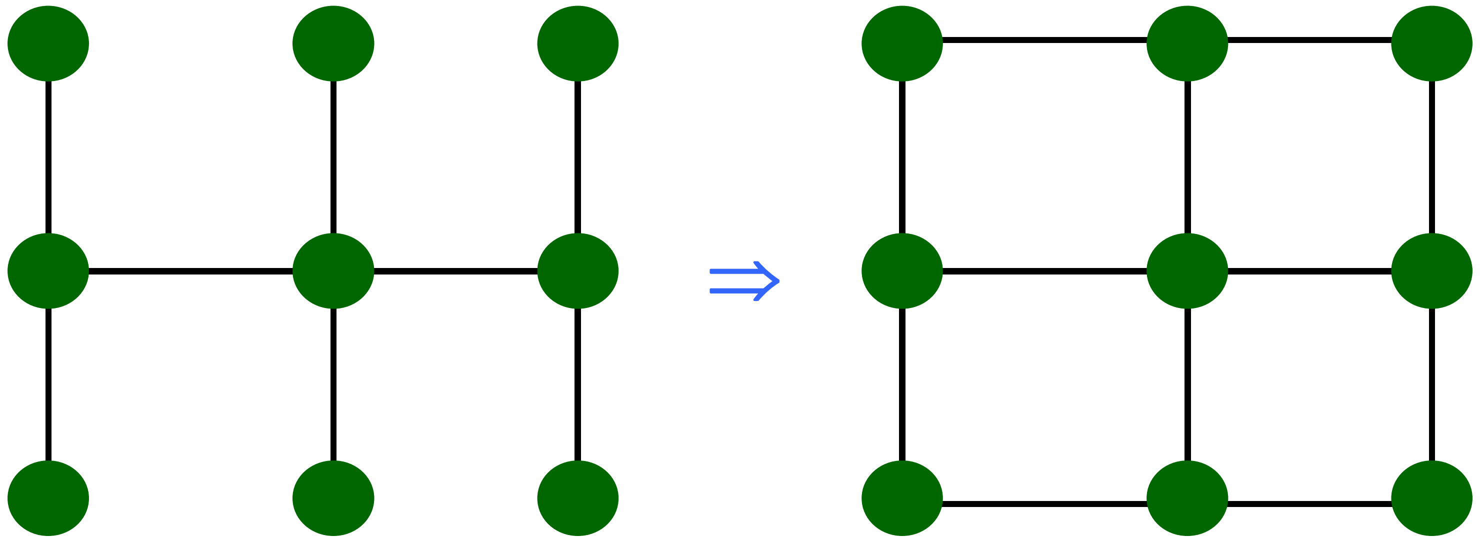


$$m_{i \to a}(x_i) = \prod_{c \in N(i) \backslash a} m_{c \to i}(x_i)$$

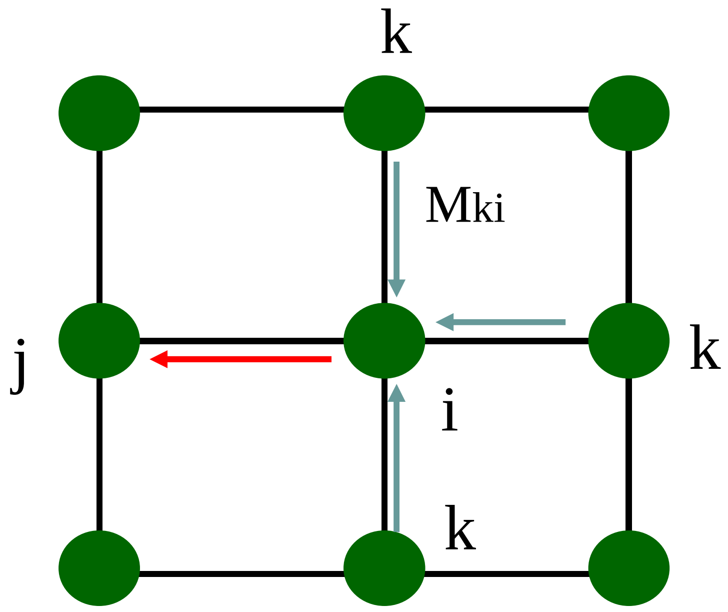
$$b_a(X_a) \propto f_a(X_a) \prod_{i \in N(a)} m_{i \to a}(x_i)$$

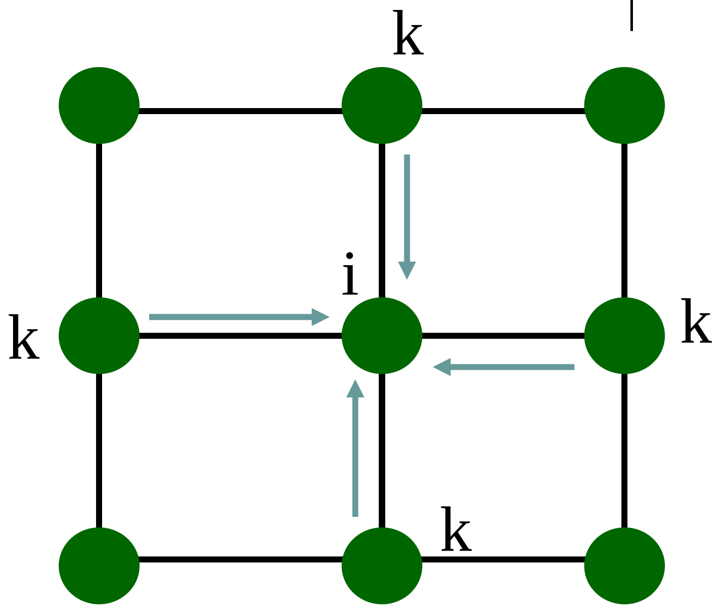
$$m_{a \to i}(x_i) = \sum_{X_a \backslash x_i} f_a(X_a) \prod_{j \in N(a) \backslash i} m_{j \to a}(x_j)$$

What if the graph is loopy?



Belief Propagation on loopy graphs





- BP Message-update Rules

$$M_{i\to j}(x_j) \propto \sum_{x_i} \psi_{ij}(x_i, x_j)\psi_i(x_i)\prod_k M_{k\to i}(x_i)$$

$$b_i(x_i) \propto \psi_i(x_i)\prod_k M_k(x_k)$$

external evidence

Compatibilities (interactions)

- May not converge or converge to a wrong solution

Loopy Belief Propagation

- A fixed point iteration procedure that tries to minimize F_{bethe}
- Start with random initialization of messages and beliefs

  - While not converged do

$$b_i(x_i) \propto \prod_{a \in N(i)} m_{a \to i}(x_i) \qquad b_a(X_a) \propto f_a(X_a) \prod_{i \in N(a)} m_{i \to a}(x_i)$$

$$m_{i \to a}^{new}(x_i) = \prod_{c \in N(i) \backslash a} m_{c \to i}(x_i) \qquad m_{a \to i}^{new}(x_i) = \sum_{X_a \backslash x_i} f_a(X_a) \prod_{j \in N(a) \backslash i} m_{j \to a}(x_j)$$

  - At convergence, stationarity properties are guaranteed
  - However, not guaranteed to converge!

