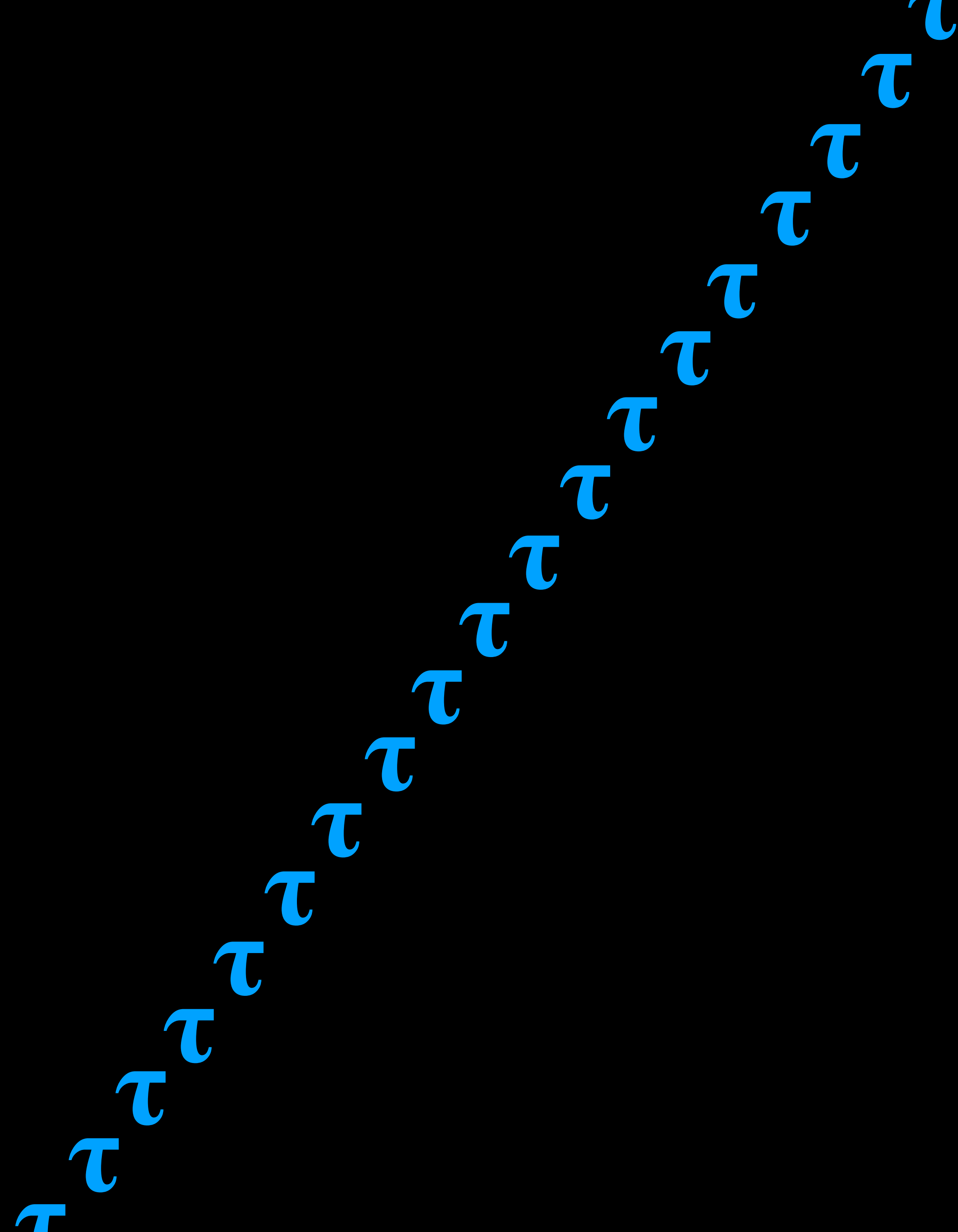


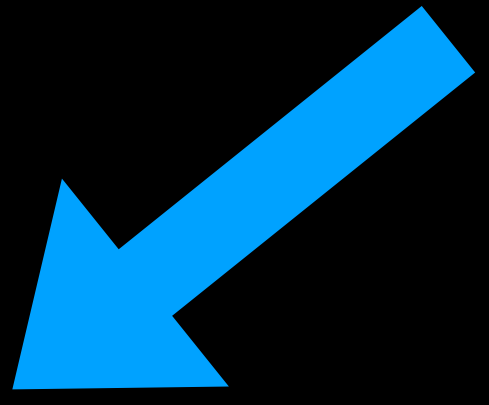
Linear Types

You only use once!

David M Kahn



once



$\lambda x. a x$

$\lambda x. x$



$\lambda f. \lambda x. \lambda y. f(x, y)$



$\lambda f. \lambda x. f x$



$\lambda x. (x, x)$



$\lambda x. ()$



$\lambda y. \lambda x. y$





can't pay with
money *twice*



can't *never*
pay debt

$$\Gamma \vdash e : \tau$$

weaken

$$\Gamma, x : \sigma \vdash e : \tau$$
$$\Gamma, x : \sigma, x : \sigma \vdash e : \tau$$

