Recap: RS code: Polynomials viewpoint

Message: $[m_0, m_1, \ldots, m_{k-1}]$ where $m_i \in GF(q)$

Consider the polynomial of degree k-1

$$P(x) = m_{k-1} x^{k-1} + \cdots + m_1 x + m_0$$

RS code: Codeword: $[P(\alpha_1), P(\alpha_2), \ldots, P(\alpha_n)]$

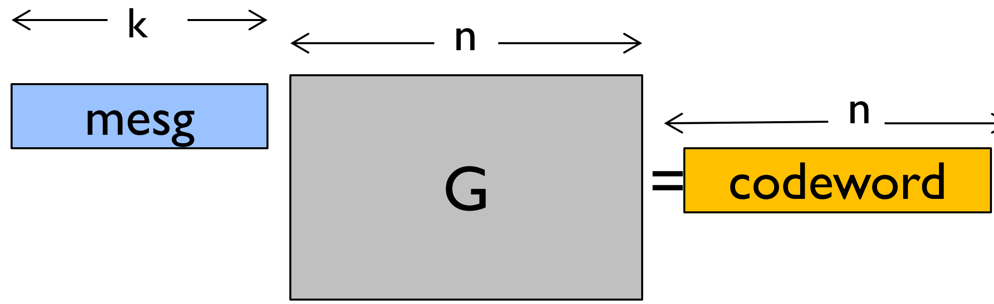
(distinct α_i's)

To make the α_i's in $P(\alpha_i)$ distinct, need field size $q \geq n$

That is, need sufficiently large field size for desired codeword length.

Recap: Generator matrix of RS code

What is the generator matrix?



$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & & \alpha_n^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & & \alpha_n^{k-1} \end{bmatrix}$$

“Vandermonde matrix”

Special property of Vandermonde matrices: Full rank (columns linearly independent)

Very useful in constructing codes.

Polynomials and their degrees

Fundamental theorem of Algebra: Any non-zero polynomial of degree m has at most m roots (over any field).

Corollary 1: If two degree-m polynomials P, Q agree on m+1 points (i.e., if $P(x_i) = Q(x_i)$ for $x_0, x_1, \ldots, x_m$), then P = Q.

~~**Corollary 2:** Given any m+1 points (x_i, y_i), there is at most one degree-m polynomial that has $P(x_i) = y_i$ for all these i.~~

Theorem: Given any m+1 points (x_i, y_i), there is exactly one degree-m polynomial that has $P(x_i) = y_i$ for all these i.

Proof: e.g., use Lagrange interpolation.

In our case, m=k-1

Minimum distance of an (n, k) RS code

Theorem: RS codes have minimum distance $\mathrm{d} = (n - k + 1)$

Proof: Any ideas?

Hint: Is it a linear code?

1. *RS is a linear code:* if we add two codewords corresponding to P(x) and Q(x), we get a codeword corresponding to the polynomial P(x) + Q(x). Similarly any linear combination..
2. *So look at the least weight codeword*. It is the evaluation of a polynomial of degree k-1 at some n points. So it can be zero on only k-1 points. Hence non-zero on at most (n-(k-1)) points. This means distance at least n-k+1

Apply Singleton bound

Meets Singleton bound: RS codes are MDS

Decoding: Recovering Erasures

Recovering from at most (d-1) erasures:

Received codeword:

$[P(\alpha_1)$, *, $P(\alpha_2)$, …,*, $P(\alpha_n)]$: at most (d-1) symbols erased (where * = erased)

Ideas?

1. At most n-k symbols erased
2. So have $P(\alpha_i)$ for at least k evaluations
3. Interpolation to recover the polynomial

Matrix viewpoint: Solving system of linear equations

Decoding: Correcting Errors

Correcting s errors: (d ≥ 2s+1 $\Rightarrow n \geq k + 2s$)

Naïve algo:

- Find k+s symbols that agree on a degree (k-1) poly P(x).
  - There must exist one: since originally k + 2s symbols agreed and at most s are in error

    (i.e., "guess" the n-s uncorrupted locations)

- Can we go wrong?