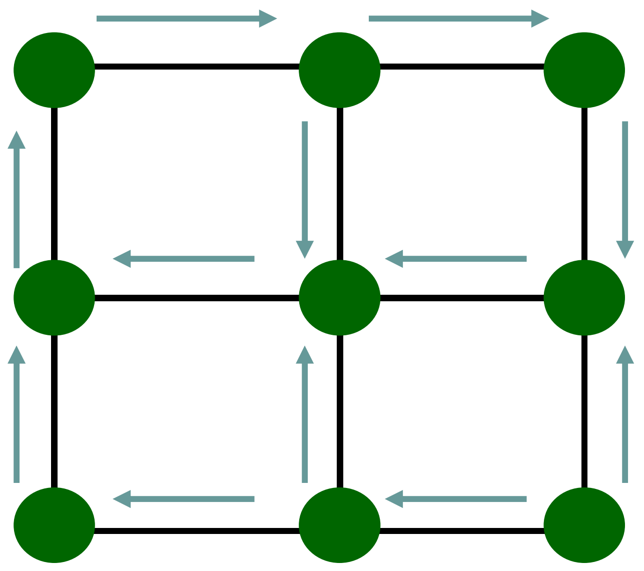
Loopy Belief Propagation

- If BP is used on graphs with loops, messages may circulate indefinitely
- But let's run it anyway and hope for the best … ☺
- Empirically, a good approximation is still achievable
  - Stop after fixed # of iterations
  - Stop when no significant change in beliefs
  - If solution is not oscillatory but converges, it usually is a good approximation

Loopy-belief Propagation for Approximate Inference: An Empirical Study
Kevin Murphy, Yair Weiss, and Michael Jordan.
UAI '99 (Uncertainty in AI). **]**

So what is going on?

- Is it a dirty hack that you bet your luck?



Approximate Inference

- Let us call the actual distribution P

$$P(X) = 1/Z \prod_{f_a \in F} f_a(X_a)$$

- We wish to find a distribution Q such that Q is a "good" approximation to P
- Recall the definition of KL-divergence

$$KL(Q_1 \| Q_2) = \sum_X Q_1(X) \log\left(\frac{Q_1(X)}{Q_2(X)}\right)$$

  - $KL(Q_1||Q_2)>=0$
  - $KL(Q_1||Q_2)=0$ iff $Q_1=Q_2$
  - We can therefore use KL as a scoring function to decide a good Q
  - But, $KL(Q_1||Q_2) \neq KL(Q_2||Q_1)$

Which KL?

- Computing KL(P||Q) requires inference!
- But KL(Q||P) can be computed without performing inference on P

$$KL(Q\|P) = \sum_X Q(X)\log(\frac{Q(X)}{P(X)})$$
$$= \sum_X Q(X)\log Q(X) - \sum_X Q(X)\log P(X)$$
$$= -H_Q(X) - E_Q \log P(X)$$

- Using $P(X) = 1/Z \prod_{f_a \in F} f_a(X_a)$

$$KL(Q\|P) = -H_Q(X) - E_Q \log(1/Z \prod_{f_a \in F} f_a(X_a))$$
$$= -H_Q(X) - \log 1/Z - \sum_{f_a \in F} E_Q \log f_a(X_a)$$

Optimization function

$$KL(Q\|P) = \underbrace{\boxed{-H_Q(X) - \sum_{f_a \in F} E_Q \log f_a(X_a)}}_{F(P,Q)} + \log Z$$

- We will call $F(P,Q)$ the "Free energy" *
- $F(P,P)$ =?

- F(P,Q) >= F(P,P)

*Gibbs Free Energy

The Energy Functional

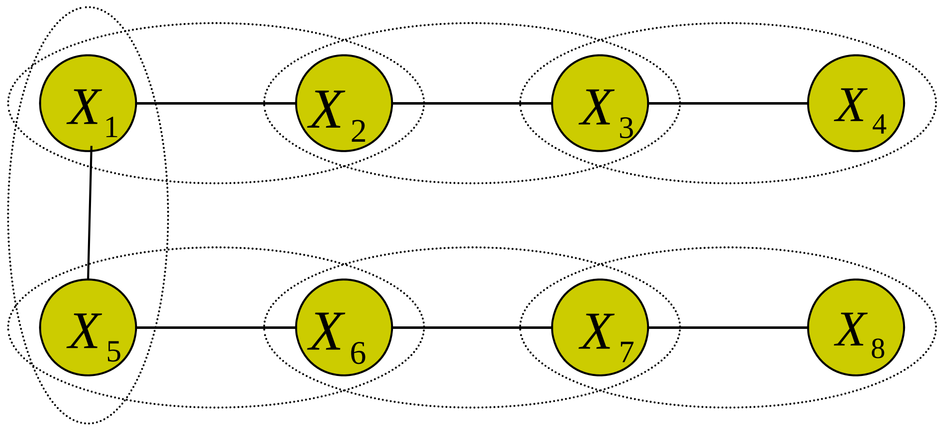
- Let us look at the functional

$$F(P,Q) = -H_Q(X) - \sum_{f_a \in F} E_Q \log f_a(X_a)$$

- $\sum_{f_a \in F} E_Q \log f_a(X_a)$ can be computed if we have marginals over each f_a
- $H_Q = -\sum_X Q(X) \log Q(X)$ is harder! Requires summation over all possible values
- Computing F, is therefore hard in general.
- Approach 1: Approximate $F(P,Q)$ with easy to compute $\hat{F}(P,Q)$

Tree Energy Functionals

- Consider a tree-structured distribution



- The probability can be written as: $b(\mathbf{x}) = \prod_a b_a(\mathbf{x}_a) \prod_i b_i(x_i)^{1-d_i}$
- $$H_{tree} = -\sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln b_a(\mathbf{x}_a) + \sum_i (d_i - 1) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i)$$
- $$F_{Tree} = \sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln \frac{b_a(\mathbf{x}_a)}{f_a(\mathbf{x}_a)} + \sum_i (1 - d_i) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i)$$
  $$= F_{12} + F_{23} + .. + F_{67} + F_{78} - F_1 - F_5 - F_2 - F_6 - F_3 - F_7$$
  - involves summation over edges and vertices and is therefore easy to compute

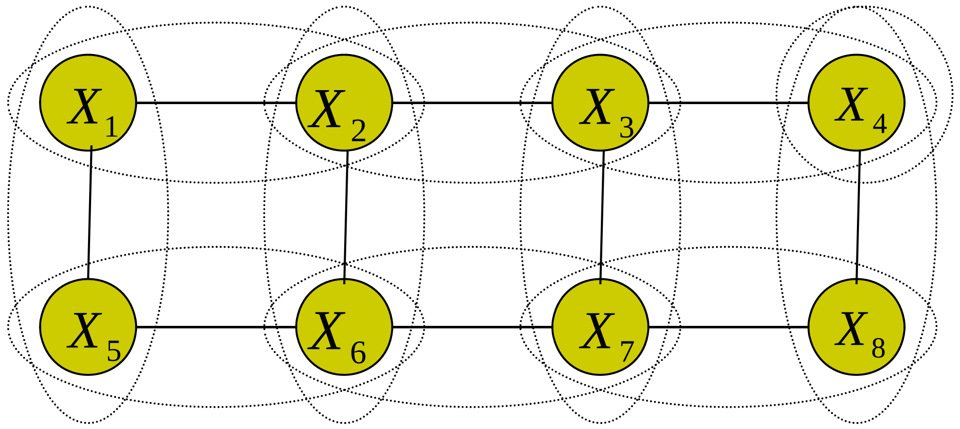
Bethe Approximation to Gibbs Free Energy