contract

$$\Gamma, x : \sigma \vdash e : \tau$$

$$\Gamma, x : \sigma, y : \rho \vdash e : \tau$$

exchange

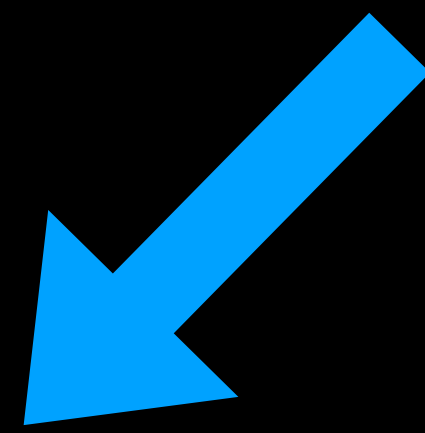
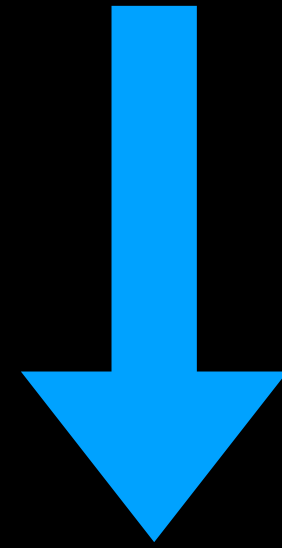
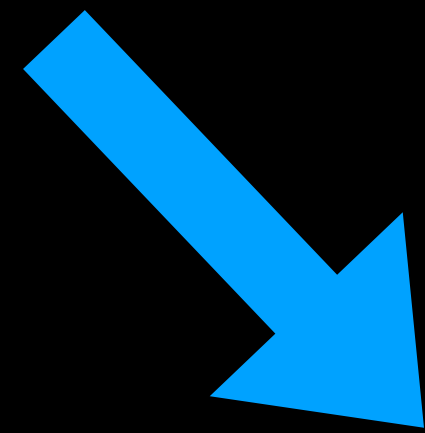
$$\Gamma, y : \rho, x : \sigma \vdash e : \tau$$

	Exchange	Weakening	Contraction	Use
Ordered	—	—	—	Exactly once in order
Linear	Allowed	—	—	Exactly once
Affine	Allowed	Allowed	—	At most once
Relevant	Allowed	—	Allowed	At least once
Normal	Allowed	Allowed	Allowed	Arbitrarily

to use

where used

linear type



Γ \parallel e $:$ τ



\times τ

\parallel



$:$

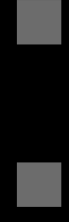
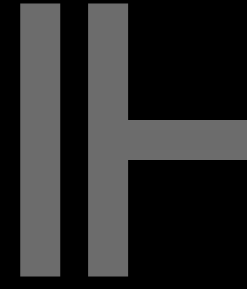
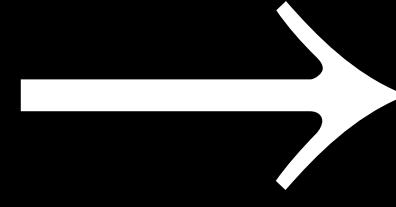
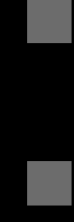
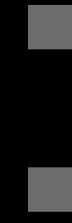
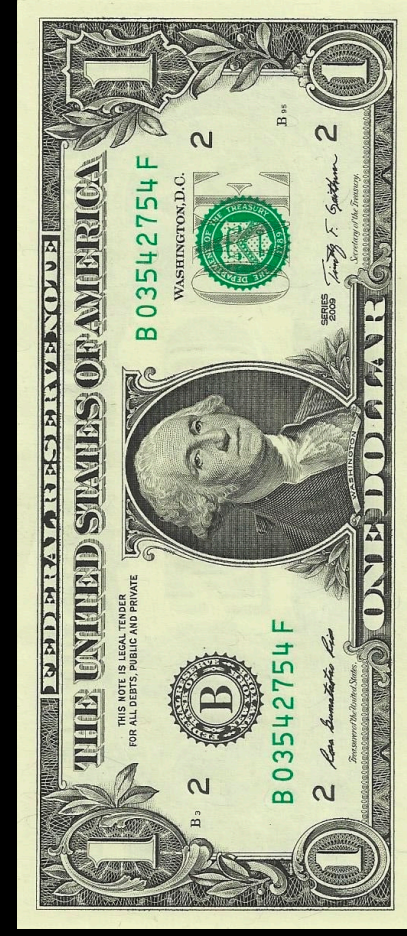
BAG OF BEANS CAFE & RESTAURANT INC
 Optd by: Bag of Beans Cafe & Restaurant Inc.
 117 Aguinaldo Highway Crossing Mendez
 West Tagaytay City Cavite 4120
 TIN: 008-117-738-000 VAT

O.R.#: 783944
 Cashier: Eloisa
 11/5/2019

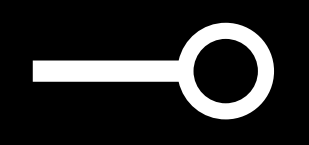
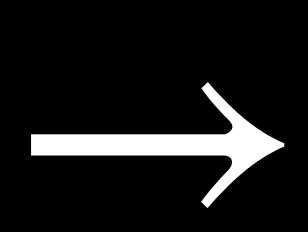
Guest: 3
 1:24:50 PM

1	Coffee Choco Banana	170.00
1	HotGreen Tea	100.00
1	D White mocha	165.00
1	Blueberry Cheesecake [Slice]	175.00

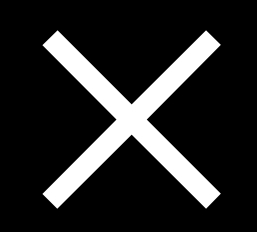
SubTotal	610.00
PreTax	544.64
Serv Charge.(10%)	54.46
12% VAT	65.36
Amount Due	664.46



