15780: Graduate AI (Spring 2017)

Homework 4: Game Theory and Social Choice

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Due: May 4, 2017

1 Swap Regret and Correlated Equilibrium [25 points]

A *modification function* $F_i : S_i \to S_i$ for each player i is a function that maps each action in S_i to an action in S_i, where S_i is the strategy space of player i. Given a modification function F_i for player i, we define the *swap regret* of player i with respect to F_i as follows:

$$regret_i(s, F_i) = u_i(F_i(s_i), s_{-i}) - u_i(s_i, s_{-i}).$$

That is, $regret_i(s, F_i)$ measures the regret player i has for playing s_i rather than $F_i(s_i)$, in the strategy profile s.

For simplicity, consider a two-player game where $N = \{1, 2\}$. A distribution p over $S_1 \times S_2$ is an ϵ-*correlated equilibrium* if and only if the following conditions are met:

- $\sum_{s_1 \in S_1} \left(\max_{s_1' \in S_1} \Big(\sum_{s_2 \in S_2} p(s_1, s_2) u_1(s_1', s_2) - \sum_{s_2 \in S_2} p(s_1, s_2) u_1(s_1, s_2) \Big) \right) \leq \epsilon.$
- $\sum_{s_2 \in S_2} \left(\max_{s_2' \in S_2} \Big(\sum_{s_1 \in S_1} p(s_1, s_2) u_2(s_1, s_2') - \sum_{s_1 \in S_1} p(s_1, s_2) u_2(s_1, s_2) \Big) \right) \leq \epsilon.$

Assume players 1 and 2 played T games and each player followed strategies with swap regret at most R. That is, for each player and *every possible* modification function, the sum of regrets over T games was at most R. Let q be the empirical distribution of the joint actions played by the players. That is, if we let $(s_1^t, s_2^t) \in S_1 \times S_2$ be the strategy profile at each t-th game, then for each strategy profile $(s_1, s_2) \in S_1 \times S_2$, $q(s_1, s_2) = \frac{1}{T}|\{t \in [T] : (s_1, s_2) = (s_1^t, s_2^t)\}|$.

Prove that the distribution q over $S_1 \times S_2$ is an (R/T)-correlated equilibrium.

Hint: The definition of an ϵ-correlated equilibrium can be rewritten using swap regret.

2 Repeated Insertion Model and the Mallows Model [25 points]

Prove that the Mallows Model with parameter ϕ is equivalent to the Repeated Insertion Model (RIM) with $p_{ij} = \phi^{i-j} \frac{1-\phi}{1-\phi^i}$, where p_{ij} is the probability of the i^{th} element being inserted in the j^{th} spot, for $i \geq j$.

Hints: As a first step, note that the RIM is equivalent to generating a vector v of m elements, where the i^{th} element is an integer in $[1, i]$ corresponding to the location at which the i^{th} ranked alternative in the true ranking is inserted in the current subranking. It may be useful to first consider the relationship between the insertion vector and the Kendall tau distance between the resulting ranking and the true ranking. In addition, you may use the fact that given a true ranking $\succ$ over m alternatives and the space of all rankings over the same alternatives Π, $(1 + \phi)(1 + \phi + \phi^2) \cdots (1 + \phi + \cdots + \phi^{m-1}) = \sum_{\succ' \in \Pi} \phi^{d_{KT}(\succ, \succ')}$.

3 Strategyproof Social Choice on a Tree [25 points]

Recall that we informally proved in class that given the single peaked preferences of a set of voters on a line, selecting the median of the points is both strategyproof and Condorcet consistent. In this problem, you will design a social choice function for a slightly more general setting that is both strategyproof and Condorcet consistent.

A city wants to build a new library, and they want to elicit the residents' preferences to determine where to build it. The locations that the city is able to build the library can be represented by a set of vertices V

which lie on a tree $T = (V, E)$ (for example, the root of the tree might be located in downtown while the leaves are on the fringes of the city).

- There are an odd number of residents (voters), indicated by the set N.
- Each voter $i \in N$ has a most preferred vertex $v_i \in V$.
- Let $d(x, y)$ denote the length (in terms of number of edges) of the unique path between vertices x and y in T. Given two vertices u and w, if $d(v_i, u) < d(v_i, w)$, then voter i prefers u to w.

Write a social choice mechanism (i.e., a function) that takes as input the most preferred vertex of each voter and outputs a single vertex $v \in V$. Your proposed mechanism should be strategyproof and Condorcet consistent. Formally prove that these properties are satisfied.

4 Programming: Stackelberg Strategies [25 points]

In a 2-player normal form game, a Stackelberg strategy is where one of the players is a leader and the other is a follower. In contrast to the default situation where both players pick their respective strategies at the same time, a Stackelberg strategy is when the leader, which is identified as player 1, first commits to a (mixed) strategy which the follower, player 2, knows. Then player 2 commits to his own strategy using his knowledge of player 1's strategy.

An optimal Stackelberg strategy would be a Stackelberg strategy where player 1's expected utility is maximized. The optimal Stackelberg strategy can be computed in polynomial time by solving multiple LPs. See Slide 14 of Lecture 19 for a description of the algorithm.

Given a 2-player normal form game, you will implement the function `stackelberg(u1, u2)` in `stackelberg.py` which will return the optimal Stackelberg strategy for the given game. You can use `cvxopt` or `cvxpy` to solve the LPs. Your function should return numpy arrays, not datatypes from these libraries. If there is a tie in the expected utility between two strategies of player one (i.e. the expected utility is off by an absolute error of 1e-5), you should return the optimal strategy induced by the lowest indexed pure strategy of player 2.

5 Submitting to Autolab

Create a tar file containing your writeup and the completed `stackelberg.py` modules for the programming problems. Make sure that your tar has these files at the root and not in a subdirectory. Use the following commands from a directory with your files to create a `handin.tgz` file for submission.

```
$ ls
stackelberg.py  writeup.pdf  [...]
$ tar cvzf handin.tgz writeup.pdf stackelberg.py
a writeup.pdf
a stackelberg.py
$ ls
handin.tgz  stackelberg.py  writeup.pdf  [...]
```