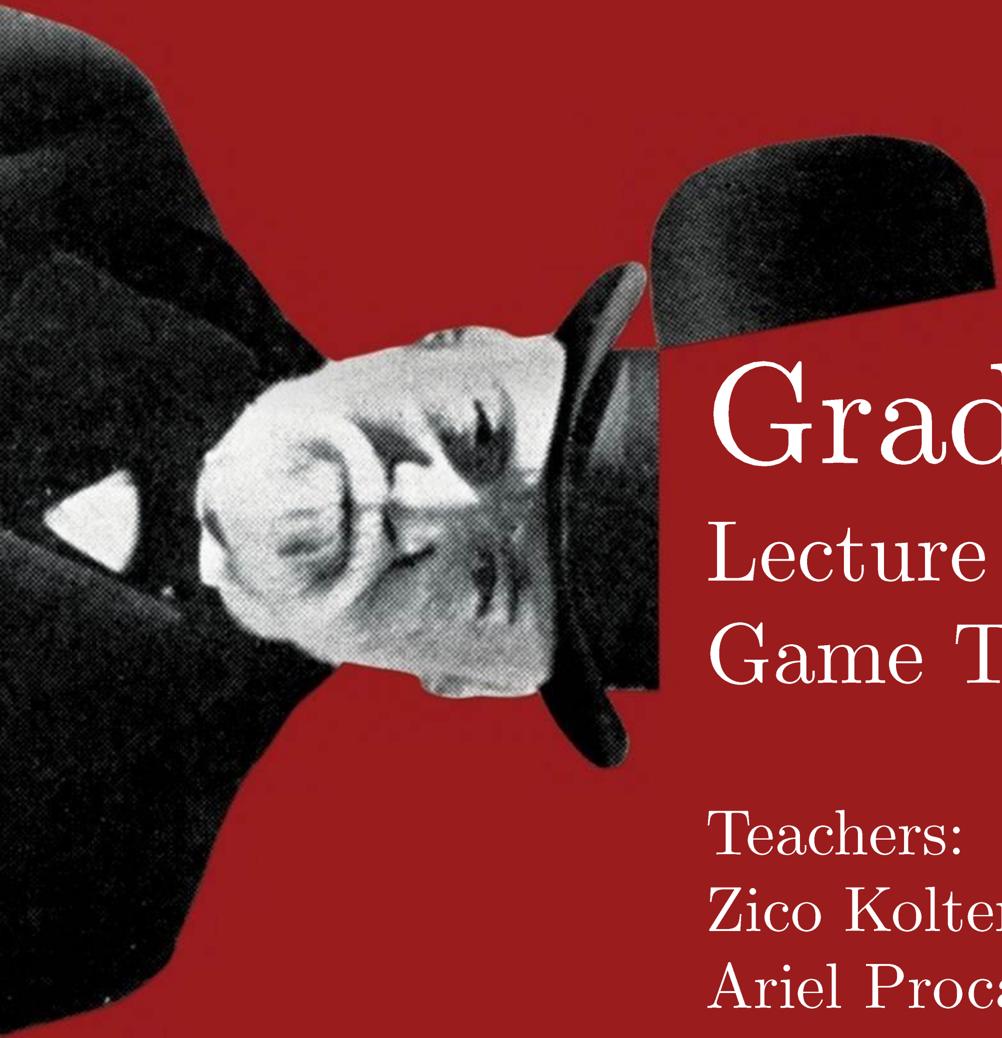


Graduate AI

Lecture 21:
Game Theory IV

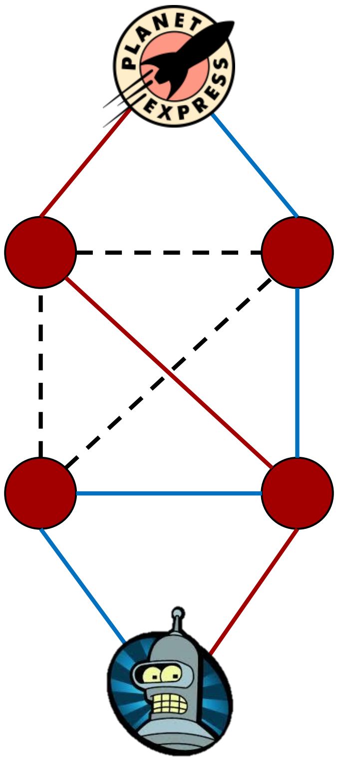
Teachers:
Zico Kolter
Ariel Procaccia (this time)

REMINDER: THE MINIMAX THEOREM

- Theorem [von Neumann, 1928]: Every 2-player zero-sum game has a unique value v such that:
  - Player 1 can guarantee value at least v
  - Player 2 can guarantee loss at most v
- We will prove the theorem via no-regret learning

HOW TO REACH YOUR SPACESHIP

- Each morning pick one of n possible routes
- Then find out how long each route took
- Is there a strategy for picking routes that does almost as well as the best fixed route in hindsight?