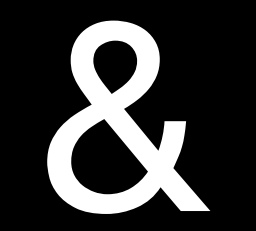
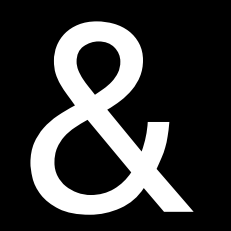
Types:



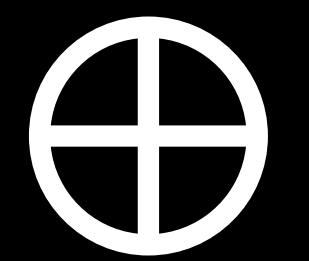
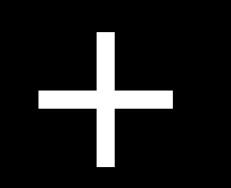
lollipop, lolli,
linear implication



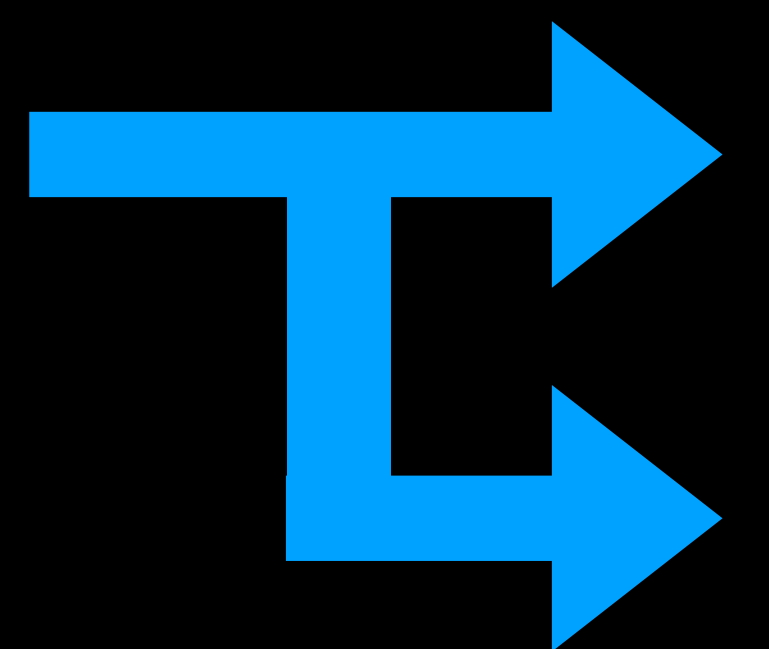
multiplicative product,
tensor



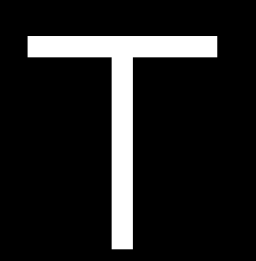
additive product,
with



additive sum,
plus



multiplicative unit



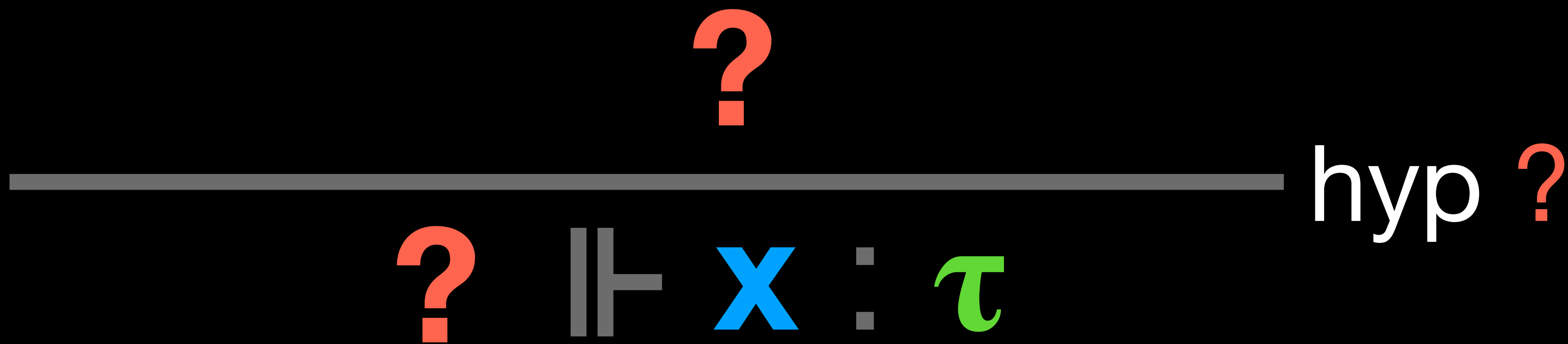
additive unit

?

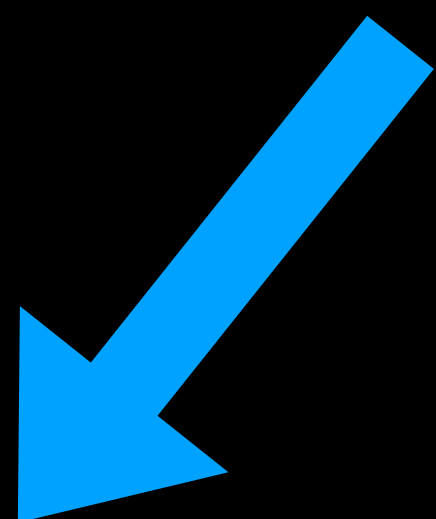
? $\Vdash \lambda x. e : \tau \multimap \sigma$ \multimap ?

$$\Gamma, x : \tau \Vdash e : \sigma$$

$$\Gamma \Vdash \lambda x. e : \tau \multimap \sigma$$
$$\multimap$$



gotta use!



