

# Graduate AI

Lecture 24:

Social Choice III

Teachers:

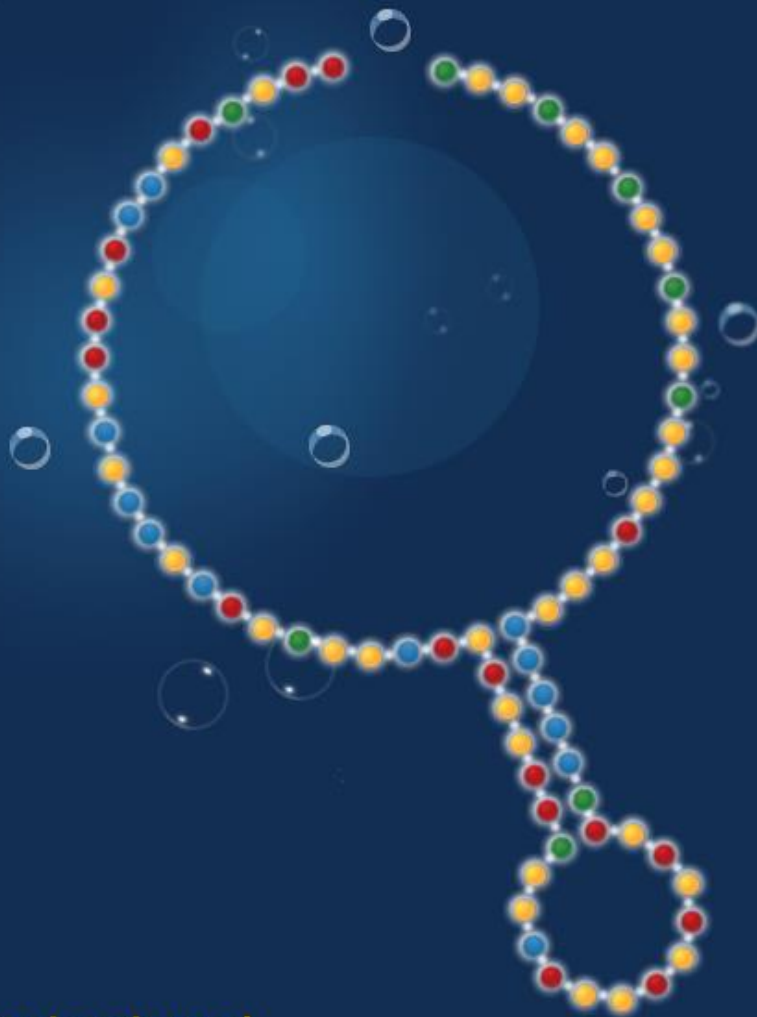
Zico Kolter

Ariel Procaccia (this time)

# CONDORCET STRIKES AGAIN

- For Condorcet [1785], the purpose of voting is not merely to balance subjective opinions; it is a collective quest for the truth
- Enlightened voters try to judge which alternative best serves society
- For  $m = 2$  the majority opinion will very likely be correct
- Realistic in trials by jury, but also in the pooling of expert opinions, or in human computation!





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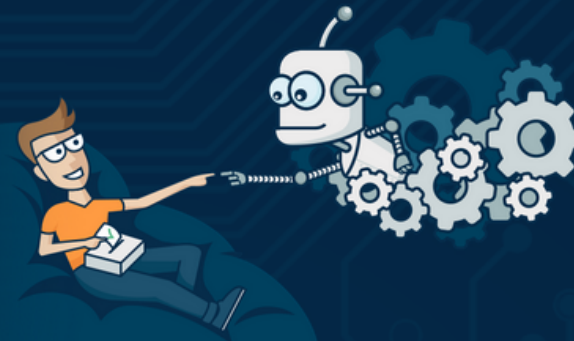
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## AI-Driven Decisions

RoboVote is a free service that helps users combine their preferences or opinions into optimal decisions. To do so, RoboVote employs state-of-the-art voting methods developed in artificial intelligence research. [Learn More](#)



## Poll Types

RoboVote offers two types of polls, which are tailored to different scenarios; it is up to users to indicate to RoboVote which scenario best fits the problem at hand.