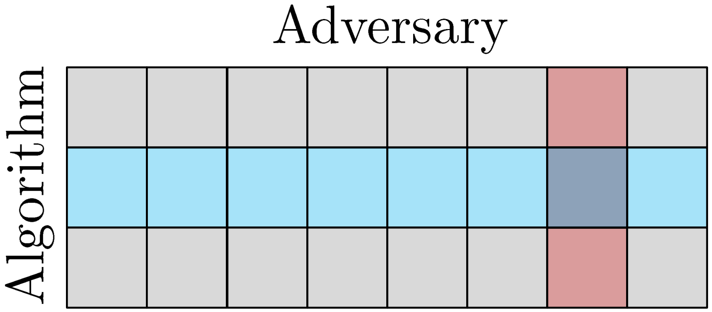
53 minutes

47 minutes

…

THE MODEL

- View as a matrix (maybe infinite #columns)



- Algorithm picks row, adversary column
- Alg pays cost of (row,column) and gets column as feedback
- Assume costs are in [0,1]

THE MODEL

- Define average regret in T time steps as (average per-day cost of alg) $-$ (average per-day cost of best fixed row in hindsight)
- No-regret algorithm: regret $\to 0$ as $T \to \infty$
- Not competing with adaptive strategy, just the best fixed row

EXAMPLE

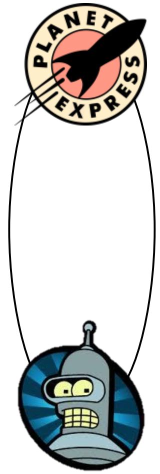
- **Algorithm 1:** Alternate between U and D
- **Poll 1:** What is algorithm 1's worst-case average regret?

1. $\Theta(1/T)$

[✓ 2.] $\Theta(1)$

3. $\Theta(T)$

4. ∞



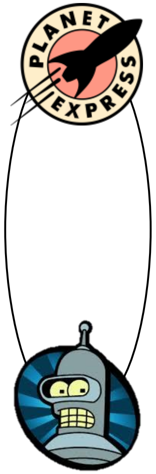
$$\begin{array}{c|c|c|} & \multicolumn{2}{c}{\text{Adversary}} \\ \hline \text{Algorithm} & 1 & 0 \\ \hline & 0 & 1 \\ \hline \end{array}$$

EXAMPLE

- **Algorithm 2:** Choose action that has lower cost so far
- **Poll 2:** What is algorithm 2's worst-case average regret?

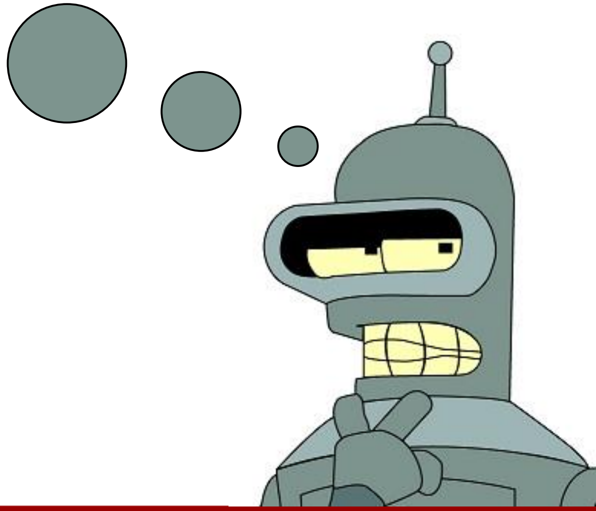
1. $\Theta(1/T)$
2. $\Theta(1/\sqrt{T})$
3. $\Theta(1/\log T)$

[✓ 4.] $\Theta(1)$

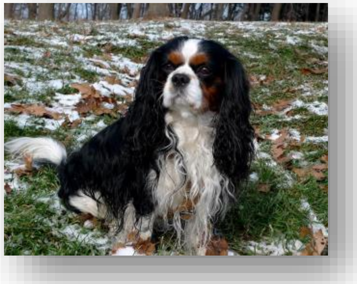


	Adversary	
Algorithm	1	0
	0	1

What can we say more generally about deterministic algorithms?



