15-496/859X: Computer Science Theory for the Information Age — Spring 2012
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PROBLEM SET 5
Due by 6pm, Friday, April 13

INSTRUCTIONS

- You should think about *each* problem by yourself for *at least 30 minutes* before choosing to collaborate with others.
- You are allowed to collaborate with fellow students taking the class in solving the problems (in groups of at most 3 people for each problem). In fact this is encouraged so that you interact with and learn from each other. However, *you must write up your solutions on your own.* If you collaborate in solving problems, you should clearly acknowledge your collaborators for each problem.
- Reference to any external material besides the course text and material covered in lecture is not allowed. In particular, you are not allowed to search for answers or hints on the web. You are encouraged to contact the instructors or the TA for a possible hint if you feel stuck on a problem and require some assistance.
- Solutions typeset in LaTeX are preferred.
- Feel free to email the instructors or the TA if you have any questions or would like any clarifications about the problems.
- You are urged to start work on the problem set early.
- Each problem is worth 10 points unless indicated otherwise.

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1. (a) Prove the following claim which we used in lower bounding the minimum escape probability of the 2-dimensional lattice. Suppose S is a subset of at most $n^2/2$ points in the $n \times n$ lattice. Show that
$$|\{(i,j) \in S \mid \text{ row } i \text{ and column } j \subseteq S\}| \leq |S|/2\ . \tag{1}$$
   (b) Show that for the $n \times n \times \cdots \times n$ grid in d dimensions, the minimum escape probability is $\Omega(1/(dn))$.
   Hint: Prove a generalization of (1) above for the d-dimensional case.

2. Prove that reducing the value of a resistor in a network cannot increase the effective resistance. Prove that increasing the value of a resistor in a network cannot decrease the effective resistance.

3. The 2SAT problem is defined as follows. An instance of 2SAT consists of n variables $x_1, x_2, \ldots, x_n$ which are supposed to take Boolean 0/1 values, and a set of constraints $C_1, C_2, \ldots, C_m$ where each constraint is of the form $\ell_i \vee \ell_j$ for some $1 \leq i < j \leq n$ where the literal ℓ_i is either x_i or $\overline{x_i}$. The goal is to determine if the instance is *satisfiable*, i.e., there is a 0-1 truth assignment to the x_i's which satisfies all the constraints of the instance simultaneously. You all have probably seen a polynomial time algorithm for 2SAT. In this problem, we will consider the following randomized algorithm for 2SAT.

   (a) Start with an arbitrary 0-1 assignment to the variables x_i.
   (b) Repeat for $10n^2$ steps:

i. If all constraints are satisfied under the current assignment, output the current assignment and halt.

ii. Otherwise, pick an arbitrary unsatisfied constraint, pick one of its two variables at random, and flip that variable's value.

(c) If still here, output that the input 2SAT instance is unsatisfiable.

Note that the algorithm is always correct on unsatisfiable instances.

Your goal in this exercise is to prove the following claim: When run on satisfiable 2SAT instances, the above randomized algorithm outputs a satisfying assignment with probability at least $1/2$.

4. Consider the following Markov chain whose state space Ω is the set of all k-element subsets of $\{1, 2, 3, \ldots, n\}$ for some k, $2 \le k \le n/2$ (so that $|\Omega| = \binom{n}{k}$):

   If the current state is $X \subseteq \{1, 2, \ldots, n\}$ with $|X| = k$:

   - With prob. $1/2$ stay at X, and with prob. $1/2$, pick $i \in X$ uniformly at random and $j \in \{1, 2, \ldots, n\} \setminus X$ uniformly at random and move to $Y := X \setminus \{i\} \cup \{j\}$.

   (a) What is the stationary distribution of the above Markov chain?

   (b) Use a coupling argument to show that the ϵ-mixing time of the Markov chain is $O(k \log(k/\epsilon))$.

5. Let $n \ge 1$ be an integer and p, $0 < p \le 1/2$, be a real. Consider the following random walk on the hypercube $\{0, 1\}^n$. From each node $x = (x_1, x_2, \ldots, x_n)$:

   - pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
   - With prob. p, flip x_i and with prob. $1 - p$ leave x_i unchanged.

   (a) What is the stationary distribution of the above Markov chain?

   (b) Prove that the Markov chain mixes (to within distance $1/3$ of its stationary distribution) in $O(n^2/p^2)$ steps.

   (c) **(Extra credit)** Can you establish a tight (up to absolute constant factors) bound on the mixing time?

6. In this exercise we will see a "routing based" approach to lower bounding the minimum escape probability. Let M be a connected Markov chain with stationary distribution π. Suppose we pick directed paths $\mathcal{P}_{x \to y}$ from x to y for every ordered pair (x, y) of states of a Markov chain M. Define the congestion of an edge $e = (u, v)$ with $u \ne v$ of the Markov chain (which has transition probability $P(u, v) > 0$) to be

$$C_e = \frac{1}{\pi(u)P(u,v)} \sum_{(x,y): \mathcal{P}_{x \to y} \ni e} \pi(x)\pi(y) ,$$

   and let C be the maximum value of C_e over all edges $e = (u, v)$, $u \ne v$, of the chain (which have $P(u, v) > 0$).

   - Prove that if Φ is the minimum escape probability of the Markov chain M, then

$$\Phi \ge \frac{1}{2C} .$$

   - **(Extra Credit)** For the Markov chain corresponding to the lazy random walk on the hypercube $\{0, 1\}^n$ (where in each step, we stay put with probability $1/2$ and flip a random bit with probability $1/2$), give a collection of paths $\mathcal{P}_{x \to y}$ with maximum congestion $O(n)$.