Intro to Econometric Theory
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Solution #2

1. You have collected a dataset on consumers' purchases of wine and cars. In particular, for each of 100 consumers, you know the price of their most recent bottle of wine and the price of their most recent car.

$$\begin{aligned}
\overline{P_{\text{wine}}} &= 6\text{in\$} \\
\hat{\sigma}^2_{\text{wine}} &= 10 \\
\\
\overline{P_{\text{car}}} &= 18\text{in\$} \\
\hat{\sigma}^2_{\text{car}} &= 10 \\
\\
\widehat{\text{Cov}}(P_w, P_c) &= 5
\end{aligned}$$

Now, suppose we wish to model the relationship between these two prices as:

$$P_c = \beta_0 + \beta_1 P_w + \epsilon \tag{1}$$

What are the OLS estimates for β_0 and β_1?

The OLS estimators are:

$$\begin{aligned}
\hat{\beta}_{1,OLS} &= \frac{\widehat{Cov}(P_c, P_w)}{\hat{V}(P_w)} \\
&= \frac{5}{10} \\
&= 0.5
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_{0,OLS} &= \overline{P_c} - \hat{\beta}_{1,OLS}\overline{P_w} \\
&= 18 - 0.5(6) \\
&= 15
\end{aligned}$$

2. Should the clever people at General Motors start distributing expensive wine?

   No. Although the setup of the CLRM essentially assumes a causation from the RHS to the LHS variable, this causation seems quite implausible here. *If* it were to be the case that drinking expensive wine causes people to buy expensive cars, then perhaps car-sellers would be wise to distribute expensive wine.

   However, it seems likely that, in fact, income and/or taste for luxury goods explains both a tendency to buy expensive wine and a tendency to buy expensive cars. So, since handing out wine is unlikely to change a persons' income or tastes much, we should expect little or no effect of handing out expensive wine on spending for cars.

3. Suppose instead we model:

$$P_w = \beta_2 + \beta_3 P_c + \epsilon \tag{2}$$

   What are the OLS estimates for β_0 and β_1?

   The OLS estimators are:

$$\begin{aligned}
\hat{\beta}_{3,OLS} &= \frac{\widehat{Cov}(P_c, P_w)}{\hat{V}(P_c)} \\
&= \frac{5}{10} \\
&= 0.5
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_{2,OLS} &= \overline{P_w} - \hat{\beta}_{1,OLS}\overline{P_c} \\
&= 6 - 0.5(18) \\
&= -3
\end{aligned}$$

4. One might expect that $\beta_{1,OLS} = \frac{1}{\beta_{3,OLS}}$. True? Why or why not?

   False, both are 0.5. We would expect $\hat{\beta}_{1,OLS} = \frac{1}{\hat{\beta}_{3,OLS}}$ if the two ways of running OLS produced the *same line*. But they should not, in general, produce the same line. The reason for this is that the two ways of running OLS embody different ideas of what a "mistake" is. In the first case, a mistake is the distance between a data point and the line in P_c. In the second case, a mistake is the distance between a data point and the line in P_w. So, since the two ways of running OLS minimize different mistake criteria, they should be expected to reach different answers.

5. Exercise 6 on page 63 of Greene.

It is pretty easy to show that the profit-maximizing level of output is:

$$q^* = \frac{1}{2}\alpha + \frac{1}{2}\beta 10 + \frac{1}{2}\epsilon$$

The expected value of this is going to be:

$$E(q^*) = \frac{1}{2}\alpha + \frac{1}{2}\beta 10$$

Or, $\alpha/2 + 5\beta$. We will use OLS to find estimates of α and β. Refer to the SAS program and output for the calculations. Using the output of the corr procedure, we can calculate the OLS coefficients as follows:

$$\begin{aligned}
\hat{\beta}_{OLS} &= \widehat{Corr}(P,Q)\frac{\sqrt{\hat{V}(Q)}}{\sqrt{\hat{V}(P)}} \\
&= -0.70\frac{5.357}{4.464} \\
&= -0.840
\end{aligned}$$

$$\begin{aligned}
\hat{\alpha}_{OLS} &= \overline{Q} - \hat{\beta}_{OLS}\overline{P} \\
&= 11.467 - (-0.84)(11.067) \\
&= 20.763
\end{aligned}$$

Then, we can calculate fitted values for Q and residuals e. This is also done in the SAS program and the results are in the SAS output. The sum of squared residuals is 204.65, so that $\hat{\sigma}^2_{OLS} = 204.65/(15-2) = 15.74$.

Using the formulae for variance in the bivariate CLRM:

$$\begin{aligned}
\hat{V}(\hat{\beta}) &= \frac{\hat{\sigma}^2}{\sum(P_i - \overline{P})^2} \\
&= \frac{\hat{\sigma}^2}{(N-1)\hat{V}(P)} \\
&= \frac{15.74}{14(4.464)^2} \\
&= 0.0564
\end{aligned}$$

$$\begin{aligned}
\hat{V}(\hat{\alpha}) &= \frac{\hat{\sigma}^2}{N}\frac{\sum P_i^2}{\sum(P_i-\overline{P})^2}\\
&= \frac{\hat{\sigma}^2}{N}\frac{N\overline{P_i^2}}{(N-1)\hat{V}(P)}\\
&= \frac{15.74}{15}\frac{15(141.07)}{14(4.464)^2}\\
&= 7.959
\end{aligned}$$

$$\begin{aligned}
\widehat{Cov}(\hat{\alpha},\hat{\beta}) &= -\hat{\sigma}^2\frac{\overline{P}}{\sum(P_i-\overline{P})^2}\\
&= -15.74\frac{11.07}{14(4.464)^2}\\
&= -0.625
\end{aligned}$$

Now, our estimate of the expected profit-maximizing quantity is $\hat{q}^* = 0.5(20.77) - 5(0.62) = 7.29$. We need to know the variance of $0.5\hat{\alpha} + 5\hat{\beta}$ in order to calculate the confidence interval. We have a formula for the variance of a sum:

$$\begin{aligned}
\hat{V}(0.5\hat{\alpha}+5\hat{\beta}) &= (0.5)^2\hat{V}(\hat{\alpha}) + (5)^2\hat{V}(\hat{\beta}) + 2(0.5)(5)\widehat{Cov}(\hat{\alpha},\hat{\beta})\\
&= 0.25(7.961) + 25(0.056) + 2(0.5)(5)(-0.625)\\
&= 0.27
\end{aligned}$$

Assuming normality for the errors and therefore for the OLS estimators, a 95% CI is then:

$$7.29 \pm 1.96\sqrt{0.27} \tag{3}$$

$$7.29 \pm 1.02 \tag{4}$$

Notice, I was able to check many of my calculations using the proc reg command at the end of the SAS program.