USING EXPERT ADVICE



- Want to predict the stock market
- Solicit advice from n experts
  - Expert = someone with an opinion

Day	Expert 1	Expert 2	Expert 3	Charlie	Truth
1	−	−	+	+	+
2	+	−	+	−	−
…	…	…	…	…	…

- Can we do as well as best in hindsight?

Simpler question

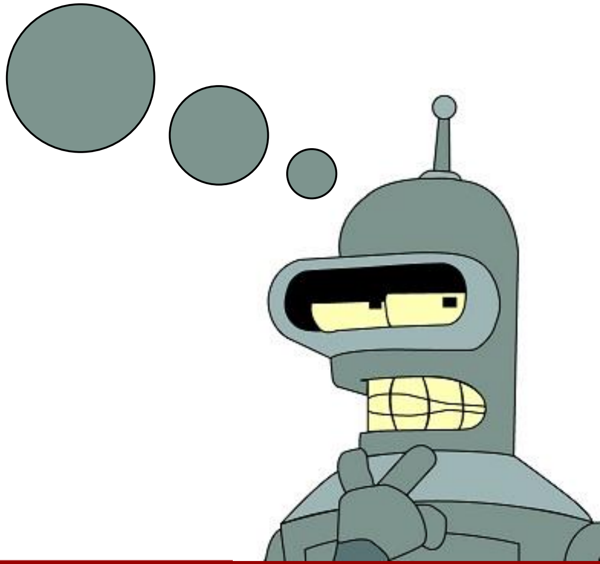
- One of the n experts never makes a mistake
- We want to find out which one
- **Algorithm 3:** Take majority vote over experts that have been correct so far
- **Poll 3:** What is algorithm 3's worst-case number of mistakes?
  1. $\Theta(1)$
  [✓ 2.] $\Theta(\log n)$
  3. $\Theta(n)$
  4. $\Theta(2^n)$

WHAT IF NO EXPERT IS PERFECT?

- Idea: Run algorithm 3 until all experts are crossed off, then repeat
- Makes at most $\log n$ mistakes per mistake of the best expert
- But this is wasteful: we keep forgetting what we've learned

Reprise:

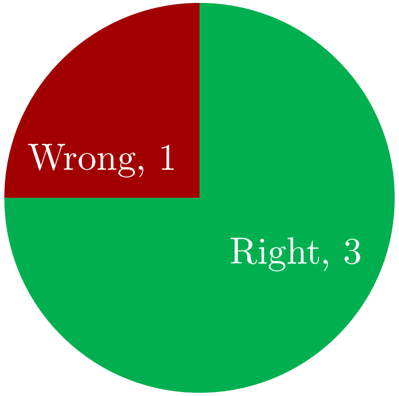
Algorithms that forget their history are doomed to repeat it!

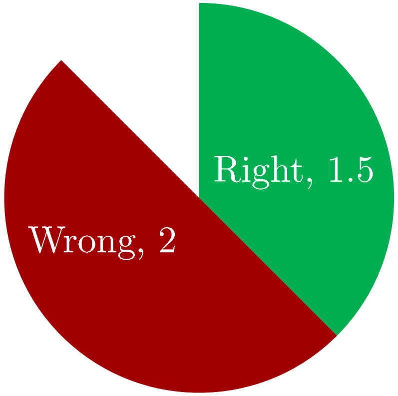


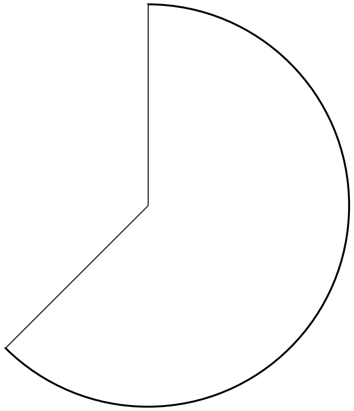
WEIGHTED MAJORITY

- Intuition: Making a mistake doesn't disqualify an expert, just lowers its weight
- Weighted Majority Algorithm:
  - Start with all experts having weight 1
  - Predict based on weighted majority vote
  - Penalize mistakes by cutting weight in half

	Expert 1	Expert 2	Expert 3	Charlie	Alg	Truth
Weights	1	1	1	1		
Prediction	−	+	+	+	+	+
Weights	0.5	1	1	1		
Prediction	+	+	−	−	−	+
Weights	0.5	1	0.5	0.5		