- For a general graph, choose $\hat{F}(P,Q) = F_{Betha}$

$$H_{Bethe} = -\sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln b_a(\mathbf{x}_a) + \sum_i (d_i - 1) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i)$$

$$F_{Bethe} = \sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln \frac{b_a(\mathbf{x}_a)}{f_a(\mathbf{x}_a)} + \sum_i (1 - d_i) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i) = -\langle f_a(\mathbf{x}_a) \rangle - H_{betha}$$

  - Called "Bethe approximation" after the physicist Hans Bethe



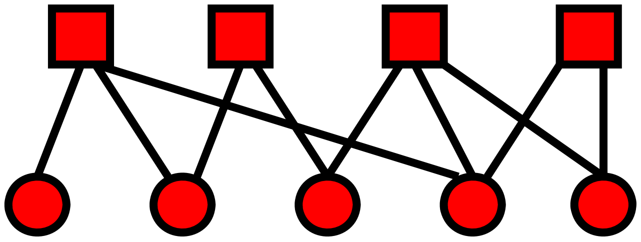
$$F_{bethe} = F_{12} + F_{23} + .. + F_{67} + F_{78} - F_1 - F_5 - 2F_2 - 2F_6 .. - F_8$$

  - Equal to the exact Gibbs free energy when the factor graph is a tree
  - In general, H_{Bethe} is **not** the same as the H of a tree

Bethe Approximation

- Pros:
  - Easy to compute, since entropy term involves sum over pairwise and single variables
- Cons:
  - $\hat{F}(P,Q) = F_{bethe}$ **may or may not** be well connected to $F(P,Q)$
  - It could, in general, be greater, equal or less than $F(P,Q)$
- Optimize each $b(\mathbf{x}_a)$'s.
  - For discrete belief, constrained opt. with *Lagrangian* multiplier
  - For continuous belief, not yet a general formula
  - Not always converge

Bethe Free Energy for FG



$$F_{Betha} = \sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln \frac{b_a(\mathbf{x}_a)}{f_a(\mathbf{x}_a)} + \sum_i (1 - d_i) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i)$$

$$H_{Bethe} = -\sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln b_a(\mathbf{x}_a) + \sum_i (d_i - 1) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i)$$

$$F_{Bethe} = -\langle f_a(\mathbf{x}_a) \rangle - H_{betha}$$

Minimizing the Bethe Free Energy

- $$L = F_{Bethe} + \sum_i \gamma_i \{1 - \sum_{x_i} b_i(x_i)\}$$
  $$+ \sum_a \sum_{i \in N(a)} \sum_{x_i} \lambda_{ai}(x_i) \left\{ b_i(x_i) - \sum_{X_a \backslash x_i} b_a(X_a) \right\}$$

- Set derivative to zero

Constrained Minimization of the Bethe Free Energy

$$L = F_{Bethe} + \sum_i \gamma_i \{\sum_{x_i} b_i(x_i) - 1\}$$

$$+ \sum_a \sum_{i \in N(a)} \sum_{x_i} \lambda_{ai}(x_i) \left\{ \sum_{X_a \setminus x_i} b_a(X_a) - b_i(x_i) \right\}$$

$$\frac{\partial L}{\partial b_i(x_i)} = 0 \quad \Longrightarrow \quad b_i(x_i) \propto \exp\left(\frac{1}{d_i - 1} \sum_{a \in N(i)} \lambda_{ai}(x_i) \right)$$

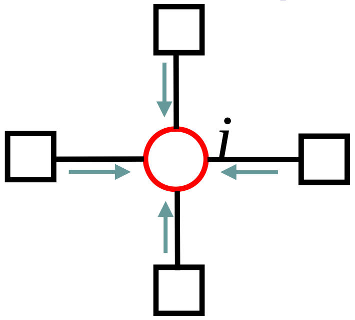
$$\frac{\partial L}{\partial b_a(X_a)} = 0 \quad \Longrightarrow \quad b_a(X_a) \propto \exp\left(-E_a(X_a) + \sum_{i \in N(a)} \lambda_{ai}(x_i) \right)$$

Bethe = BP on FG

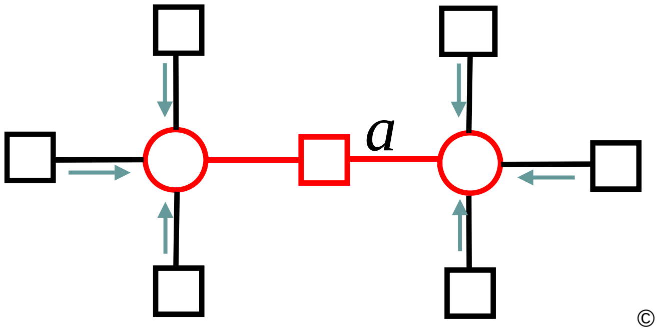
- We had:

$$b_i(x_i) \propto \exp\left(\frac{1}{d_i - 1}\sum_{a\in N(i)} \lambda_{ai}(x_i)\right) \qquad b_a(X_a) \propto \exp\left(-\log f_a(X_a) + \sum_{i\in N(a)} \lambda_{ai}(x_i)\right)$$

- Identify $\lambda_{ai}(x_i) = \log(m_{i\to a}(x_i)) = \log \prod_{b\in N(i)\neq a} m_{b\to i}(x_i)$
- to obtain BP equations:



$$\underset{\text{“beliefs”}}{b_i(x_i)} \propto f_i(x_i) \prod_{a\in N(i)} \underset{\text{“messages”}}{m_{a\to i}(x_i)}$$



$$b_a(X_a) \propto f_a(X_a) \prod_{i\in N(a)} \prod_{c\in N(i)\backslash a} m_{c\to i}(x_i)$$

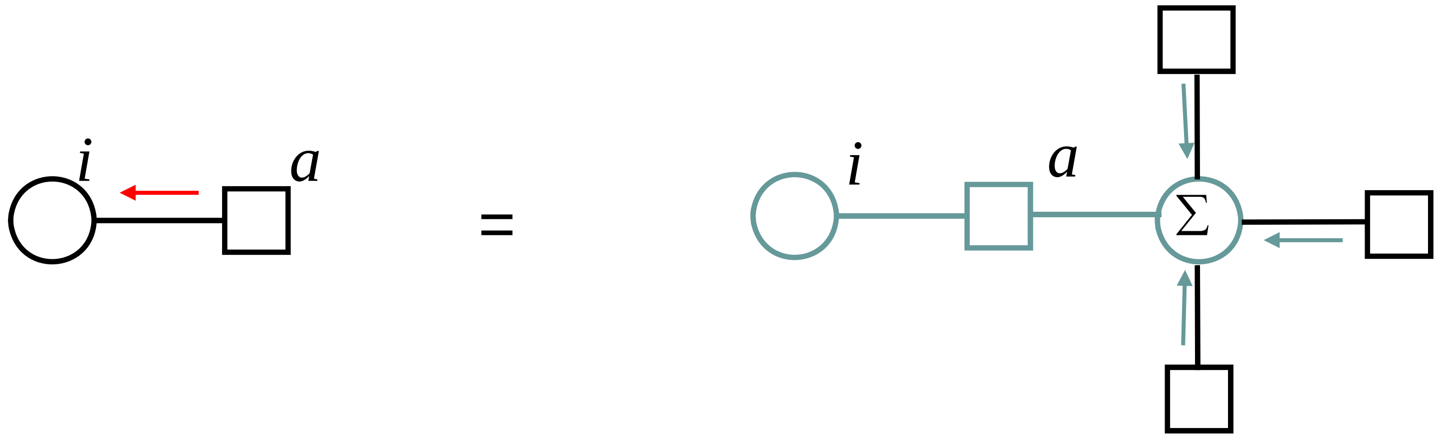
The “belief” is the BP approximation of the marginal probability.

