Problem Set 1 Solution

by Yuan Zhou

1. (a) $\text{Vol}(S) = (1/2)^{d/4}\text{Vol}(K)$.

   (b) Let V_{d-1} denote the volume of the unit ball under ℓ_4 norm in $(d-1)$ dimension. We first upper bound the volume outside the slab by

$$\begin{aligned}
&2\int_{c/d^{1/4}}^{1}(1-x^4)^{(d-1)/4}V_{d-1}dx\\
\leq&2\int_{c/d^{1/4}}^{+\infty}(1-x^4)^{(d-1)/4}V_{d-1}dx\\
\leq&2V_{d-1}\int_{c/d^{1/4}}^{+\infty}\exp(-x^4(d-1)/4)dx\\
\leq&2V_{d-1}\int_{c/d^{1/4}}^{+\infty}\frac{x^3}{(c/d^{1/4})^3}\exp(-x^4(d-1)/4)dx\\
=&2V_{d-1}\cdot\frac{d^{3/4}}{c^3}\cdot\frac{1}{d-1}\cdot\left(-\exp(-x^4(d-1)/4)\right)\Big|_{c/d^{1/4}}^{+\infty}\\
\leq&\frac{3V_{d-1}}{c^3(d-1)^{1/4}}\exp(-c^4/4). \qquad \text{(for large } d\text{'s)}
\end{aligned}$$

   Now we lower bound the volume of K.

$$\begin{aligned}
\text{Vol}(K) =&2\int_0^1(1-x^4)^{(d-1)/4}V_{d-1}dx\\
\geq&2V_{d-1}\int_0^{1/(d-1)^{1/4}}(1-x^4)^{(d-1)/4}dx\\
\geq&\frac{2V_{d-1}}{(d-1)^{1/4}}\left(1-\frac{1}{d-1}\right)^{(d-1)/4}dx\\
\geq&\frac{2V_{d-1}}{(d-1)^{1/4}}(1-\frac{1}{d-1}\cdot d-14) \qquad \text{(for large } d\text{'s)}\\
\geq&\frac{V_{d-1}}{(d-1)^{1/4}}.
\end{aligned}$$

   Therefore, the fraction of volume of K outside the slab $|x_1| \leq c/d^{1/4}$ is at most

$$\frac{\frac{3V_{d-1}}{c^3(d-1)^{1/4}}\exp(-c^4/4)}{\frac{V_{d-1}}{(d-1)^{1/4}}} = \frac{3}{c^3}\exp(-c^4/4).$$

2. (a) $\mathbf{E}[x_i] = 0$, therefore $\mathbf{E}[\boldsymbol{x}] = \mathbf{0}$.

   (b) By symmetry we have

$$\mathbf{E}[x_i^2] = \frac{1}{d}\mathbf{E}\left[\sum_{i=1}^{d} x_i^2\right] = 1/d.$$

   Therefore

$$\mathbf{Var}[x_i] = \mathbf{E}[x_i^2] - \mathbf{E}[x_i]^2 = 1/d.$$

(c)

$$\begin{aligned}
\mathbf{Var}[\boldsymbol{u}^T\boldsymbol{x}] &= \mathbf{E}\left[(\boldsymbol{u}^T\boldsymbol{x})^2\right] - \mathbf{E}\left[\boldsymbol{u}^T\boldsymbol{x}\right]^2 \\
&= \mathbf{E}\left[(\boldsymbol{u}^T\boldsymbol{x})^2\right] \qquad\qquad (\text{since } \mathbf{E}[x_i] = 0 \text{ for all } i) \\
&= \sum_{i,j} \mathbf{E}\left[u_i u_j x_i x_j\right] \\
&= \frac{1}{d}\sum_i u_i^2 \qquad\qquad (\text{since } \mathbf{E}[x_i x_j] = 0 \text{ when } i \neq j \text{ and } \mathbf{E}[x_i^2] = 1/d) \\
&= \frac{1}{d}.
\end{aligned}$$

The standard deviation of $\boldsymbol{u}^T\boldsymbol{x}$ is $\sqrt{\mathbf{Var}[\boldsymbol{u}^T\boldsymbol{x}]} = 1/\sqrt{d}$.