WEIGHTED MAJORITY: ANALYSIS

- M = #mistakes we've made so far
- m = #mistakes of best expert so far
- W = total weight (starts at n)
- For each mistake, W drops by at least 25%
  $\Rightarrow$ after M mistakes: $W \le n(3/4)^M$
- Weight of best expert is $(1/2)^m$

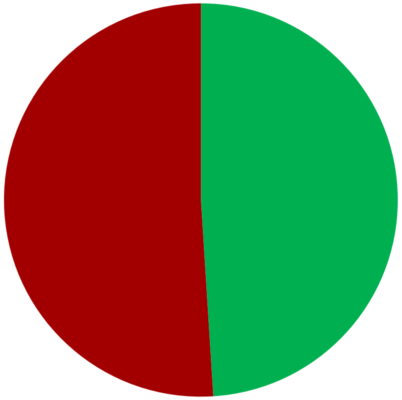
$$\left(\frac{1}{2}\right)^m \le n\left(\frac{3}{4}\right)^M \Rightarrow \left(\frac{4}{3}\right)^M \le n2^m \Rightarrow M \le 2.5(m + \lg n)$$

RANDOMIZED WEIGHTED MAJORITY

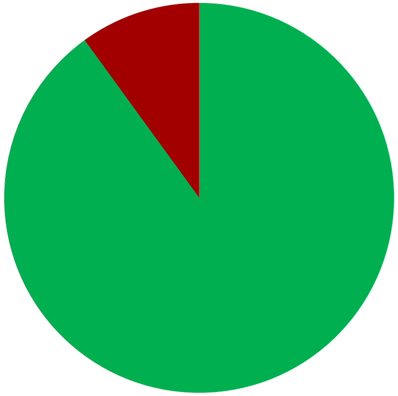
- Randomized Weighted Majority Algorithm:
  - Start with all experts having weight 1
  - Predict proportionally to weights: the total weight of + is w_+ and the total weight of − is w_-, predict + with probability $\frac{w_+}{w_+ + w_-}$ and − with probability $\frac{w_-}{w_+ + w_-}$
  - Penalize mistakes by removing ϵ fraction of weight

Randomized Weighted Majority

Idea: smooth out the worst case



The worst-case is ~50-50: now we have a 50% chance of getting it right



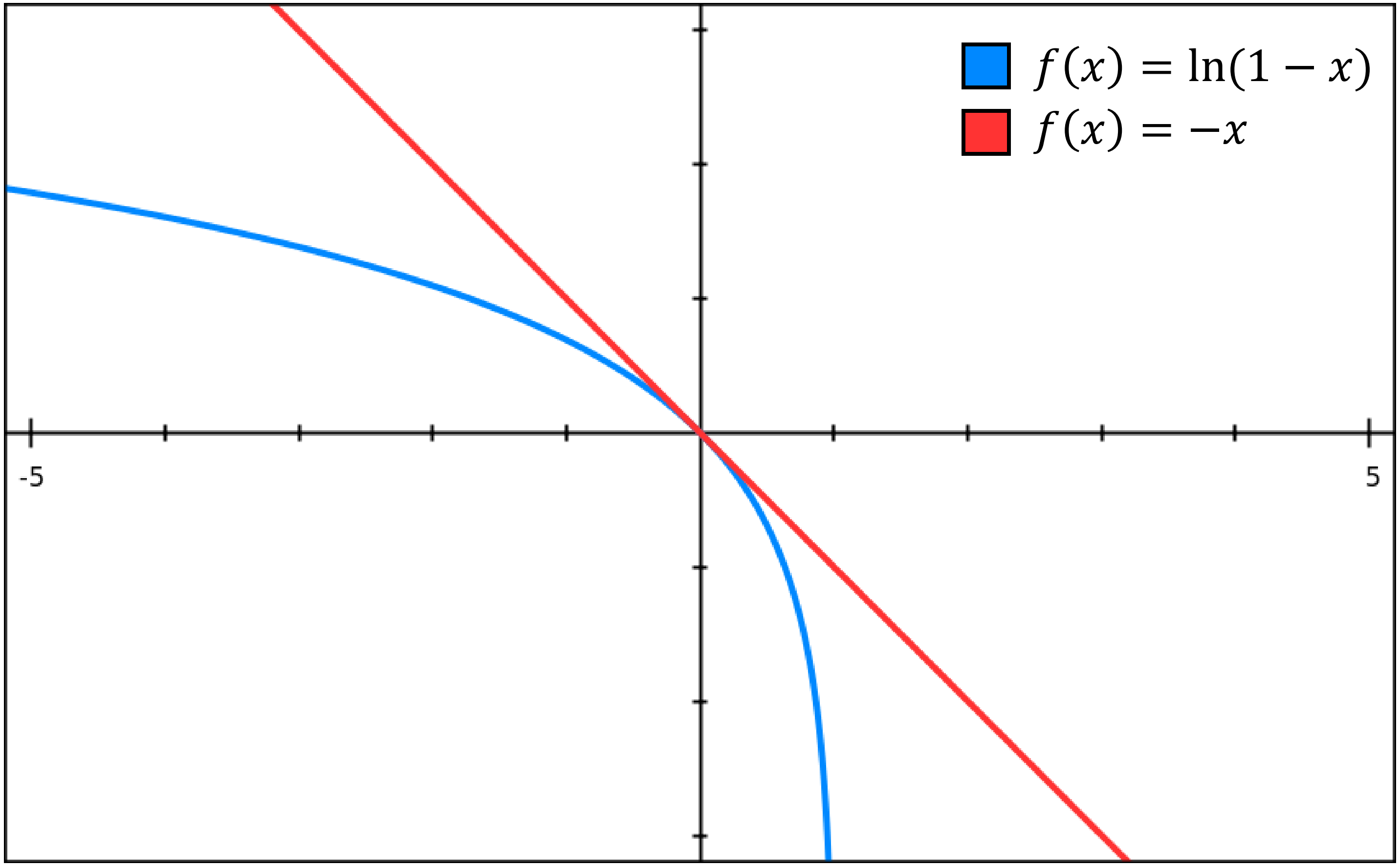
What about 90-10? We're very likely to agree with the majority