BP Message-update Rules

Using $b_{a\to i}(x_i) = \sum_{X_a \backslash x_i} b_a(X_a)$, we get

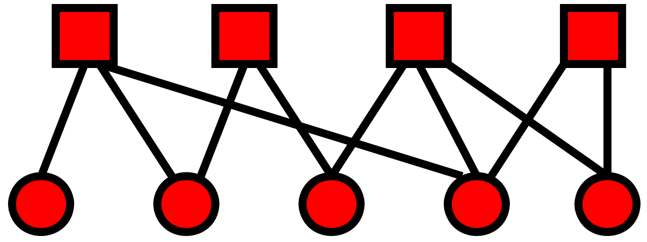
$$\boxed{m_{a\to i}(x_i) = \sum_{X_a \backslash x_i} f_a(X_a) \prod_{j\in N(a)\backslash i} \prod_{b\in N(j)\backslash a} m_{b\to j}(x_j)}$$

(A sum product algorithm)



Summary so far

- $P(X) = 1/Z \prod_{f_a \in F} f_a(X_a)$

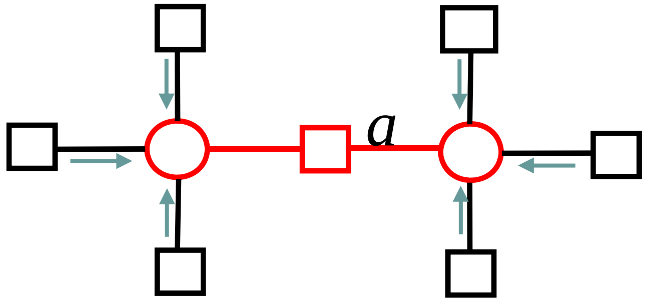


$$F(P,Q) = -H_Q(X) - \sum_{f_a \in F} E_Q \log f_a(X_a)$$

$$\hat{F}(P,Q) = \sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \log \frac{f_a(\mathbf{x}_a)}{b_a(\mathbf{x}_a)} + \sum_i (1-d_i) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \log b_i(\mathbf{x}_i)$$

$$b_a(X_a) \propto \exp\left(-\log f_a(X_a) + \sum_{i \in N(a)} \lambda_{ai}(x_i) \right)$$

$$b_i(x_i) \propto \exp\left(\frac{1}{d_i - 1} \sum_{a \in N(i)} \lambda_{ai}(x_i) \right)$$



The Theory Behind LBP

- For a distribution $p(\mathrm{X}|\theta)$ associated with a complex graph, computing the marginal (or conditional) probability of arbitrary random variable(s) is intractable

- Variational methods
  - formulating probabilistic inference as an optimization problem:

$$q^* = \arg\min_{q \in S} \left\{ F_{Betha}(p,q) \right\}$$

$$F_{Bethe} = \sum_a \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln \frac{b_a(\mathbf{x}_a)}{f_a(\mathbf{x}_a)} + \sum_i (1-d_i) \sum_{\mathbf{x}_i} b_i(\mathbf{x}_i) \ln b_i(\mathbf{x}_i) = -\langle f_a(\mathbf{x}_a) \rangle - H_{bethe}$$

q : a (tractable) probability distribution

The Theory Behind LBP

- But we do not optimize $q(\mathbf{X})$ explicitly, focus on the set of beliefs
  - $e.g., \quad b = \{ b_{i,j} = \tau(x_i, x_j), \;\; b_i = \tau(x_i) \}$
- Relax the optimization problem
  - approximate objective: $H_q \approx F(b)$
  - relaxed feasible set: $\mathcal{M} \to \mathcal{M}_o \quad (\mathcal{M}_o \supseteq \mathcal{M})$

$$b^* = \arg\min_{b \in \mathcal{M}_o} \left\{ \langle E \rangle_b + F(b) \right\}$$

- The loopy BP algorithm:
  - a fixed point iteration procedure that tries to solve *b**

The Theory Behind LBP

- But we do not optimize *q*(**X**) explicitly, focus on the set of beliefs
  - $e.g., \quad b = \{b_{i,j} = \tau(x_i, x_j), \;\; b_i = \tau(x_i)\}$
- Relax the optimization problem
  - approximate objective: $H_{Betha} = H(b_{i,j}, b_i)$
  - relaxed feasible set: $\mathcal{M}_o = \left\{ \tau \geq 0 \mid \sum_{x_i} \tau(x_i) = 1, \sum_{x_i} \tau(x_i, x_j) = \tau(x_j) \right\}$

$$b^* = \arg\min_{b \in \mathcal{M}_o} \left\{ \langle E \rangle_b + F(b) \right\}$$

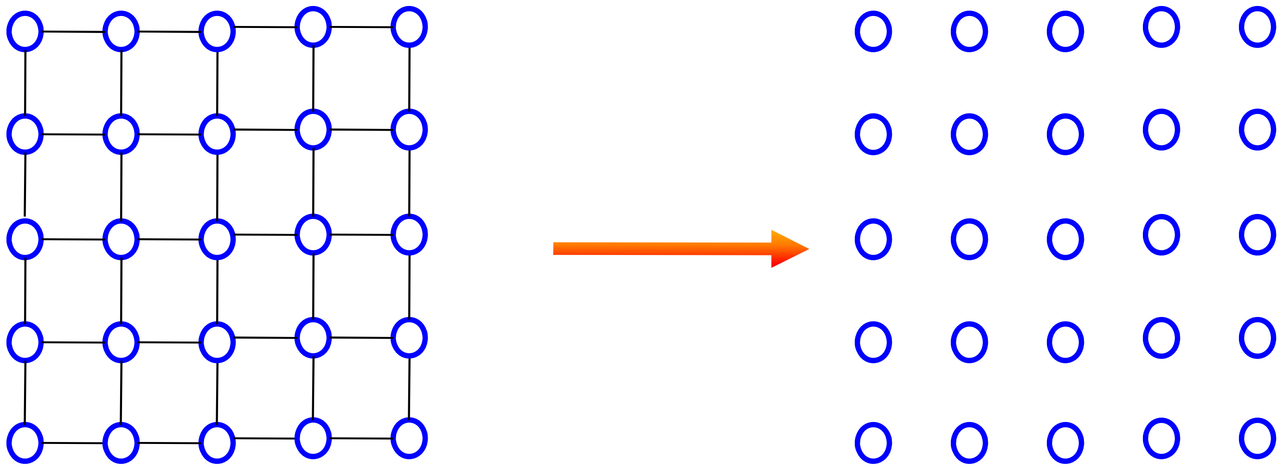
- The loopy BP algorithm:
  - a fixed point iteration procedure that tries to solve *b**

