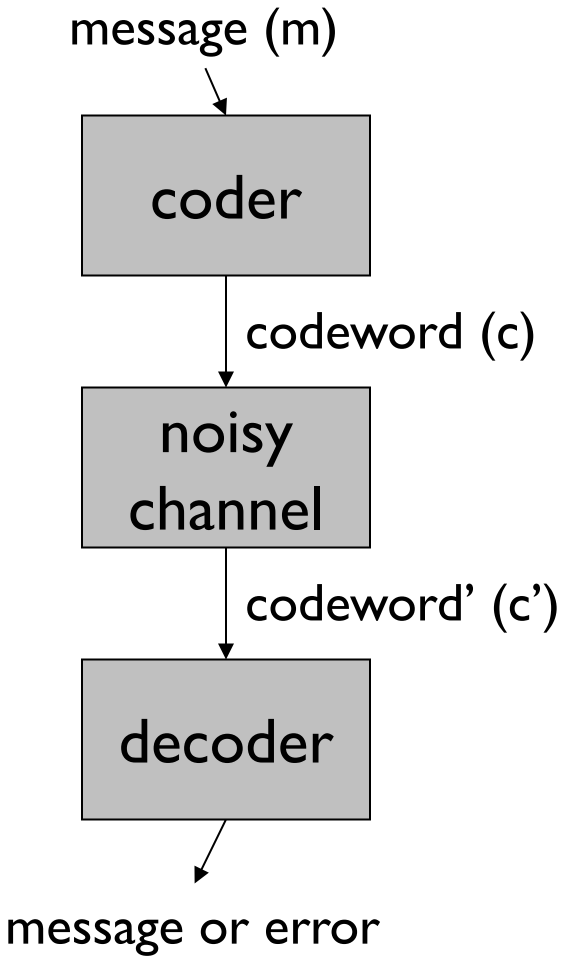
Recap: Block Codes



Each message and codeword is of fixed size

Σ = codeword alphabet

k $=|m|$ **n** $= |c|$ **q** $= |\Sigma|$

C = "code" = set of codewords

C $\subseteq \Sigma^n$ (codewords)

Δ**(x,y)** = number of positions s.t. $x_i \neq y_i$

d $= \min\{\Delta(x,y) : x,y \in C, x \neq y\}$

Code described as: $\mathbf{(n,k,d)_q}$

Recap: Role of Minimum Distance

Theorem:

A code C with minimum distance “d” can:

1. detect any (d-1) errors
2. recover any (d-1) erasures
3. correct any $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors

Stated another way:

For s-bit error detection $d \geq s + 1$

For s-bit error correction $d \geq 2s + 1$

To correct a erasures and b errors if $d \geq a + 2b + 1$

Desired Properties

We look for codes with the following properties:

1. Good rate: k/n should be high (low overhead)
2. Good distance: d should be large (good error correction)
3. Small block size k (helps with latency)
4. Fast encoding and decoding
5. Others: want to handle bursty/random errors, local decodability, ...

Q:

If no structure in the code, how would one perform encoding?

Gigantic lookup table!

If no structure in the code, encoding is highly inefficient.

A common kind of structure added is **linearity**

Linear Codes

If Σ is a finite field, then Σ^n is a vector space

Definition: C is a linear code if it is a linear subspace of Σ^n of dimension k.

This means that there is a set of k independent vectors $v_i \in \Sigma^n$ $(1 \leq i \leq k)$ that span the subspace.

i.e. every codeword can be written as:

$$c = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k \quad \text{where } a_i \in \Sigma$$

"Basis (or spanning) Vectors"

Some Properties of Linear Codes

1. Linear combination of two codewords is a codeword.

2. Minimum distance (d) = weight of least weight (non-zero) codewords

(Weight of a vector refers to the Hamming weight of a vector, which is equal to the number of non-zero symbols in the vector)

$$d = \min_{\substack{c_i, c_j \in C \\ i \neq j}} |c_i - c_j|$$

$$= \min_{\substack{c \in C \\ c \neq 0}} |c|$$

Generator and Parity Check Matrices

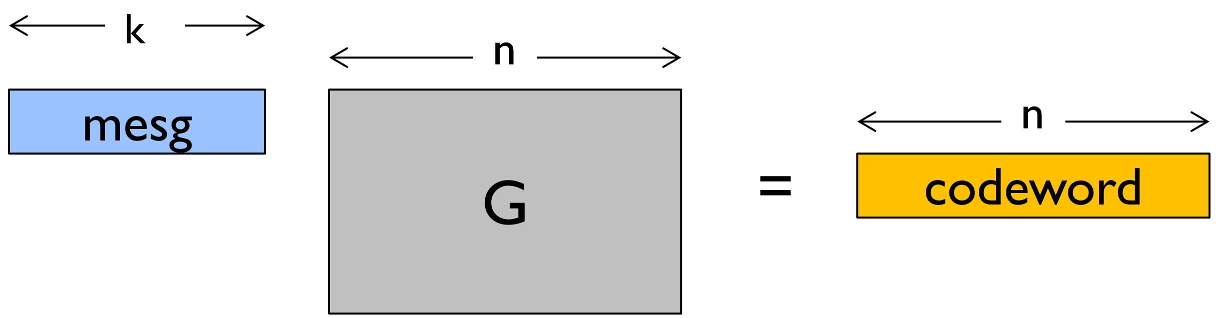
3. Every linear code has two matrices associated with it.