(d) The ratio between the volume of intersection and the volume of each unit ball equals 2 times the fraction of the hemisphere above the plane $x_1 = a/2$ (of a unit ball centered at origin), which, by Lemma 1.2 in the book, is at most

$$2 \cdot \frac{2}{\sqrt{d-1} \cdot a/2} \exp\left(-\frac{(a/2)^2(d-1)}{2}\right) = \frac{8}{a\sqrt{d-1}} \exp\left(-\frac{a^2(d-1)}{8}\right).$$

(e) For $a = c/\sqrt{d-1}$ (think of $c \gg 1$ and note that we assume that the radius $r = 1$), the fraction of the intersection is at most $\frac{8}{c}\exp\left(-\frac{c^2}{8}\right)$, which is exponentially small in c.

3. (a) $\mathbf{E}[\langle \boldsymbol{u}, \boldsymbol{r}\rangle] = \sum_i \mathbf{E}[u_i r_i] = 0$.

(b) By rotational symmetry, we can assume w.l.o.g. that $\boldsymbol{u} = (1, 0, 0, \cdots, 0)$. Therefore $\mathbf{E}[|\langle \boldsymbol{u}, \boldsymbol{r}\rangle|] = \mathbf{E}[|r_1|] = \sqrt{2/\pi}$.

(c)

$$\begin{aligned}
&\mathbf{E}[\langle \boldsymbol{u}, \boldsymbol{r}\rangle \cdot \langle \boldsymbol{v}, \boldsymbol{r}\rangle] \\
=& \mathbf{E}\left[\left(\sum_i u_i r_i\right)\left(\sum_j v_j r_j\right)\right] \\
=& \sum_{i,j} u_i v_j \, \mathbf{E}[r_i r_j] \\
=& \sum_i u_i v_i \qquad\qquad (\text{since } \mathbf{E}[r_i r_j] = 0 \text{ for } i \neq j \text{ and } \mathbf{E}[r_i^2] = 1) \\
=& \langle \boldsymbol{u}, \boldsymbol{v}\rangle.
\end{aligned}$$

(d) By rotational symmetry, we can assume w.l.o.g. that the vectors $\boldsymbol{u}, \boldsymbol{v}$ have non-zero entries only at their first 2 coordinates, i.e. $\boldsymbol{u} = (u_1, u_2, 0, 0, \cdots, 0)$ and $\boldsymbol{v} = (v_1, v_2, 0, 0, \cdots, 0)$. Note that $\langle \boldsymbol{u}, \boldsymbol{r}\rangle = u_1 r_1 + u_2 r_2$ and $\langle \boldsymbol{v}, \boldsymbol{r}\rangle = v_1 r_1 + v_2 r_2$ only involve the first two gaussian coordinates r_1, r_2 – the whole problem is reduce to the 2-dimensional problem.

In the 2-dimensional problem, the event $\text{sign}(\langle \boldsymbol{u}, \boldsymbol{r}\rangle) \neq \text{sign}(\langle \boldsymbol{v}, \boldsymbol{r}\rangle)$ is equivalent to the event that $\boldsymbol{u}$ and $\boldsymbol{v}$ are on different sides of the line going through the origin, and with $\boldsymbol{r}$ as its norm vector. Since $\boldsymbol{r}$ is uniformly random in its direction, we know that

$$\Pr[\text{sign}(\langle \boldsymbol{u}, \boldsymbol{r}\rangle) \neq \text{sign}(\langle \boldsymbol{v}, \boldsymbol{r}\rangle)] = \frac{\arccos\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\pi}.$$

Therefore,

$$\Pr[\mathrm{sign}(\langle \boldsymbol{u}, \boldsymbol{r}\rangle) = \mathrm{sign}(\langle \boldsymbol{v}, \boldsymbol{r}\rangle)] = 1 - \frac{\arccos\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\pi}.$$

4. Since $k < d$, we know that A has a non-trivial null space. Then there are two vectors $\boldsymbol{x} \neq \boldsymbol{y}$ such that $A\boldsymbol{x} = A\boldsymbol{y}$. Now we have $\|\boldsymbol{x} - \boldsymbol{y}\| \neq 0$ and $\|A\boldsymbol{x} - A\boldsymbol{y}\| = 0$, which implies unbounded distortion.

