

PROBLEM SET 5 SOLUTION
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1. (a) There can be at most $|S|/n$ rows (or columns) with all points in S . So there are at most $|S|/n$ choices of each coordinate, giving at most

$$\left(\frac{|S|}{n}\right)^2 = \left(\frac{|S|}{n^2}\right) \cdot |S| \leq \frac{|S|}{2}$$

such points.

- (b) Let S be a set of at most $n^d/2$ vertices. Observe that for each coordinate, S can completely cover at most $|S|/n^{d-1}$ “coordinate rows”. Therefore, the number of vertices in S that has a vertex in \bar{S} that differs at exactly 1 coordinate is at least

$$|S| - \left(\frac{|S|}{n^{d-1}}\right)^d \geq \frac{|S|}{2}. \quad (\text{since } |S| \leq n^d/2)$$

Therefore, the number of vertices in S that has a neighbor in \bar{S} is at least $|S|/(2n)$ – that is, the value above divided by n . Since each edge has a contribution $1/(dn^d)$ to the flow between S and \bar{S} , we lower bound the escape probability by

$$\Phi(S) \geq \frac{(|S|/2n) \cdot (1/(dn^d))}{|S|/n^d} = \Omega(1/dn).$$

2. Let L' be the Laplacian L with an edge conductance increased (that is, resistance decreased). We have $L' - L \succeq 0$. Therefore we have $L \preceq L'$. Let L^+, L'^+ be the pseudo-inverse of L, L' respectively. We have $L^+ \succeq L'^+$. Therefore

$$(\mathbf{e}_u - \mathbf{e}_v)^T L^+ (\mathbf{e}_u - \mathbf{e}_v)^T \geq (\mathbf{e}_u - \mathbf{e}_v)^T L'^+ (\mathbf{e}_u - \mathbf{e}_v)^T,$$

which means that, the effective resistance between any pair of vertices u, v can never increase.

3. Fix a satisfying $x_1^*, x_2^*, \dots, x_n^*$. Let X be the number of variables of x_1, x_2, \dots, x_n that are different from their counterparts in $x_1^*, x_2^*, \dots, x_n^*$. Note that X is a random variable from $\{0, 1, 2, \dots, n\}$. Once $X = 0$, we reached the $x_1^*, x_2^*, \dots, x_n^*$ solution and the algorithm terminates successfully.

At each step of the algorithm in (b), since the chosen clause is not satisfied, at least one of the two variables in the clause is different from x^* . Therefore, with probability at least $1/2$, a variable that is different from x^* is chosen and flipped, which means that X decreases by 1. On the other hand, in the rest case (which happens with probability at most $1/2$), X increases by 1.

Now consider another random variable X' which starts with the same value as X . Whenever X increases by X , X' also increases by 1. But at each step, X' increases by 1 with probability *exactly* $1/2$, and with the other $1/2$ probability, it decreases by 1. It is clear by definition that X' is always no less than X . Therefore,

$$\mathbf{E}[\text{\#steps taken for } X \text{ to reach } 0] \leq \mathbf{E}[\text{\#steps taken for } X' \text{ to reach } 0].$$

On the other hand, X' can be viewed as a random walk on the integers (with equal probabilities to go left or right). We have

$$\mathbf{E}[\text{\#steps taken for } X' \text{ to reach } 0] \leq n^2.$$

In all, we have

$$\mathbf{E}[\text{\#steps taken for } X \text{ to reach } 0] \leq n^2,$$

and therefore

$$\Pr[\text{the algorithm finds a satisfying solution when there is one}] \geq 9/10,$$

by a Markov bound.

4. (a) The stationary distribution puts $1/\binom{n}{k}$ probability mass on each set.
- (b) Given two states X, X' , suppose w.l.o.g. that $|X \cap X'| = \ell$ and $X = \{1, 2, \dots, \ell, \ell + 1, \dots, k\}$, $X' = \{1, 2, \dots, \ell, k + 1, \dots, 2k - \ell\}$. For any $(i, j) \in X \times \bar{X}$, we define a bijection from (i, j) to $(i', j') \in X' \times \bar{X}'$ in the following way.
- i. If $i \in \{1, \dots, \ell\}, j \in \{2k - \ell + 1, \dots, n\}$, then let $i' = i, j' = j$.
 - ii. If $i \in \{\ell + 1, \dots, k\}, j \in \{2k - \ell + 1, \dots, n\}$, then let $i' = i + (k - \ell), j' = j$.
 - iii. If $i \in \{1, \dots, \ell\}, j \in \{k + 1, \dots, 2k - \ell\}$, then let $i' = i, j' = j - (k - \ell)$.
 - iv. If $i \in \{\ell + 1, \dots, k\}, j \in \{k + 1, \dots, 2k - \ell\}$, then let $i' = j, j' = i$.