  Are there k+s symbols that agree on the wrong degree (k-1) polynomial P'(x)? No.
  - Any subset of k symbols will define P'(x)
  - Since at most s out of the k+s symbols are in error, P'(x) = p(x)

Decoding: Correcting Errors

Correcting s errors: ($d \geq 2s+1$)

Naïve algo:

- Find k+s symbols that agree on a degree (k-1) poly P(x).
  - There must exist one: since originally k + 2s symbols agreed and at most s are in error
    (i.e., "guess" the n-s uncorrupted locations)

This suggests a brute-force approach, very inefficient.

"guess" = "enumerate", so time is (n choose s) ~ n^s.

More efficient algorithms exist: <polynomial>

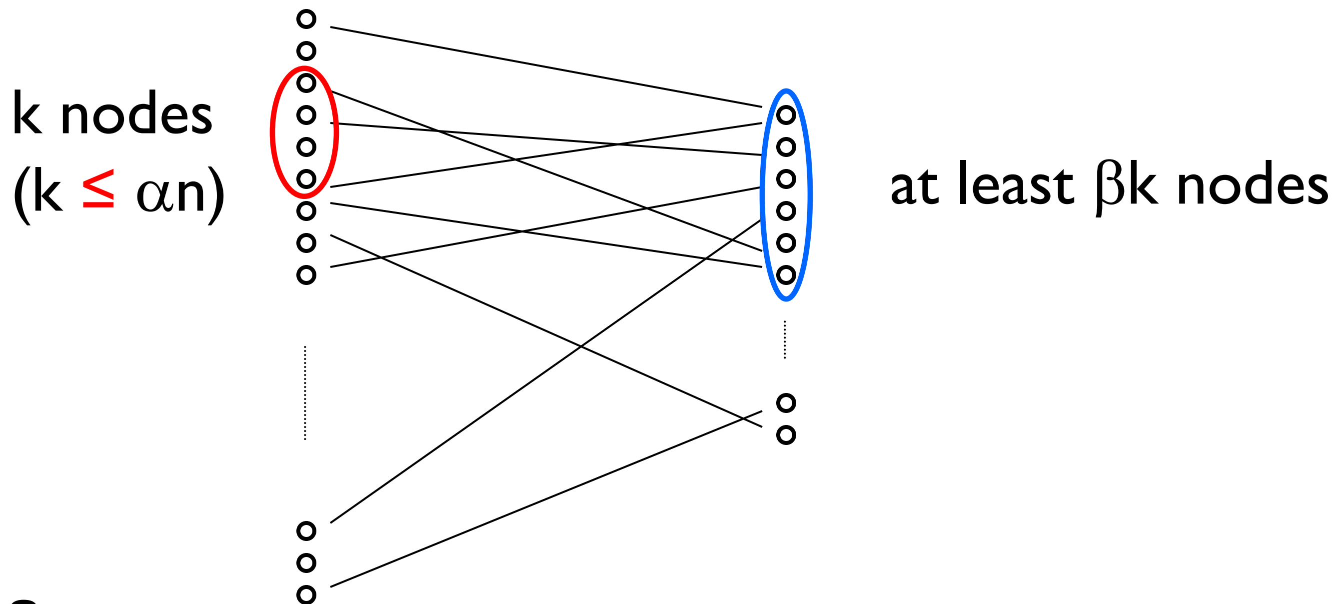
"The Berlekamp Welch Algorithm" (results in solving a system of n linear equations; uses "error" polynomials)

$O(n^2)$ algorithms exist

Codes based on graphs

- Optimized for fast (de)coding
- Based on graphical constructions
- Constructions based on properties of expander graphs

(α, β) Expander Graphs (bipartite)



Properties

- **Expansion:** every small subset ($k \le \alpha n$) on left has many ($\ge \beta k$) neighbors on right
- **Low degree –** not technically part of the definition, but typically assumed

d-regular graphs

An undirected graph is **d-regular** if every vertex has d neighbors.