5. (a) The total volume of the ϵ-neighborhoods of the points in COVER should be at least $1(= 1^d)$. Therefore,

$$|\mathsf{COVER}| \geq \frac{1}{\text{volume of a } d\text{-dimensional ball with radius } \epsilon} \geq \frac{(d/2)!}{(\pi\epsilon^2)^{d/2}}.$$

   (b) Consider adding all points whose coordinates are multiples of $\epsilon/\sqrt{d}$, i.e. let

$$\mathsf{COVER} = \{(i_1\epsilon/\sqrt{d}, i_2\epsilon/\sqrt{d}, \cdots, i_d\epsilon/\sqrt{d} | i_1, i_2, \cdots, i_d \in \{0, 1, 2, \cdots, \lfloor\sqrt{d}/\epsilon\rfloor\}\}.$$

   One can verify that COVER is a valid (d, ϵ)-covering set, and

$$|\mathsf{COVER}| = \left(\frac{\sqrt{d}}{\epsilon} + 1\right)^d \leq \left(\frac{2d}{\epsilon^2}\right)^{d/2}.$$

6. (a) Fix $x \neq y$. Let X_i be the indicator variable for the event $C(x)_i = C(y)_i$. We see that $\{X_i\}$ is a set of mutually independent variables, $\mathbf{E}[X_i] = 1/2$ and $X_i \in [0, 1]$. By Hoeffding's inequality, we have

$$\Pr[\mathrm{avg}(X_i) - 1/2] \leq \exp\left(-2\left(\frac{2}{5}\right)^2 n\right) = \exp(-3.2m) < 2^{-3m}.$$

   (b) By a union bound, we have

$$\begin{aligned}
&\Pr[\forall x \neq y, C(x) \text{ and } C(y) \text{ differ on more than } n/10 \text{ positions}] \\
\geq & 1 - \sum_{x \neq y} \Pr[C(x) \text{ and } C(y) \text{ differ on no more than } n/10 \text{ positions}] \\
\geq & 1 - 2^{2m} \cdot 2^{-3m} = 1 - 2^{-m}.
\end{aligned}$$

7. (a) Since there are $|V|$ vectors, the optimal solution must reside in a $|V|$-dimensional space.

   (b) Let $k = \frac{10^8}{\epsilon^2}\log(1/\epsilon)$, and let $\{x_u^*\}_{u \in V}$ be the optimal solution (in a $|V|$-dimensional space) of value Θ. Now we project $\{x_u^*\}_{u \in V}$ to a random k-dimensional subspace, denote the projection vectors by $\{\tilde{x}_u\}_{u \in V}$. Finally, let $x_u = \tilde{x}_u / \|\tilde{x}_u\|$ for all $u \in V$.

   It is easy to see that $\{x_u\}_{u \in V}$ is a set of unit vectors. Now we are going to show that the value of (1) obtained by these vectors is at least $\Theta - \epsilon$. We are going to prove the following claim.

   Claim 1

$$\forall u, v \in V, \Pr\left[\|x_u - x_v\|^2 < \|x_u^* - x_v^*\|^2 - \frac{\epsilon}{2}\right] < \frac{\epsilon}{8}.$$

Once we have Claim 1, we finish the proof as follows. For all $u, v \in V$, we have

$$\begin{aligned}&\mathbf{E}\left[\|x_u^* - x_v^*\|^2 - \|x_u - x_v\|^2\right] \\ &\quad < \frac{\epsilon}{2} + \Pr\left[\|x_u - x_v\|^2 < \|x_u^* - x_v^*\|^2 - \frac{\epsilon}{2}\right] \cdot \|x_u^* - x_v^*\|^2 < \frac{\epsilon}{2} + \frac{\epsilon}{8} \cdot 4 = \epsilon.\end{aligned}$$

Finally, by linearity of expectation,

$$\mathbf{E}\left[\frac{1}{|E|} \sum_{(u,v)\in E} \|x_u - x_v\|^2\right] \geq \frac{1}{|E|} \sum_{(u,v)\in E} \|x_u^* - x_v^*\|^2 - \epsilon = \Theta - \epsilon.$$

Now we are going to prove Claim 1.