### Objective Opinions

In this scenario, some alternatives are objectively better than others, and the opinion of a participant reflects an attempt to estimate the correct order. RoboVote's proposed outcome is guaranteed to be as close as possible — based on the available information — to the best outcome. Examples include deciding which product prototype to develop, or which company to invest in, based on a metric such as projected revenue or market share. [Try the demo.](#)



### Subjective Preferences

In this scenario participants' preferences reflect their subjective taste; RoboVote proposes an outcome that mathematically makes participants as happy as possible overall. Common examples include deciding which restaurant or movie to go to as a group, which destination to choose for a family vacation, or whom to elect as class president. [Try the demo.](#)

Ready to get started?

CREATE A POLL

# CONDORCET'S NOISE MODEL

- True ranking of the alternatives
- Voting pairwise on alternatives, each comparison is correct with prob.  $p > 1/2$
- Results are tallied in a voting matrix

	<i>a</i>	<i>b</i>	<i>c</i>
<i>a</i>	-	8	6
<i>b</i>	5	-	11
<i>c</i>	7	2	-

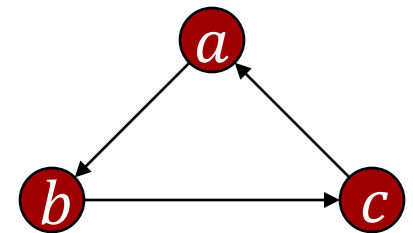
- **Poll 1:** What is the Borda score of alternative *b*?
  1. 5
  2. 8
  3. 10
  4. 16



# CONDORCET'S 'SOLUTION'

- Condorcet's goal: find “the most probable” ranking
- Condorcet suggested: take the majority opinion for each comparison; if a cycle forms, “successively delete the comparisons that have the least plurality”
- In example, we delete  $c \succ a$  to get  $a \succ b \succ c$

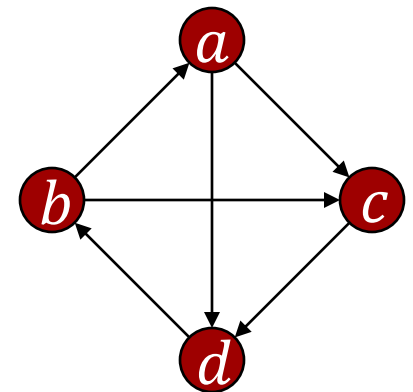
	<i>a</i>	<i>b</i>	<i>c</i>
<i>a</i>	-	8	6
<i>b</i>	5	-	11
<i>c</i>	7	2	-



# CONDORCET'S 'SOLUTION'

- With four alternatives we get ambiguities
- In example, order of strength is  $c \succ d$ ,  $a \succ d$ ,  $b \succ c$ ,  $a \succ c$ ,  $d \succ b$ ,  $b \succ a$
- Delete  $b \succ a \Rightarrow$  still cycle
- Delete  $d \succ b \Rightarrow$  either  $a$  or  $b$  could be top-ranked

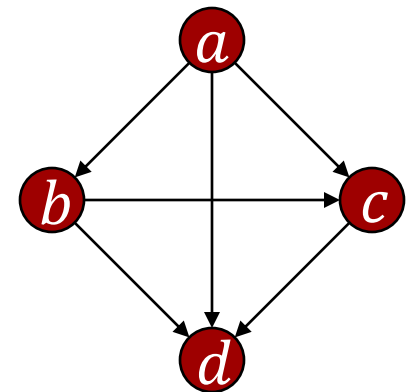
	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>a</i>	-	12	15	17
<i>b</i>	13	-	16	11
<i>c</i>	10	9	-	18
<i>d</i>	8	14	7	-