Mean Field Approximation

Naïve Mean Field

- Fully factorized variational distribution

$$q(x) = \prod_{s \in V} q(x_s)$$



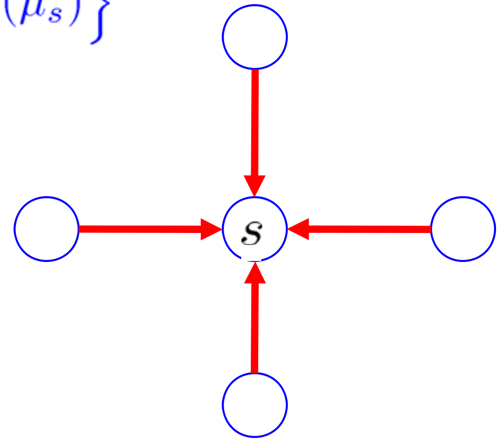
Naïve Mean Field for Ising Model

- Optimization Problem

$$\max_{\mu \in [0,1]^m} \left\{ \sum_{s \in V} \theta_s \mu_s + \sum_{(s,t) \in E} \theta_{st} \mu_s \mu_t + \sum_{s \in V} H_s(\mu_s) \right\}$$

- Update Rule

$$\mu_s \leftarrow \sigma\left(\theta_s + \sum_{t \in N(s)} \theta_{st} \mu_t\right)$$



- $\mu_t = p(X_t = 1) = \mathbb{E}_p[X_t]$ resembles "message" sent from node t to s
- $\{\mathbb{E}_p[X_t], t \in N(s)\}$ forms the "mean field" applied to s from its neighborhood

Mean field methods

- Optimize $q(\mathbf{X}_H)$ in the space of tractable families
  - $i.e.$, subgraph of G_p over which exact computation of H_q is feasible

- Tightening the optimization space
  - exact objective: H_q
  - tightened feasible set: $\mathcal{Q} \rightarrow \mathcal{T} \quad (\mathcal{T} \subseteq \mathcal{Q})$

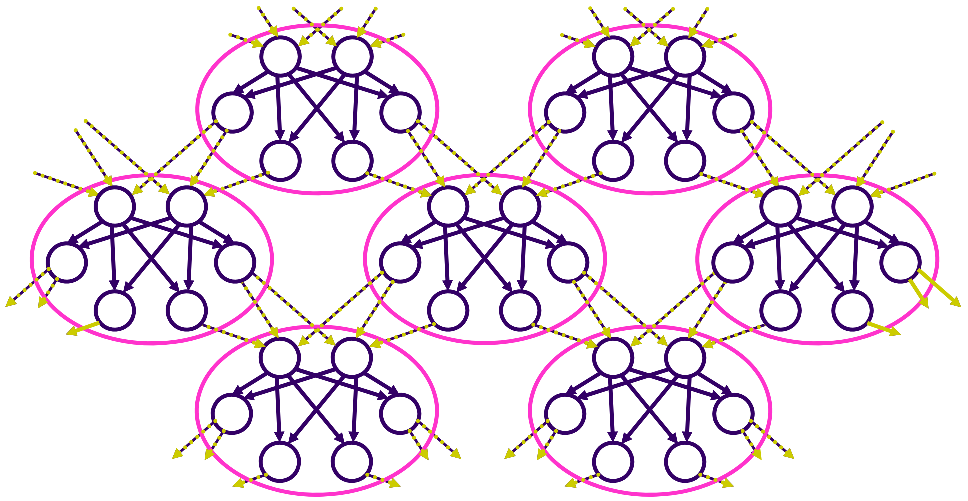
$$q^* = \arg\min_{q \in \mathcal{T}} \langle E \rangle_q - H_q$$

Cluster-based approx. to the Gibbs free energy

(Wiegerinck 2001, Xing *et al* 03,04)

Exact: $G[p(X)]$ *(intractable)*

Clusters: $G[\{q_c(X_c)\}]$



Mean field approx. to Gibbs free energy

- Given a disjoint clustering, {C$_1$, … , C$_I$}, of all variables
- Let
$$q(\mathbf{X}) = \prod_i q_i(\mathbf{X}_{C_i}),$$
- Mean-field free energy

$$G_{\mathrm{MF}} = \sum_i \sum_{\mathbf{x}_{C_i}} \prod_i q_i(\mathbf{x}_{C_i}) E(\mathbf{x}_{C_i}) + \sum_i \sum_{\mathbf{x}_{C_i}} q_i(\mathbf{x}_{C_i}) \ln q_i(\mathbf{x}_{C_i})$$

e.g., $G_{\mathrm{MF}} = \sum_{i<j} \sum_{x_i x_j} q(x_i) q(x_j) \phi(x_i x_j) + \sum_i \sum_{x_i} q(x_i) \phi(x_i) + \sum_i \sum_{x_i} q(x_i) \ln q(x_i)$ (naïve mean field)

  - Will **never** equal to the exact Gibbs free energy no matter what clustering is used, but it does **always** define a lower bound of the likelihood

- Optimize each $q_i(x_c)$'s.
  - Variational calculus …
  - Do inference in each $q_i(x_c)$ using any tractable algorithm

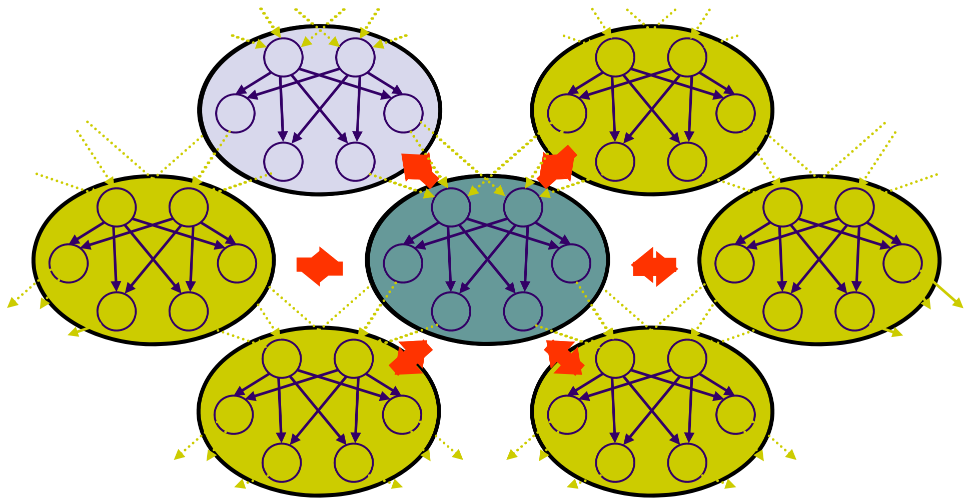
The Generalized Mean Field theorem

Theorem: The optimum GMF approximation to the cluster marginal is isomorphic to the cluster posterior of the original distribution given internal evidence and its generalized mean fields:

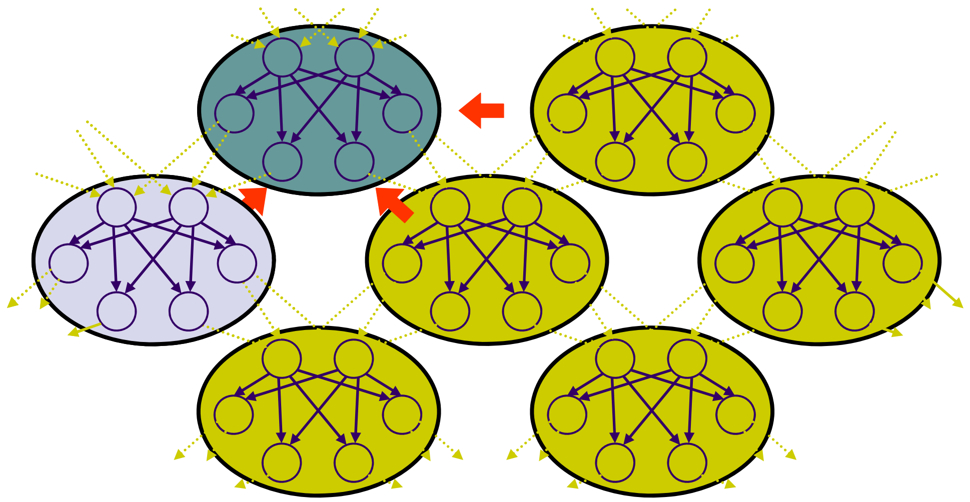
$$q_i^*(\mathbf{X}_{H,C_i}) = p(\mathbf{X}_{H,C_i} \mid \mathbf{x}_{E,C_i}, \langle \mathbf{X}_{H,MB_i} \rangle_{q_{j \neq i}})$$

GMF algorithm: Iterate over each q_i

A generalized mean field algorithm [xing *et al*. UAI 2003]



A generalized mean field algorithm [xing *et al*. UAI 2003]



Convergence theorem

Theorem: The GMF algorithm is guaranteed to converge to a local optimum, and provides a lower bound for the likelihood of evidence (or partition function) the model.

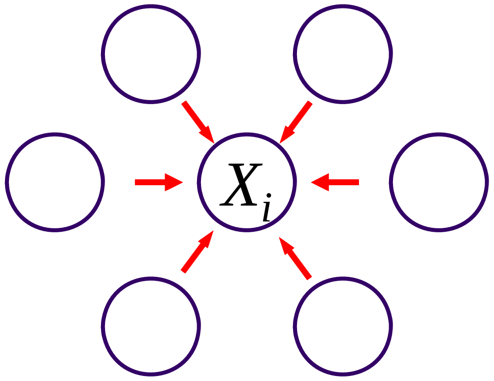
The naive mean field approximation

- Approximate $p(\mathbf{X})$ by fully factorized $q(\mathbf{X})=\prod_i q_i(X_i)$
- For Boltzmann distribution $p(X)=\exp\{\sum_{i<j} q_{ij}X_iX_j+q_{io}X_i\}/Z$:

mean field equation:

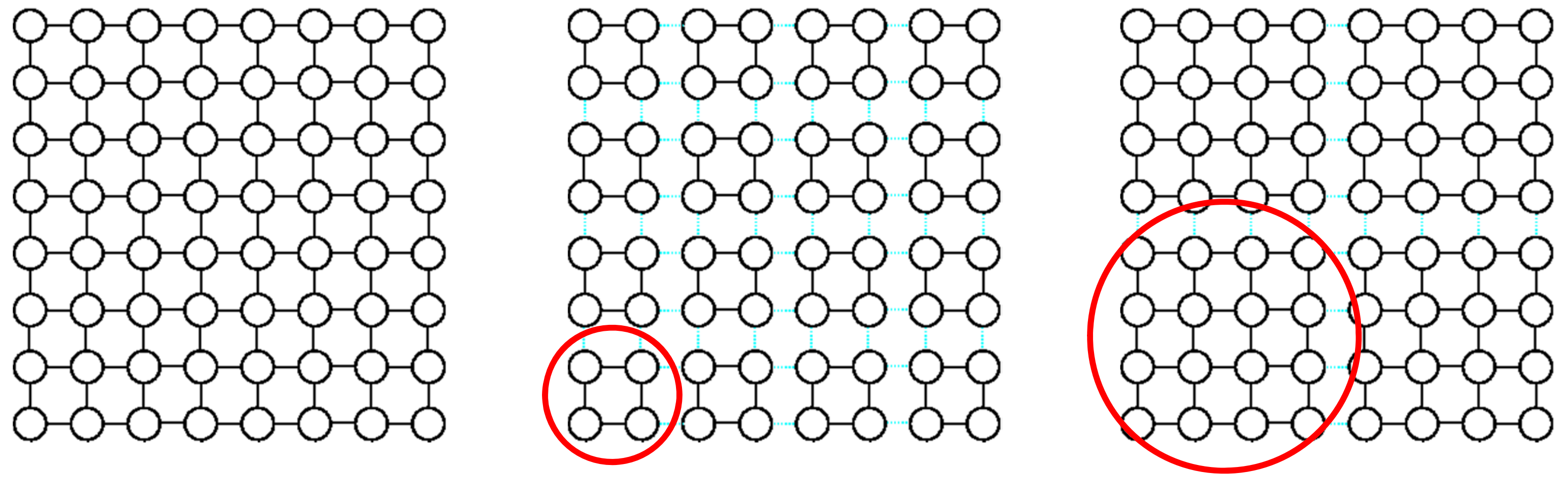
$$q_i(X_i) = \exp\left\{\theta_{i0}X_i + \sum_{j\in\mathcal{N}_i}\theta_{ij}X_i\langle X_j\rangle_{q_j} + A_i\right\}$$

$$= p(X_i \mid \{\langle X_j\rangle_{q_j} : j\in\mathcal{N}_i\})$$



- $\langle X_j\rangle_{q_j}$ resembles a "message" sent from node *j* to *i*
- $\{\langle X_j\rangle_{q_j} : j\in\mathcal{N}_i\}$ forms the "mean field" applied to X_i from its neighborhood