1. Generator Matrix:

A k x n matrix **G** such that: $C = \{ xG \mid x \in \Sigma^k \}$

(Note: Here vectors are "row vectors".)

Made from stacking the spanning vectors

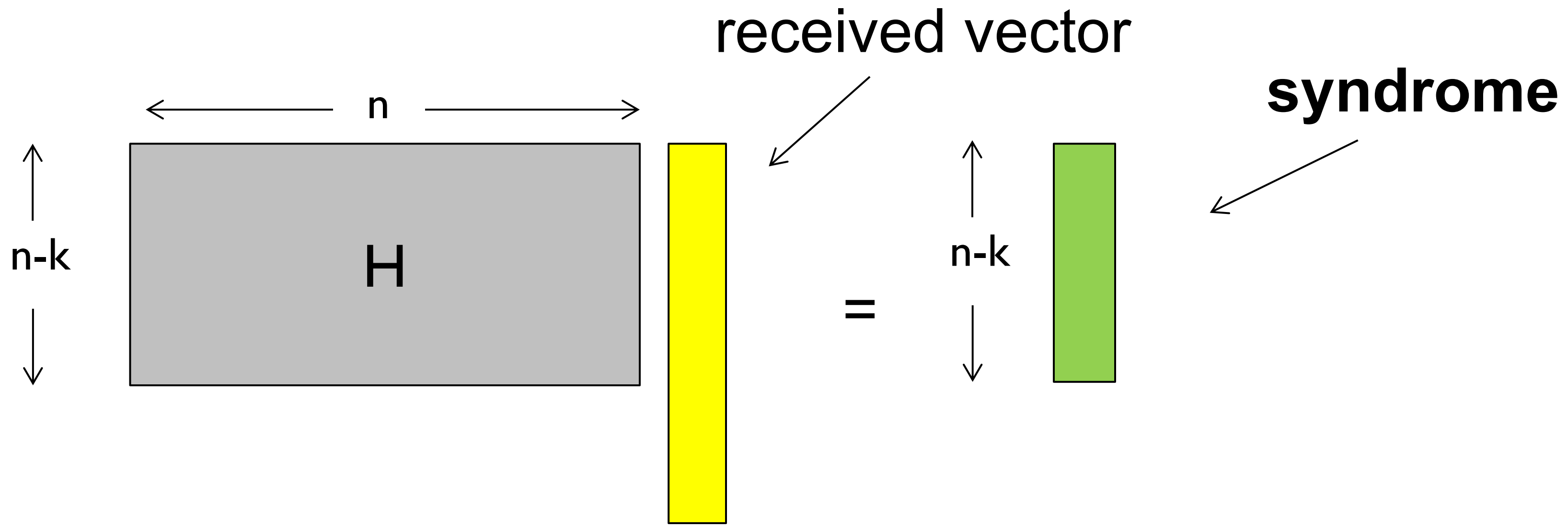


Generator and Parity Check Matrices

2. Parity Check Matrix:

An (n – k) x n matrix **H** such that: $C = \{y \in \Sigma^n \mid Hy^T = 0\}$

(Codewords are the null space of H.)



if syndrome = 0, received vector = codeword

else have to use syndrome to get back codeword ("decode")

Advantages of Linear Codes

- Encoding is efficient (vector-matrix multiply)
- Error detection is efficient (vector-matrix multiply)
- **Syndrome** (Hy^T) has error information
- How to decode? In general, have q^{n-k} sized table for decoding (one for each syndrome). Useful if n-k is small, else want (and there exist) other more efficient decoding algorithms.

The d of linear codes

Theorem: Linear codes have distance d if
every set of (d-1) columns of **H** are linearly independent,
but there is a set of d columns that are linearly dependent.

Proof sketch: Ideas?

For linear codes, distance equals least weight
of non-zero codeword.
Each codeword gives some collection of columns that
must sum to zero.

Example and "Standard Form"

"Standard form" of G for systematic codes: $[I_k\, A]$.

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

(7,4,3) Hamming code

Relationship of G and H

Theorem: For binary codes, if G is in standard form $[I_k\ A]$ then $H = [-A^T\ I_{n-k}]$

Example of (7,4,3) Hamming code:

transpose

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Relationship of G and H

Proof:

Two parts to prove: (exercise)

1. Suppose that x is a message. Then $H(xG)^T = 0$.

2. Conversely, suppose that $Hy^T = 0$. Then y is a codeword.

Singleton bound

Theorem: For every $(n, k, d)_q$ code, $n \geq (k + d - 1)$

Another way to look at this: $\mathrm{d} \leq (n - k + 1)$

(We will not go into the proof of this theorem in this course due to limited time on this topic.)

Codes that meet Singleton bound with equality are called **Maximum Distance Separable (MDS)**

Maximum Distance Separable (MDS)

Only two binary MDS codes!

Q: What are they?

1. Repetition codes (k = 1)
2. Single-parity check codes (n-k = 1)

Need to go beyond the binary alphabet.

Finite fields!