A **bipartite** graph is **d-left-regular** if every vertex on the left has d neighbors on the right.

We consider only d-left-regular constructions.

(And call it d-regular with abuse of notation.)

Expander Graphs: Constructions

Important parameters: size (n), degree (d), expansion (β)

Randomized constructions

– A random d-regular graph is an expander with a high probability
– Time consuming and cannot be stored compactly

Explicit constructions

– Cayley graphs, Ramanujan graphs etc
– Typical technique – start with a small expander, apply operations to increase its size

Expander Graphs: Constructions

Theorem: For every constant $0 < c < 1$, can construct bipartite graphs with

n nodes on left,
cn on right,
d-regular (left),

that are $(\alpha, 3d/4)$ expanders, for constants α and d that are functions of c alone.

"Any set containing at most alpha fraction of the left has (3d/4) times as many neighbors on the right"

Tornado codes

(Luby Mitzenmacher Shokrollahi Spielman 2001)

Goal: low (linear-time) complexity encoding and decoding

We will focus on **erasure** recovery

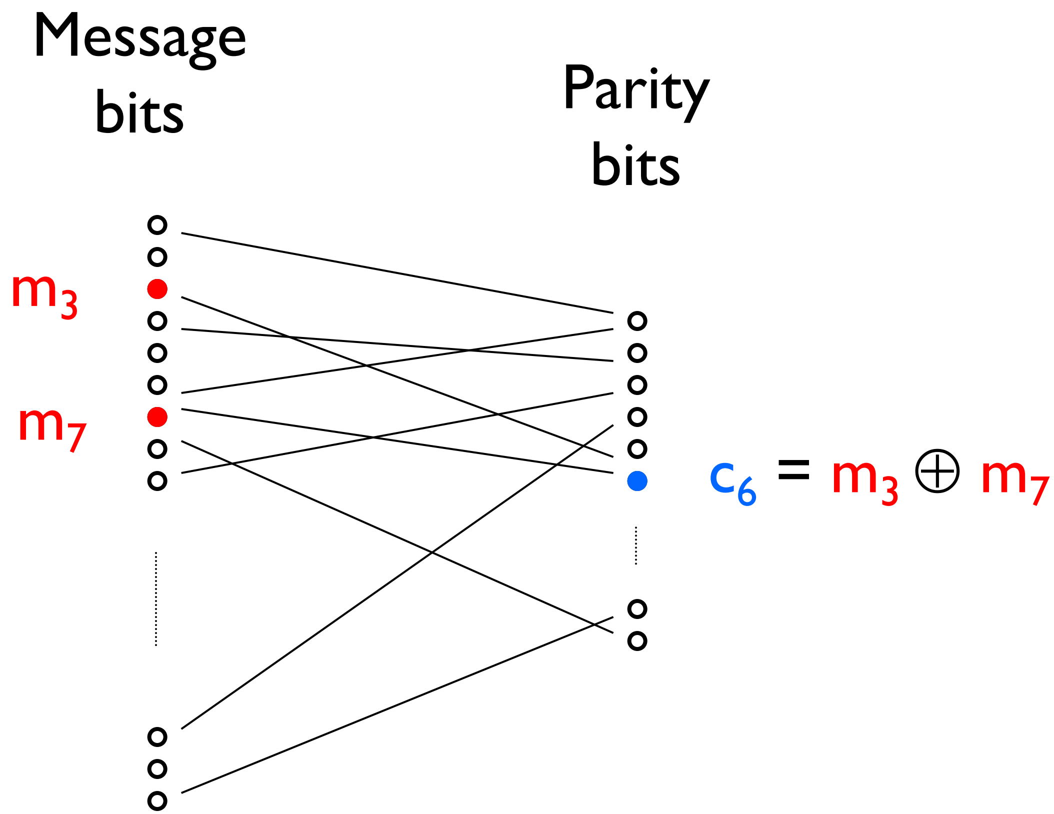
- Each bit either reaches intact, or is lost.
- We know the positions of the lost bits.