Example 1: Generalized MF approximations to Ising models

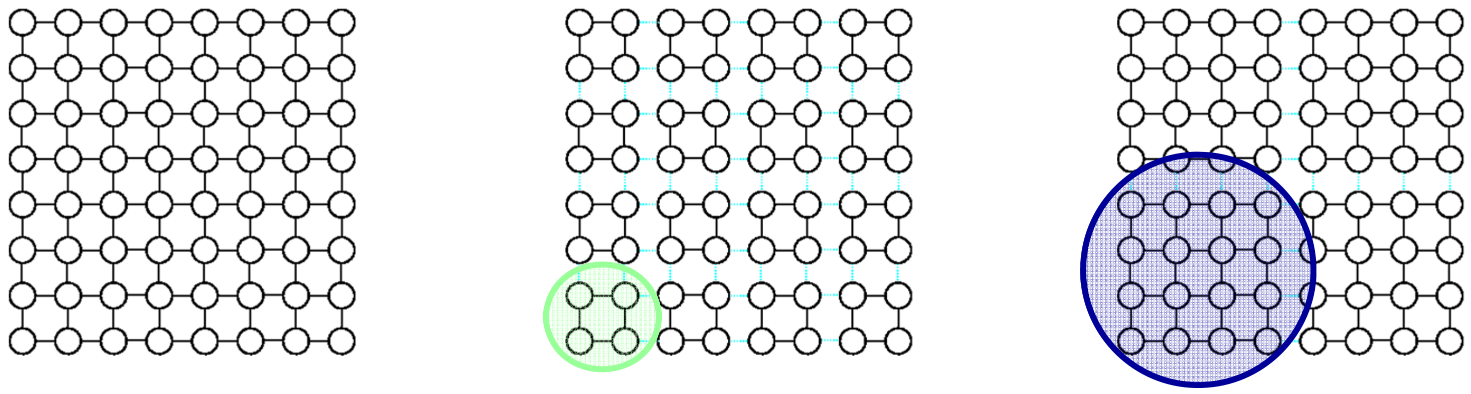


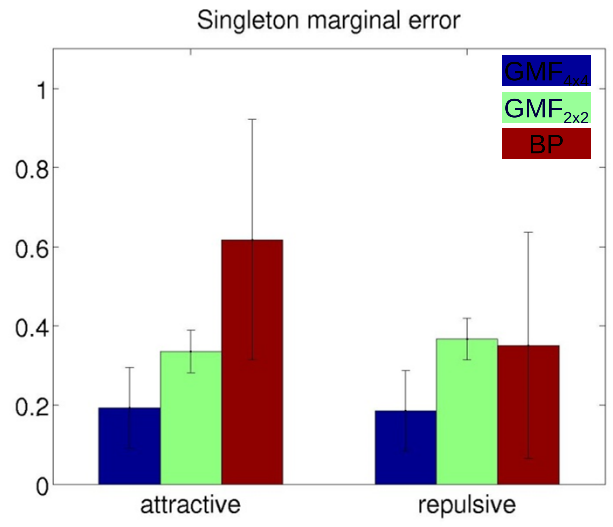
Cluster marginal of a square block C_k:

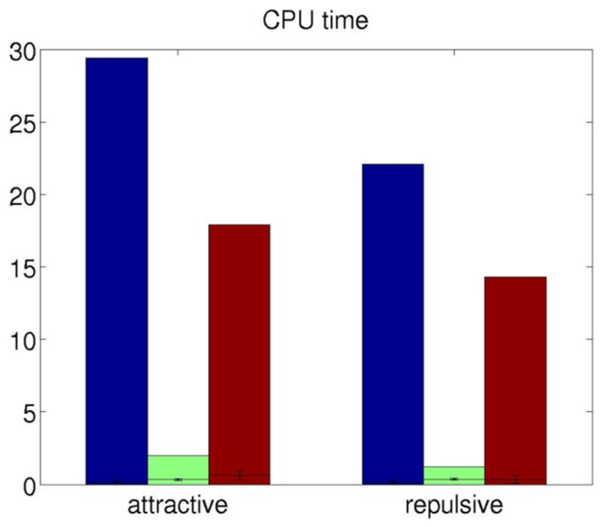
$$q(X_{C_k}) \propto \exp\left\{ \sum_{i,j\in C_k} \theta_{ij} X_i X_j + \sum_{i\in C_k} \theta_{i0} X_i + \sum_{\substack{i\in C_k,\, j\in MB_k, \\ k'\in MBC_k}} \theta_{ij} X_i \left\langle X_j \right\rangle_{q(X_{C_{k'}})} \right\}$$

Virtually a reparameterized Ising model of small size.