Γ

x

:

τ

\Vdash

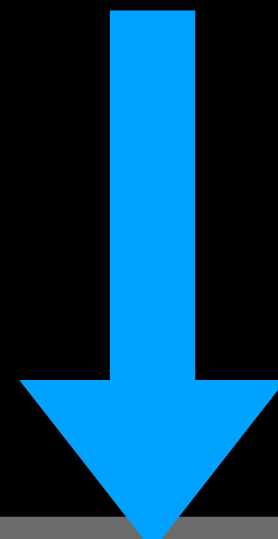
x

:

τ

hyp ?

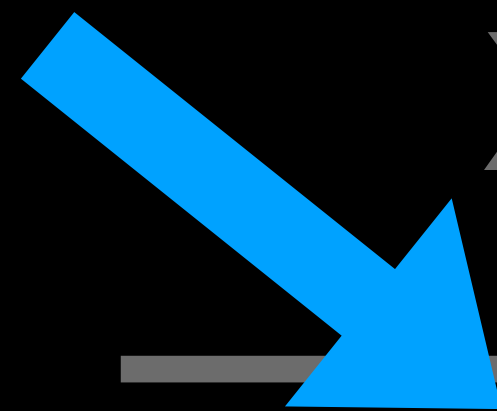
gotta use!



hyp ?

$x : \tau, y : \sigma \Vdash x : \tau$

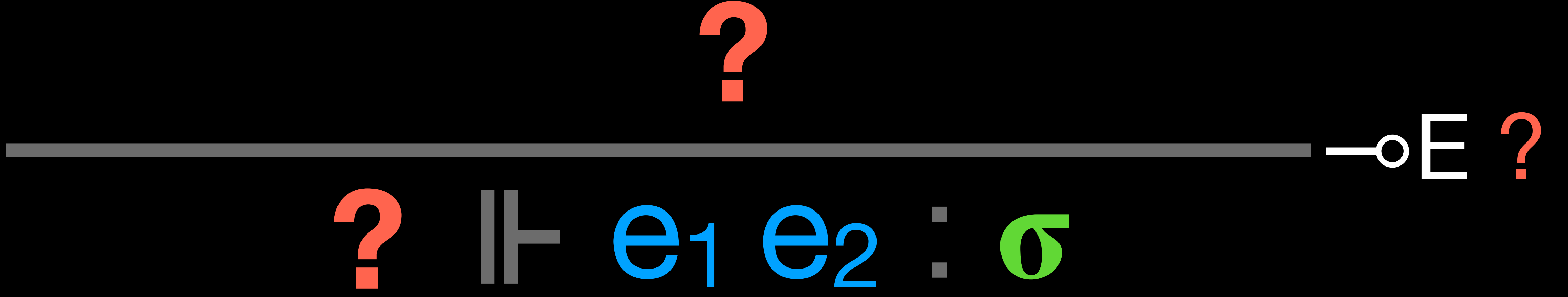
nonlinear



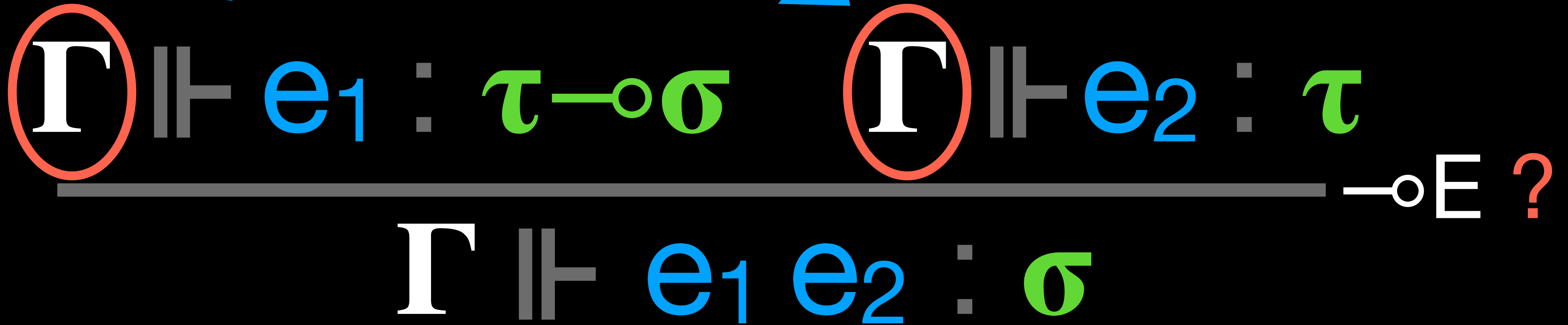
$x : \tau \Vdash \lambda y. x : \sigma \multimap \tau$