We define the movement of X' as follows. If X stays the same, then X' stays the same; if X moves to $Y = X \setminus \{i\} \cup \{j\}$, then X' moves to $Y' = X' \setminus \{i'\} \cup \{j'\}$.

Note that conditioned on a movement, in Cases ii., and iii., we have $|Y \cap Y'| = \ell + 1$, while in the other two cases, we have $|Y \cap Y'| = \ell$.

Therefore, with probability

$$\frac{1}{2} \cdot \frac{(k - \ell)(n - 2k + \ell) + \ell(k - \ell)}{k(n - k)} = \frac{(k - \ell)(n - 2(k - \ell))}{2k(n - k)} \quad (*),$$

the cardinality of the intersection of the two sets X', Y' increases by 1 – while with the remaining probability, it stays the same.

When $k \leq n/4$, we have that

$$(*) \geq \frac{k - \ell}{4k}.$$

Using the analysis of the coupon collector problem, we know that if we run $ck \log(k/\epsilon)$ steps of the random process (for some large enough c), the two states would have intersection k – i.e. become the same state. Therefore, the ϵ -mixing time is $O(k \log(k/\epsilon))$.

When $n/4 < k \leq n/2$, we split the random process into to parts –

- When $\ell \leq k/2$, the probability that $|X \cap X'|$ increases is at least $\frac{n - 2(k - \ell)}{4n}$.
- When $\ell > k/2$, the probability that $|X \cap X'|$ increases is at least $\frac{k - \ell}{2k}$.

Both parts can be analyzed as a coupon collector problem. For some large enough c , we know that it needs at most $cn \log(n/\epsilon) = O(k \log(k/\epsilon))$ steps for $\ell = |X \cap X'|$ to go greater than $k/2$, and at most $ck \log(k/\epsilon)$ steps for $\ell = |X \cap X'|$ to reach k (once it gets greater than $k/2$). Therefore, the ϵ -mixing time in this case is also $O(k \log(k/\epsilon))$.

5. (a) The stationary distribution puts $1/2^n$ probability mass on each vertex.
 (b) We use the coupling method to show that the $1/3$ -mixing time of this Markov chain is at most $O(n \log n/p)$.

Given two states $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$, we define the following random walk for (x, y) .

- Pick $i \in \{1, 2, \dots, n\}$ uniformly at random.
- With probability $1 - 2p$, do nothing. For the remaining probability $2p$, if $x_i = y_i$, flip x_i, y_i (simultaneously) with probability p ; if $x_i \neq y_i$, flip x_i with probability p , flip y_i with the remaining probability p .

Using the analysis of the coupon collector problem, we know that x and y become the same state with probability at least $2/3$ after $c \cdot n \log n/p$ steps, for some large enough c . Therefore, the $1/3$ -mixing time of the Markov chain is $O(n \log n/p)$.

6. (a) It suffices to show that for any S such that $\pi(S) \leq 1/2$, we have

$$\frac{\text{flow}(S, \bar{S})}{\pi(S)} \geq \frac{1}{2C}.$$

Since each path from S to \bar{S} goes through an edge from (S, \bar{S}) , we have

$$\begin{aligned} & 2C \cdot \text{flow}(S, \bar{S}) \\ &= 2C \sum_{u \in S, v \in \bar{S}} \pi(U)P(u, v) \\ &\geq 2 \sum_{u \in S, v \in \bar{S}} C_{(u,v)} \pi(u)P(u, v) \\ &= 2 \sum_{u \in S, v \in \bar{S}} \sum_{(x,y): \mathcal{P}_{x \rightarrow y} \ni (u,v)} \pi(x)\pi(y) \\ &\geq 2 \sum_{x \in S, y \in \bar{S}} \pi(x)\pi(y) \\ &= 2\pi(S)\pi(\bar{S}) \\ &\geq \pi(S). \quad (\text{since } \pi(\bar{S}) \geq \frac{1}{2}) \end{aligned}$$

This finishes the proof.

- (b) For each x, y , we use the most natural path – starting from x , change the first coordinate that is different from y , then the second, the third, and so on. Now consider an arbitrary edge $(b_1, \dots, b_{i-1}, 0, b_{i+1}, \dots, b_n) - (b_1, \dots, b_{i-1}, 1, b_{i+1}, \dots, b_n)$, the edge is taken by the path from x to y only when the first $i - 1$ coordinates of y is b_1, \dots, b_{i-1} and the last $n - i$ coordinates of x is b_{i+1}, \dots, b_n . Therefore, there are at most $2^{i-1} \cdot 2^{n-i} = 2^{n-1}$ such x, y pairs. Therefore, for each edge e , using $P(u, v) = 1/(2n)$ and $\pi(u) = 1/2^n$, we have

$$C_e = \frac{1}{(1/2^n) \cdot (1/(2n))} \cdot 2^{n-1} \cdot \left(\frac{1}{2^n}\right)^2 = O(n).$$