GMF approximation to Ising models

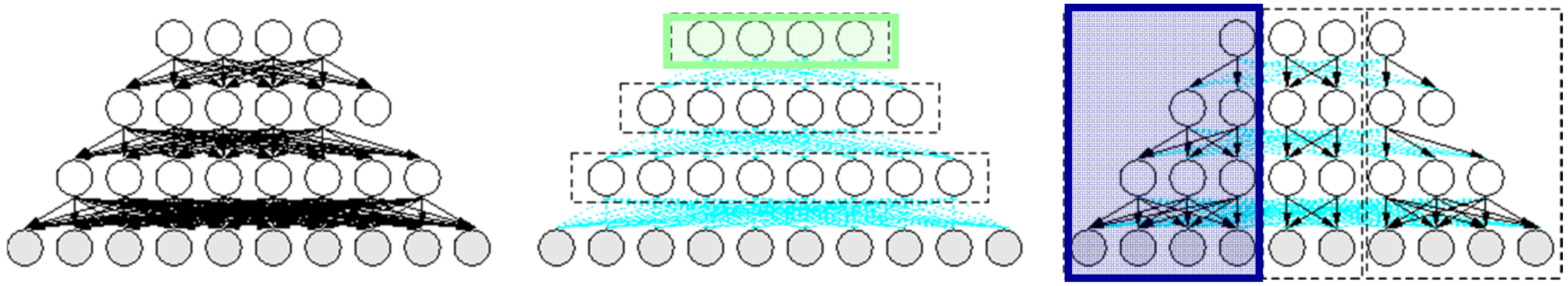


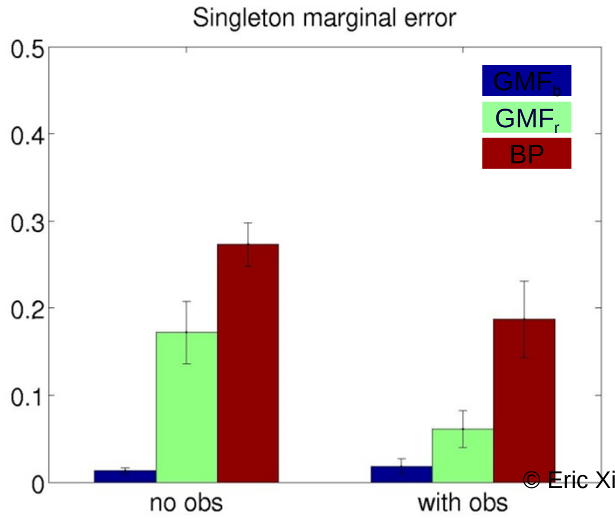


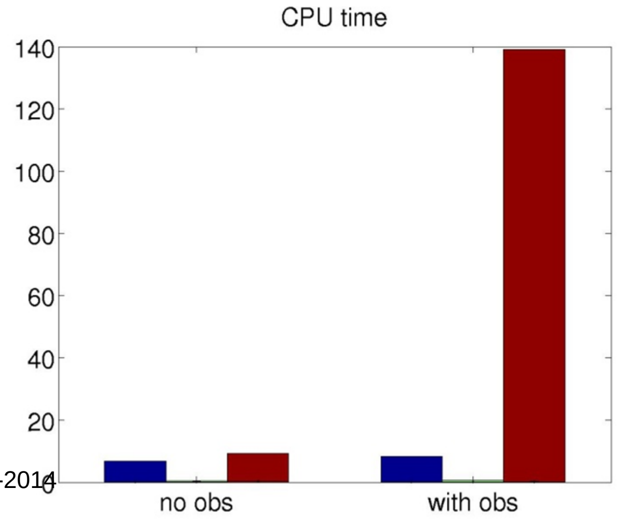


Attractive coupling: positively weighted
Repulsive coupling: negatively weighted

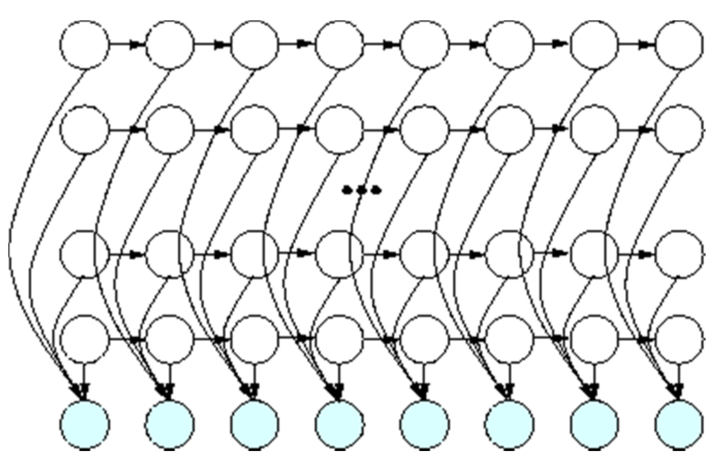
Example 2: Sigmoid belief network

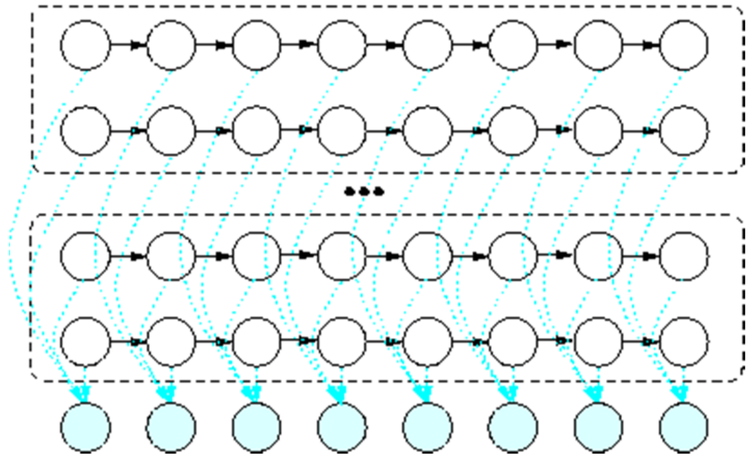


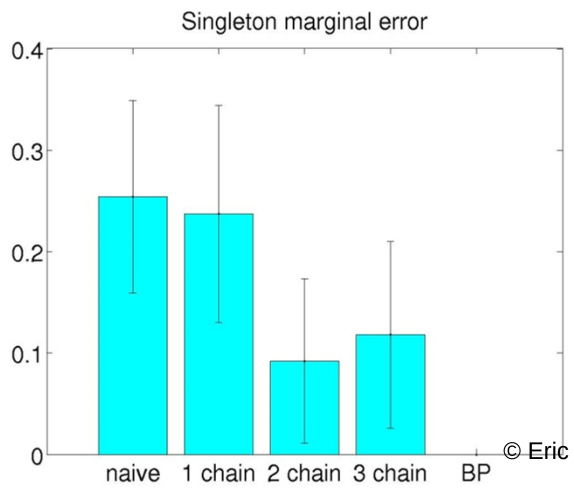


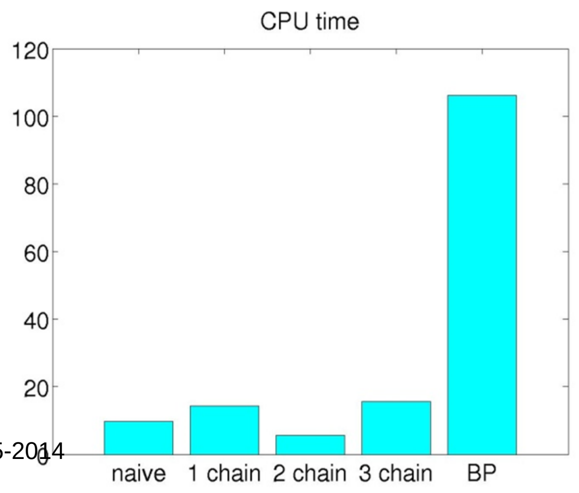


Example 3: Factorial HMM

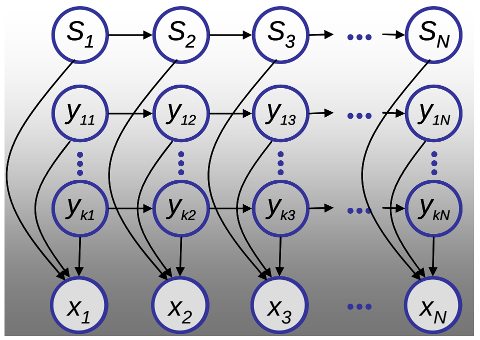




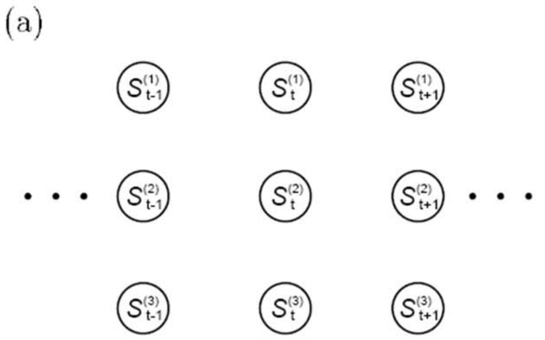


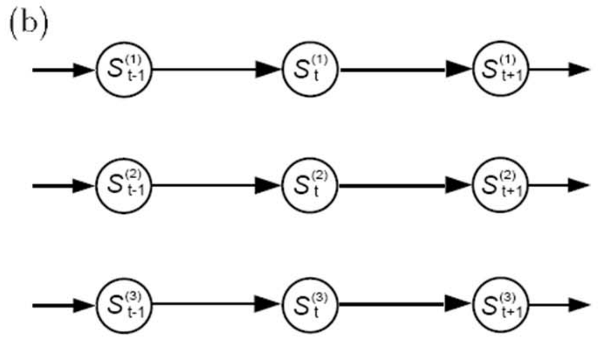


Automatic Variational Inference



$\Rightarrow$





fHMM **Mean field approx.** **Structured variational approx.**

- Currently for each new model we have to
  - derive the variational update equations
  - write application-specific code to find the solution
- Each can be time consuming and error prone
- Can we build a general-purpose inference engine which automates these procedures?

Cluster-based MF (e.g., GMF)

- a general, iterative message passing algorithm
- clustering completely defines approximation
  - preserves dependencies
  - flexible performance/cost trade-off
  - clustering automatable
- recovers model-specific structured VI algorithms, including:
  - fHMM, LDA
  - variational Bayesian learning algorithms
- easily provides new structured VI approximations to complex models