$\Vdash \lambda x. \lambda y. x : \tau \multimap \sigma \multimap \tau$

$X : \tau \Vdash X : \tau$ hyp

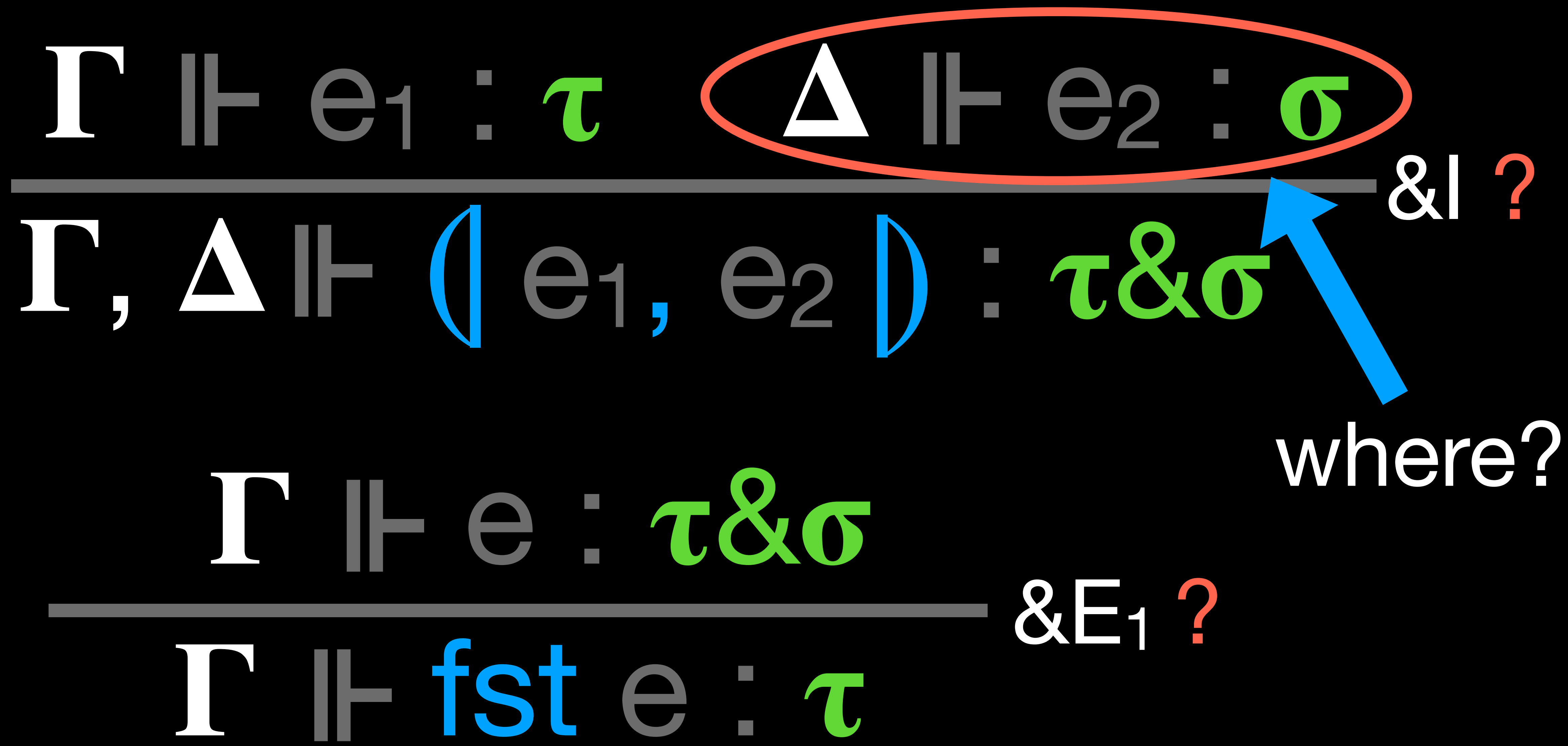


reuse!

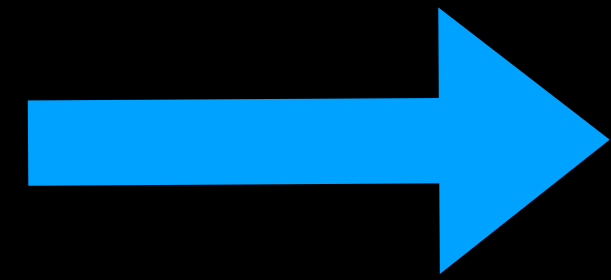


$$\frac{\Gamma \Vdash e_1 : \tau \quad \Delta \Vdash e_2 : \tau}{\Gamma, \Delta \Vdash e_1 e_2 : \sigma} \text{E}$$

$$\Gamma \Vdash e_1 : \tau \quad \Delta \Vdash e_2 : \sigma$$
 $\otimes I$
$$\Gamma, \Delta \Vdash (e_1, e_2) : \tau \otimes \sigma$$
$$\Gamma \Vdash e_1 : \tau \otimes \sigma \quad \Delta, x:\tau, y:\sigma \Vdash e_2 : \rho$$
 $\otimes E$
$$\Gamma, \Delta \Vdash \text{case } e_1 ((x, y) \Rightarrow e_2) : \rho$$