The random erasure model

Random erasure model:

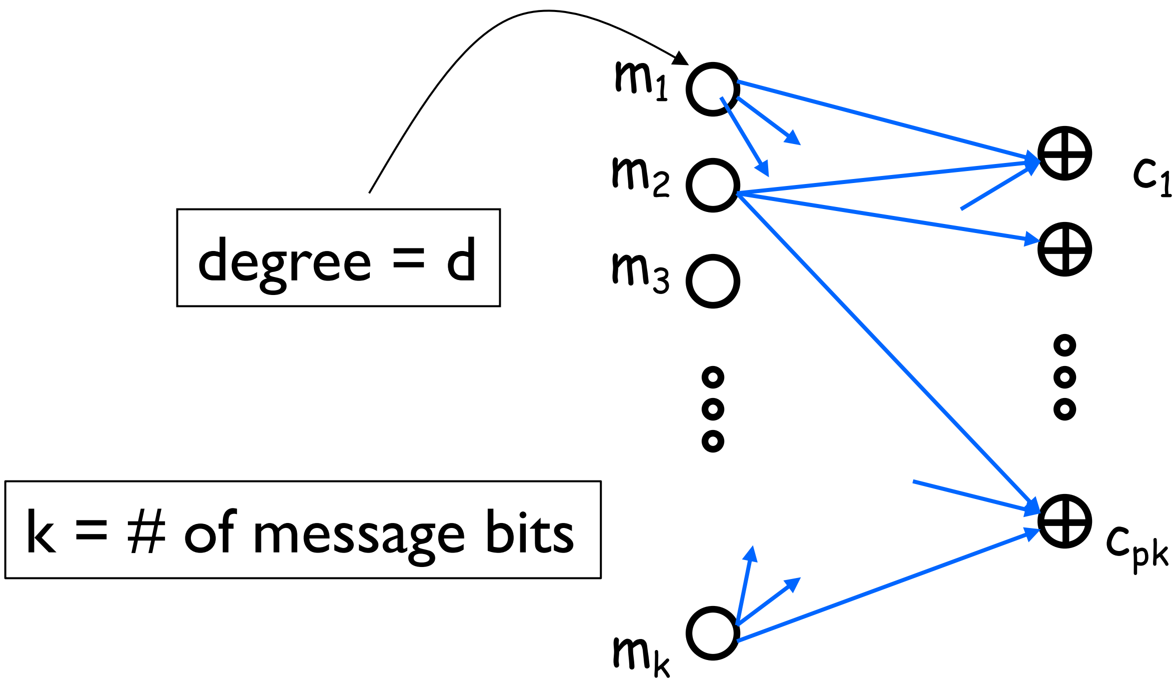
- Each bit is erased with some probability p (say ½ here)
- Known: a random linear code with rate < 1-p works (why?)

For simplicity.


Message
bits
Parity
bits
m_3
m_7
$c_6 = m_3 \oplus m_7$

Tornado codes

- Have d-left-regular bipartite graphs with k nodes on the left and pk on the right.

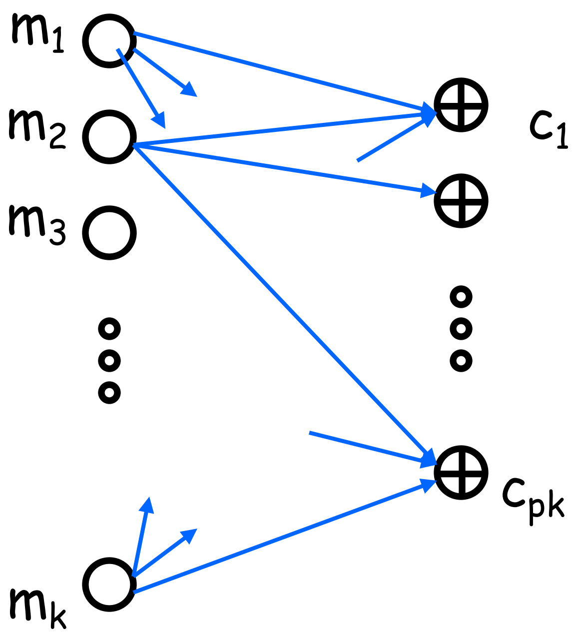


- Let's again assume 3d/4-expansion.