Proof of Claim 1. For notational convenience, let $\tilde{x}'_u = \frac{|V|}{d} \tilde{x}_u$ for all $u \in V$. By the Random Projection Theorem and our choice of k (and assuming $\epsilon \leq 0.9$), each of the following 3 events happen with probability at most $\epsilon/24$.

- $|\|\tilde{x}'_u\|^2 - 1| > \frac{\epsilon}{100}$.
- $|\|\tilde{x}'_v\|^2 - 1| > \frac{\epsilon}{100}$.
- $|\|\tilde{x}'_u - \tilde{x}'_v\|^2 - \|x_u^* - x_v^*\|^2| > \frac{\epsilon}{100} \cdot \|x_u^* - x_v^*\|^2$.

By a union bound, with probability at least $1 - \epsilon/8$, none of the 3 events above happens. In this case, we are going to show that we have $|\|x_u^* - x_v^*\|^2 - \|x_u - x_v\|^2| \leq \frac{\epsilon}{2}$. By triangle inequality,

$$|\|x_u^* - x_v^*\|^2 - \|x_u - x_v\|^2| \leq |\|x_u^* - x_v^*\|^2 - \|\tilde{x}'_u - \tilde{x}'_v\|^2| + |\|\tilde{x}'_u - \tilde{x}'_v\|^2 - \|x_u - x_v\|^2|.$$

For the first term, we upper bound it by $\frac{\epsilon}{100} \cdot \|x_u^* - x_v^*\|^2 \leq \frac{\epsilon}{100} \cdot 4 = \frac{\epsilon}{25}$.

For the second term, we use

$$\begin{aligned}
&|\|\tilde{x}'_u - \tilde{x}'_v\|^2 - \|x_u - x_v\|^2| \\
=&|\langle (\tilde{x}'_u - \tilde{x}'_v) - (x_u - x_v), (\tilde{x}'_u - \tilde{x}'_v) + (x_u - x_v)\rangle| \\
=&|\langle (\tilde{x}'_u - x_u) - (\tilde{x}'_v - x_v), (\tilde{x}'_u - \tilde{x}'_v) + (x_u - x_v)\rangle| \\
\leq&\|(\tilde{x}'_u - x_u) - (\tilde{x}'_v - x_v)\|\|(\tilde{x}'_u - \tilde{x}'_v) + (x_u - x_v)\| && \text{(Cauchy-Schwartz)} \\
\leq&4\|(\tilde{x}'_u - x_u) - (\tilde{x}'_v - x_v)\| \\
\leq&4\left(\|(\tilde{x}'_u - x_u)\| + \|(\tilde{x}'_v - x_v)\|\right). && \text{(triangle inequality)}
\end{aligned}$$

Since $x_u = \tilde{x}'_u / \|\tilde{x}'_u\|$, we have

$$\|(\tilde{x}'_u - x_u)\| = |\|\tilde{x}'_u\| - 1| = \frac{|\|\tilde{x}'_u\|^2 - 1|}{\|\tilde{x}'_u\| + 1} \leq \frac{\epsilon}{100}.$$

Similarly, we can show that $\|(\tilde{x}'_u - x_u)\| \leq \frac{\epsilon}{100}$. In all, we upper bound $|\|\tilde{x}'_u - \tilde{x}'_v\|^2 - \|x_u - x_v\|^2|$ by $2 \cdot \frac{\epsilon}{100} = \frac{\epsilon}{50}$.

Therefore, we have

$$|\|x_u^* - x_v^*\|^2 - \|x_u - x_v\|^2| \leq \frac{\epsilon}{25} + \frac{\epsilon}{50} < \frac{\epsilon}{2}.$$