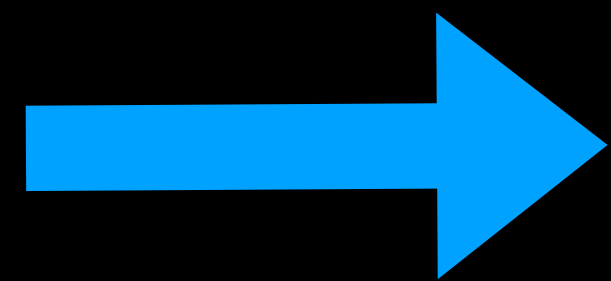
nonlinear



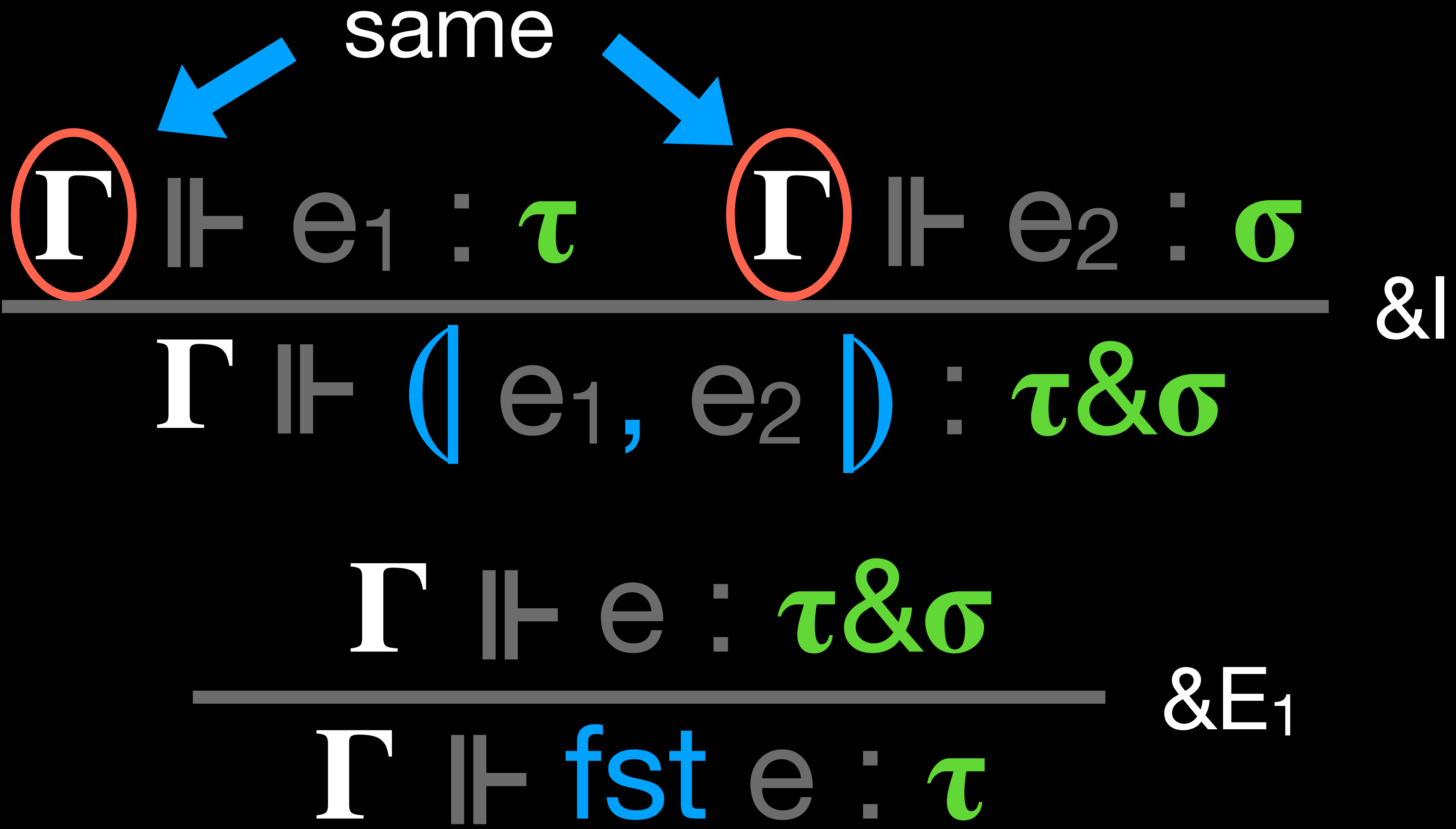
$\lambda x. \lambda y. x$



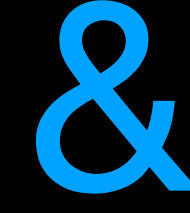
linear ?



$\lambda x. \lambda y. \text{fst } (| x, y |)$







\$1


no!

\$2

He :



$x : \text{\$} \Vdash \textit{buyC} :$ 

$x : \text{\$} \Vdash \textit{buyK} :$ 

&

$x : \text{\$} \Vdash (\textit{buyC}, \textit{buyK}) :$  **&** 

⌋

$\Vdash \lambda x. (\textit{buyC}, \textit{buyK}) :$  **⌋**  **&** 

$A \otimes B$: “both A and B simultaneously”

$A \& B$: “your choice of either A or B ”

$A \dashv\dashv B$: “turns A into B with no leftover”

$$\Gamma \Vdash e : \tau \oplus \sigma$$
$$\Delta, x : \tau \Vdash e_1 : \rho \quad \Delta, x : \sigma \Vdash e_2 : \rho$$

$$\Gamma, \Delta \Vdash \text{case } e (L \cdot x \Rightarrow e_1 \mid R \cdot x \Rightarrow e_2) : \rho \quad \oplus E$$
$$\Gamma \Vdash e : \tau$$

$$\Gamma \Vdash L \cdot e : \tau \oplus \sigma \quad \oplus I_1$$

$A \otimes B$: “both A and B simultaneously”

$A \& B$: “your choice of either A or B ”

$A \rightarrow B$: “turns A into B with no leftover”