ANALYSIS

- At time t we have a fraction F_t of weight on experts that made a mistake
- Prob. F_t of making a mistake, remove ϵF_t fraction of total weight
- $W_{final} = n \prod_t (1 - \epsilon F_t)$
- $\ln W_{final} = \ln n + \sum_t \ln(1 - \epsilon F_t)$

$$\underset{\substack{\uparrow \\ \ln(1-x) \le -x \\ \text{(next slide)}}}{\le} \ln n - \epsilon \sum_t F_t = \ln n - \epsilon M$$

ANALYSIS



ANALYSIS

- Weight of best expert is $W_{best} = (1-\epsilon)^m$
- $\ln n - \epsilon M \geq \ln W_{final} \geq \ln W_{best} = m \ln(1-\epsilon)$
- By setting $\epsilon = \sqrt{\ln n / m}$ and solving, we get
$$M \leq m + 2\sqrt{m \ln n}$$
- Since $m \leq T$, $M \leq m + 2\sqrt{T \ln n}$
- Average regret is $\left(2\sqrt{T \ln n}\right)/T \to 0$ ■

MORE GENERALLY

- Each expert is an action with cost in [0,1]
- Run Randomized Weighted Majority
  - Choose expert i with probability w_i/W
  - Update weights: $w_i \leftarrow w_i(1 - c_i\epsilon)$
- Same analysis applies:
  - Our expected cost: $\sum_j c_j w_j/W$
  - Fraction of weight removed: $\epsilon \sum_j c_j w_j/W$
  - So, fraction removed $= \epsilon \cdot$ (our cost)

PROOF OF THE MINIMAX THM

- Suppose for contradiction that zero-sum game G has $V_C > V_R$ such that:
    - If column player commits first, there is a row that guarantees row player at least V_C
    - If row player commits first, there is a column that guarantees row player at most V_R
- Scale matrix so that payoffs to row player are in $[-1,0]$, and let $V_C = V_R + \delta$

PROOF OF THE MINIMAX THM

- Row player plays RWM, and column player responds optimally to current mixed strategy
- After T steps
    - ALG $\geq$ best row in hindsight $-2\sqrt{T\log n}$
    - ALG $\leq T \cdot V_R$
    - Best row in hindsight $\geq T \cdot V_C$
- It follows that $T \cdot V_R \geq T \cdot V_C - 2\sqrt{T\log n}$
- $\delta T \leq 2\sqrt{T\log n}$ – contradiction for large enough T ■

SUMMARY

- Terminology:
    - Regret
    - No-regret learning
- Algorithms:
    - Randomized weighted majority
- Big ideas:
    - It is possible to achieve no-regret learning guarantees!
    - Connections between game theory and learning theory