# CONDORCET'S 'SOLUTION'

- Did Condorcet mean we should **reverse** the weakest comparisons?
- Reverse  $b > a$  and  $d > b \Rightarrow$  we get  $a > b > c > d$ , with 89 votes
- $b > a > c > d$  has 90 votes (only reverse  $d > b$ )

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>a</i>	-	12	15	17
<i>b</i>	13	-	16	11
<i>c</i>	10	9	-	18
<i>d</i>	8	14	7	-





# EXASPERATION?

- “The general rules for the case of any number of candidates as given by Condorcet are stated so briefly as to be hardly intelligible . . . and as no examples are given it is quite hopeless to find out what Condorcet meant” [Black 1958]
- “The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ... **no amount of examples can convey an adequate impression of the evils**” [Todhunter 1949]



# YOUNG'S SOLUTION

- $M$  = matrix of votes
- Suppose true ranking is  $a \succ b \succ c$ ;  
prob of observations  $\Pr[M \mid \succ]$ :

$$\binom{13}{8} p^8 (1-p)^5 \cdot \binom{13}{6} p^6 (1-p)^7 \cdot \binom{13}{11} p^{11} (1-p)^2$$

- For  $a \succ c \succ b$ ,  $\Pr[M \mid \succ]$  is

$$\binom{13}{8} p^8 (1-p)^5 \cdot \binom{13}{6} p^6 (1-p)^7 \cdot \binom{13}{2} p^2 (1-p)^{11}$$

- Coefficients are identical, so  
 $\Pr[M \mid \succ] \propto p^{\#agree} (1-p)^{\#disagree}$

	$a$	$b$	$c$
$a$	-	8	6
$b$	5	-	11
$c$	7	2	-



# YOUNG'S SOLUTION

- $\Pr[\succ | M] = \frac{\Pr[M|\succ] \cdot \Pr[\succ]}{\Pr[M]}$
- Assume uniform prior over  $\succ$ ,  $\Pr[\succ] = \frac{1}{m!}$
- Must maximize  $\Pr[M | \succ]$
- The optimal rule maximizes #agreements with voters on pairs of alternatives
- This rule is called the **Kemeny rule**



# THE KEMENY RULE

- The **Kendall tau** distance between  $\succ$  and  $\succ'$  is
$$d_{KT}(\succ, \succ') = |\{(a, b) \in A^2 \mid (a \succ b) \wedge (b \succ' a)\}|$$
- The Kemeny rule chooses the ranking that minimizes the sum of Kendall tau distances to the preference profile
- **Theorem [Bartholdi, Tovey, Trick 1989]:**  
Computing the Kemeny ranking is NP-hard



# THE KEMENY RULE

- Typically formulated as an IP: for every  $a, b \in A$ ,  
 $x_{(a,b)} = 1$  iff  $a$  is ranked above  $b$ , and  
 $w_{(a,b)} = |\{i \in N \mid a \succ_i b\}|$

Minimize  $\sum_{(a,b)} x_{(a,b)} w_{(b,a)}$

Subject to

For all distinct  $a, b \in A$ ,  $x_{(a,b)} + x_{(b,a)} = 1$

For all distinct  $a, b, c \in A$ ,  $x_{(a,b)} + x_{(b,c)} + x_{(c,a)} \leq 2$

For all distinct  $a, b \in A$ ,  $x_{(a,b)} \in \{0,1\}$

# THE MALLOWS MODEL

- Same as Condorcet's model, but votes are rankings
- Defined by parameter  $\phi \in (0,1]$
- Probability of a voter casting the vote  $\succ'$  given true ranking  $\succ$  is