$A \oplus B$: “either A or B , NOT your choice”

$$\Gamma \Vdash e : \tau \& \sigma$$
$$\&E_1$$

$$\Gamma \Vdash \text{fst } e : \tau$$
$$\Gamma \Vdash e : \tau$$
$$\oplus I_1$$

$$\Gamma \Vdash L \cdot e : \tau \oplus \sigma$$

$$\Gamma \Vdash e_1 : \tau \quad \Gamma \Vdash e_2 : \sigma$$

&I

$$\Gamma \Vdash (e_1, e_2) : \tau \& \sigma$$

~~$\Gamma \Vdash () : \tau \oplus \sigma$~~

$$\Delta, x : \tau \Vdash e_1 : \rho \quad \Delta, x : \sigma \Vdash e_2 : \rho$$

$\oplus E$

~~Γ~~ , $\Delta \Vdash \text{case } e (L \cdot x \Rightarrow e_1 \mid R \cdot x \Rightarrow e_2) : \rho$

$$\Gamma \Vdash e_1 : \tau \quad \Delta \Vdash e_2 : \sigma$$
 \otimes
$$\Gamma, \Delta \Vdash (e_1, e_2) : \tau \otimes \sigma$$
$$\Gamma \Vdash e_1 : \tau \multimap \sigma \quad \Delta \Vdash e_2 : \tau$$
 \multimap
$$\Gamma, \Delta \Vdash e_1 e_2 : \sigma$$

$$\frac{\cancel{\Gamma \Vdash e_1 : \tau \otimes \sigma} \quad \Delta, x:\tau, y:\sigma \Vdash e_2 : \rho}{\Gamma, \Delta \Vdash \text{case } e_1 ((x, y) \Rightarrow e_2) : \rho} \otimes E$$

$$\frac{\Gamma, x:\tau \Vdash e : \sigma}{\Gamma \Vdash \lambda x. e : \tau \multimap \sigma} \multimap I$$

$$\Gamma \Vdash e_1 : \tau \quad \Delta \Vdash e_2 : \sigma$$
 \otimes

$$\Gamma, \Delta \Vdash (e_1, e_2) : \tau \otimes \sigma$$

$$\cdot \Vdash () : 1$$
 11

$$\frac{\Gamma \Vdash e_1 : \tau \otimes \sigma \quad \Delta, x:\tau, y:\sigma \Vdash e_2 : \rho}{\Gamma, \Delta \Vdash \text{case } e_1 ((x, y) \Rightarrow e_2) : \rho} \otimes E$$

$$\frac{\Gamma \Vdash e_1 : 1 \quad \Delta \Vdash e_2 : \rho}{\Gamma, \Delta \Vdash \text{case } e_1 (() \Rightarrow e_2) : \rho} 1E$$

$$\frac{\Gamma \Vdash e_1 : \tau \quad \Gamma \Vdash e_2 : \sigma}{\Gamma \Vdash (e_1, e_2) : \tau \& \sigma} \quad \&I$$

$$\frac{}{\Gamma \Vdash () : \top} \quad \top I$$

$\Gamma \Vdash e : \tau \& \sigma$



$\Gamma \Vdash \text{fst } e : \tau$

$\&E_1$

none

$A \rightarrow B$: “turns A into B with no leftover”

$A \otimes B$: “both A and B simultaneously”

$A \& B$: “your choice of either A or B ”

$A \oplus B$: “either A or B , NOT your choice”

1 : “object of no information”

T : “permanent garbage pit”

$!A$: “any amount of A ”

$\Delta; \Gamma \Vdash e : \tau$

$\Delta; x : \tau \Vdash x : \tau$ hyp₁

$\Delta, x : \tau; \cdot \Vdash x : \tau$ hyp₂

$$\frac{\Omega; \Gamma \Vdash e_1 : \tau \quad \Omega; \Delta \Vdash e_2 : \sigma}{\Omega; \Gamma, \Delta \Vdash (e_1, e_2) : \tau \otimes \sigma} \otimes I$$

$\Delta; \cdot \Vdash e : \tau$



$\Delta; \cdot \Vdash e : !\tau$

!!

$\Delta, x : \sigma; \Gamma \Vdash e : \tau$



$\Delta; \Gamma, x : !\sigma \Vdash e : \tau$

!shift

fix $x.e \mapsto [\mathbf{fix} \ x.e/x]e$

fix $x.f \ x \mapsto f (\mathbf{fix} \ x.f \ x)$

$f \ f (\mathbf{fix} \ x.f \ x)$

$f \ f \ f (\mathbf{fix} \ x.f \ x)$

$\Delta, x : \tau; \cdot \Vdash e : \tau$



$\Delta; \cdot \Vdash \text{fix } x. e : \tau$

rec

$$\Delta; \Gamma \Vdash e : [\mu\alpha . \tau / \alpha] \tau$$
 μI
$$\Delta; \Gamma \Vdash \text{fold } e : \mu\alpha . \tau$$
$$\Delta; \Gamma \Vdash e : \mu\alpha . \tau$$
 μE
$$\Delta; \Gamma \Vdash \text{unfold } e : [\mu\alpha . \tau / \alpha] \tau$$

flip : bits \rightarrow bits

flip = **fix** f. λ b. case unfold b (
 b0·y \Rightarrow fold (b1·f y)
 | b1·y \Rightarrow fold (b0·f y)
 | e·y \Rightarrow fold (e·y)
)