Tornado codes: Encoding

Why is it linear time?

(Hint: Look at the number of edges)



Computes the sum modulo 2 of its neighbors

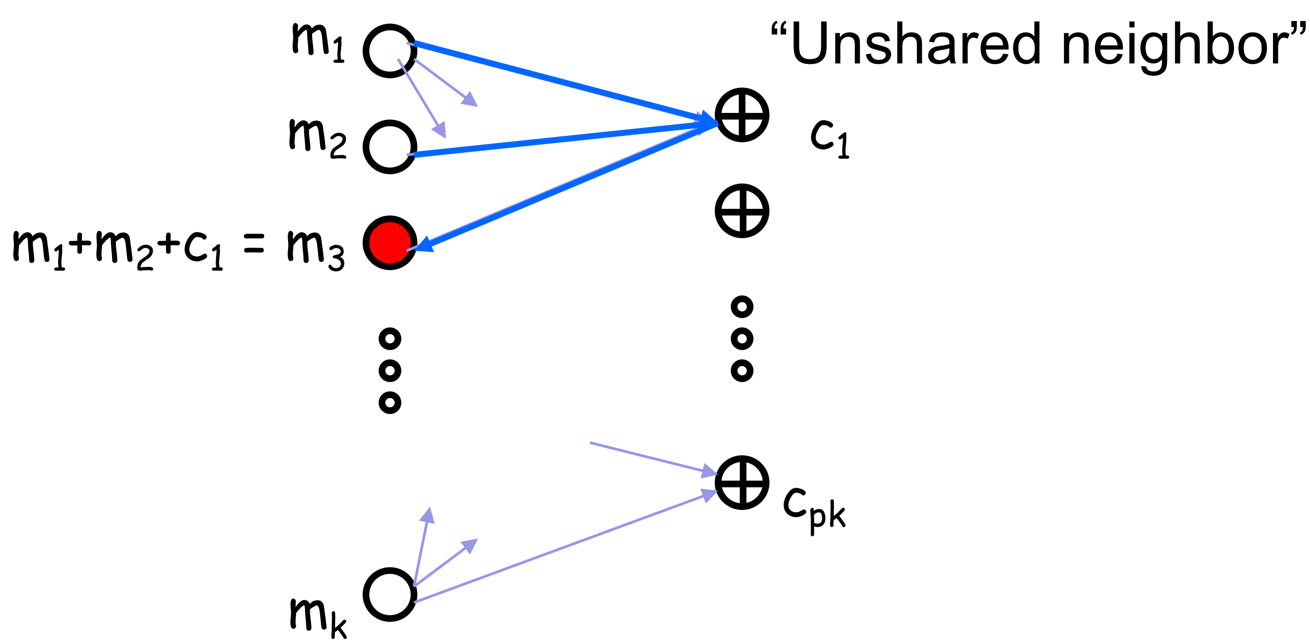
Number of edges = kd

Tornado codes: Decoding

First, assume that all the parity bits are intact

Find a parity bit such that only one of its neighbors is erased (an *unshared neighbor*)

Fix the erased bit, and repeat.



