ME 24-731
Conduction and Radiation Heat Transfer

Application of Duhamel's Superposition Principle to 1-D Unsteady Conduction in a Slab

The problem being solved is:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

with

$$T(x,0) = 0; T(0,t) = T_w(t); T(L,t) = 0.0$$

For constant T_w, the solution is:

$$T(x,t) = T_w\left(1 - \frac{x}{L} - \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{\sin(\lambda_n x)}{\lambda_n}\right)$$

Thus f(x,t) is given by

$$f(x,t) = \left(1 - \frac{x}{L} - \frac{2}{L}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\frac{\sin(\lambda_n x)}{\lambda_n}\right)$$

and therefore

$$\frac{\partial f}{\partial t}(t-\tau) = \frac{2\alpha}{L}\sum_{n=1}^{\infty}\lambda_n e^{-\alpha\lambda_n^2(t-\tau)}\sin(\lambda_n x)$$

so that

$$T(x,t) = \frac{2\alpha}{L}\sum_{n=1}^{\infty}\lambda_n \sin(\lambda_n x)\int_0^t T_w(\tau)e^{-\alpha\lambda_n^2(t-\tau)}d\tau$$

It is not clear by looking at this solution that the boundary condition at x=0 is satisfied. One way to make this explicitly visible is to use integration by parts. Following Arpaci, we write

$$T(x,t) = \frac{2\alpha}{L}\sum_{n=1}^{\infty}\lambda_n \sin(\lambda_n x) I(t) \tag{1}$$

where

$$\begin{aligned}
I(t) &= \int_0^t T_w(\tau)e^{\left(-\alpha\lambda_n^2(t-\tau)\right)}d\tau \\
&= \frac{e^{\left(-\alpha\lambda_n^2 t\right)}}{\alpha\lambda_n^2}\int_0^t T_w(\tau)d(e^{\alpha\lambda_n^2\tau}) \\
&= \frac{e^{\left(-\alpha\lambda_n^2 t\right)}}{\alpha\lambda_n^2}\left(T_w(\tau)e^{\alpha\lambda_n^2\tau}|_0^t - \int_0^t e^{\alpha\lambda_n^2\tau}d(T_w)\right) \\
&= \frac{1}{\alpha\lambda_n^2}\left(T_w(t) - T_w(0)e^{-\alpha\lambda_n^2 t} - \int_0^t e^{-\alpha\lambda_n^2(t-\tau)}d(T_w)\right)
\end{aligned}$$

Substituting I(t) into Equation 1 we get

$$T(x,t) = \frac{2}{L} T_w(t) \sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)}{\lambda_n} - \frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)}{\lambda_n} \left(T_w(0) e^{-\alpha \lambda_n^2 t} + \int_0^t e^{-\alpha \lambda_n^2 (t-\tau)} d(T_w) \right)$$

However

$$\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)}{\lambda_n} = 1 - \frac{x}{L}$$

so that the complete solution is:

$$T(x,t) = T_w(t)(1 - \frac{x}{L}) - \frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)}{\lambda_n} \left(T_w(0) e^{-\alpha \lambda_n^2 t} + \int_0^t e^{-\alpha \lambda_n^2 (t-\tau)} d(T_w) \right)$$

From this it is clear that at x=0, $T = T_w(t)$ at all times.