$P, Q ::= x \leftarrow y \mid x \leftarrow P; Q$
| $x.l; P \mid x.r; P$
| $\text{case } x \{ l \Rightarrow P \mid r \Rightarrow Q \}$
| $\text{close } x \mid \text{wait } x; P$
| $\text{send } x y; P$
| $x \leftarrow \text{recv } y; P$

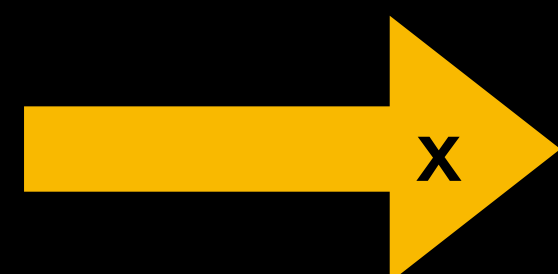
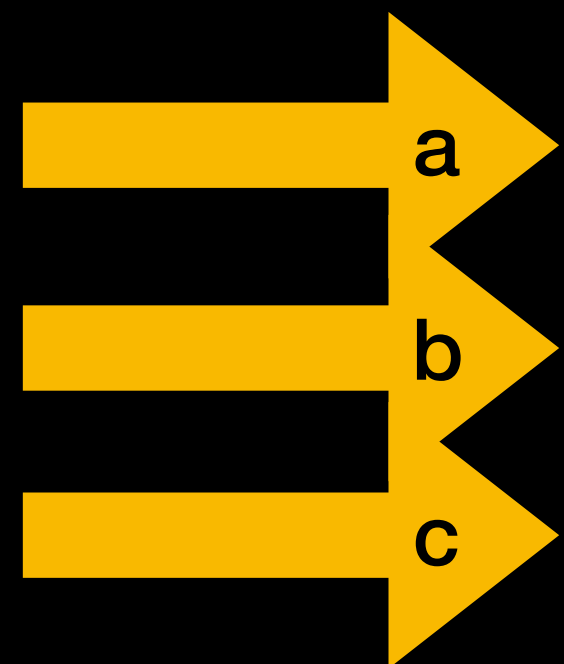
to use

where used

linear type

$\Gamma \Vdash P :: (x : \tau)$

Γ



provided channel

1. channels aren't ignored
2. channels are contained
3. dualities \sim client/server
4. progress = no deadlocks

id

$$\mathbf{y} : \boldsymbol{\tau} \Vdash \mathbf{x} \leftarrow \mathbf{y} :: (\mathbf{x} : \boldsymbol{\tau})$$

$$\Gamma \Vdash P :: (\mathbf{x} : \boldsymbol{\sigma}) \quad \Delta, \mathbf{x} : \boldsymbol{\sigma} \Vdash Q :: (\mathbf{y} : \boldsymbol{\tau})$$

cut

$$\Gamma, \Delta \Vdash \mathbf{x} \leftarrow P; Q :: (\mathbf{y} : \boldsymbol{\tau})$$

$$\frac{\Gamma, x : \tau \Vdash P :: (y : \rho) \quad \Gamma, x : \sigma \Vdash Q :: (y : \rho)}{\Gamma, x : \tau \oplus \sigma \Vdash \text{case } x (l \Rightarrow P \mid r \Rightarrow Q) :: (y : \rho)} \oplus L$$

$$\frac{\Gamma \Vdash P :: (x : \sigma)}{\Gamma \Vdash x.r; P :: (x : \tau \oplus \sigma)} \oplus R_2$$

$$\Gamma, \mathbf{x} : \sigma \Vdash P :: (\mathbf{y} : \rho)$$

$$\Gamma, \mathbf{x} : \tau \& \sigma \Vdash \mathbf{x}.r; P :: (\mathbf{y} : \rho) \quad \&L_2$$
$$\Gamma \Vdash P :: (\mathbf{x} : \tau)$$
$$\Gamma \Vdash Q :: (\mathbf{x} : \sigma)$$

$$\Gamma \Vdash \text{case } \mathbf{x} (l \Rightarrow P \mid r \Rightarrow Q) :: (\mathbf{x} : \tau \& \sigma) \quad \&R$$

$$\Gamma, x : \sigma \Vdash P :: (z : \rho)$$

$$\Gamma, x : \tau \multimap \sigma, y : \tau \Vdash \text{send } x \ y; P :: (z : \rho)$$
$$\Gamma, x : \tau \Vdash P :: (y : \sigma)$$

$$\Gamma \Vdash x \leftarrow \text{recv } y; P :: (y : \tau \multimap \sigma)$$

$$\Gamma, x : \sigma, y : \tau \Vdash P :: (z : \rho)$$

$$\Gamma, x : \tau \otimes \sigma \Vdash y \leftarrow \text{recv } x; P :: (z : \rho) \quad \otimes L$$
$$\Gamma \Vdash P :: (x : \sigma)$$

$$\Gamma, y : \tau \Vdash \text{send } x y; P :: (x : \tau \otimes \sigma) \quad \otimes R$$

$$\Gamma \Vdash P :: (y : \tau)$$

 $1L$
$$\Gamma, x : 1 \Vdash \text{wait } x; P :: (y : \tau)$$

 $1R$
$$\cdot \Vdash \text{close } x :: (x : 1)$$

$A \multimap B$: “receives A , continues as B ”

$A \otimes B$: “sends A , continues as B ”

$A \& B$: “offers a choice of either A or B ”

$A \oplus B$: “is one of A or B ”

1 : “a finished channel”

≤ 1

≥ 1

resources and obligations

money

debts

channel access

channel handling

no cloning

no deleting