Tornado codes: Decoding

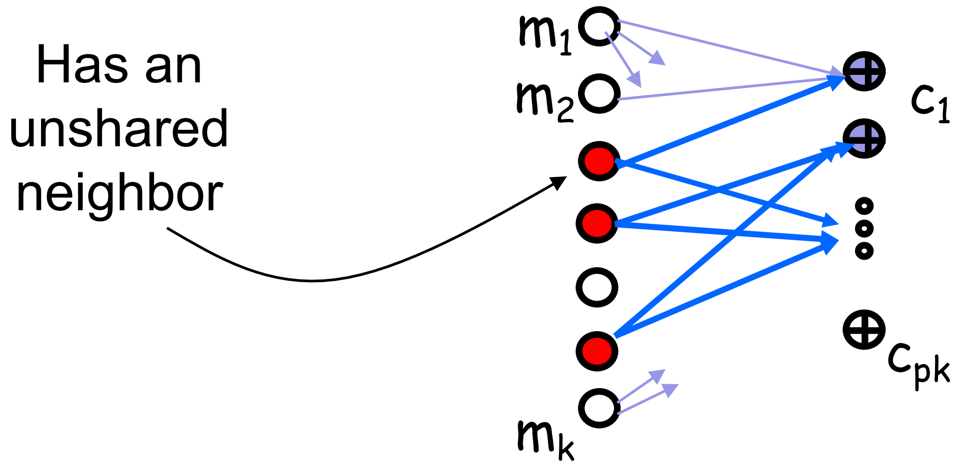
Intuition:

Want to always find such a parity bit with “Unshared neighbor” property.

Consider the set of corrupted message bit and their neighbors.

(Suppose this set is small.)

=> at least one message bit has an unshared neighbor.



Tornado codes: Decoding

Can we always find unshared neighbors?

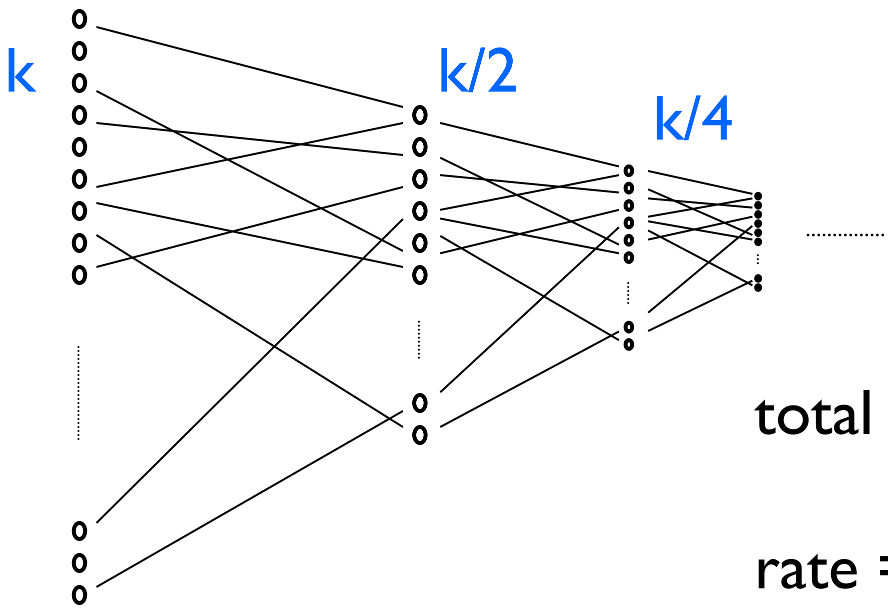
Expander graphs give us this property if expansion > d/2 (similar argument to one above)

Also, [Luby et al] show that if we construct the graph from a specific kind of degree distribution, then we can always find unshared neighbors.

What if parity bits are lost?

Cascading

- Use another bipartite graph to construct another level of parity bits for the parity bits
- Final level is encoded using RS or some other code



stop when $k/2^t$ "small enough"

total bits $n \leq k(1 + \frac{1}{2} + \frac{1}{4} + \ldots)$

$= 2k$

rate $= k/n = \frac{1}{2}$.

(assuming p =1/2)

Tornado codes enc/dec complexity

Encoding time?

– for the first t stages : $|E| = d \times |V| = O(k)$

– for the last stage: poly(last size) = $O(k)$ by design.

Decoding time?

– start from the last stage and move left

– Last stage is $O(k)$ by design

– Rest proportional to $|E| = O(k)$

So get very fast (linear-time) coding and decoding.

100s-10,000 times faster than RS