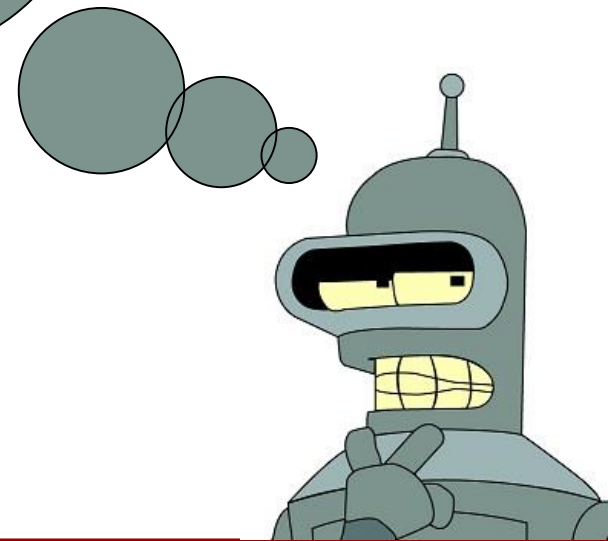
$$\Pr[\succ' \mid \succ] = \frac{\phi^{d_{KT}(\succ', \succ)}}{\sum_{\succ''} \phi^{d_{KT}(\succ'', \succ)}}$$

- Kemeny still gives the MLE ranking



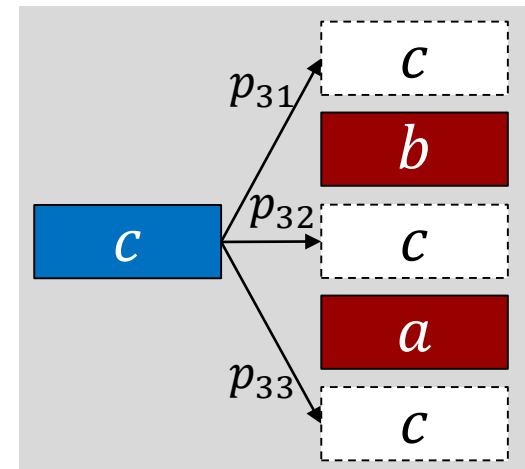
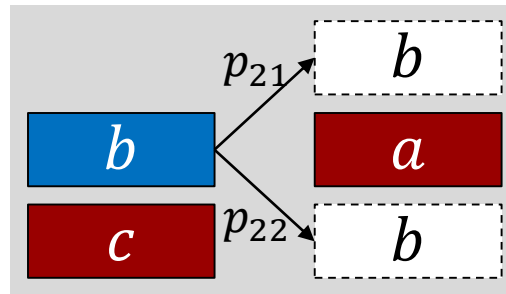
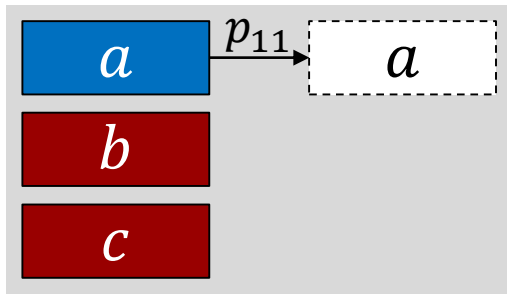
$$\Pr[\succ' \mid \succ] = \frac{\phi^{d_{KT}(\succ', \succ)}}{\sum_{\succ''} \phi^{d_{KT}(\succ'', \succ)}}$$

What is the relation between  $\phi$  in the Mallows model, and  $p$  in the Condorcet model?



# THE MALLOWS MODEL

- How can we sample a vote?
- Suppose the true ranking is  $a \succ b \succ c$
- Repeated insertion model:



- Theorem [Doignon et al. 2004]:

By setting  $p_{ij} = \phi^{i-j} \cdot \frac{1-\phi}{1-\phi^i}$  for  $j \leq i \leq m$ , RIM induces the same distribution over rankings as Mallows





# IS MALLOWS REALISTIC?

Drag these down to the gray area below.

5	7	2
8	1	3
	4	6

A

7	4	6
1		2
8	5	3

B

7	5	1
2	3	6
8		4

C

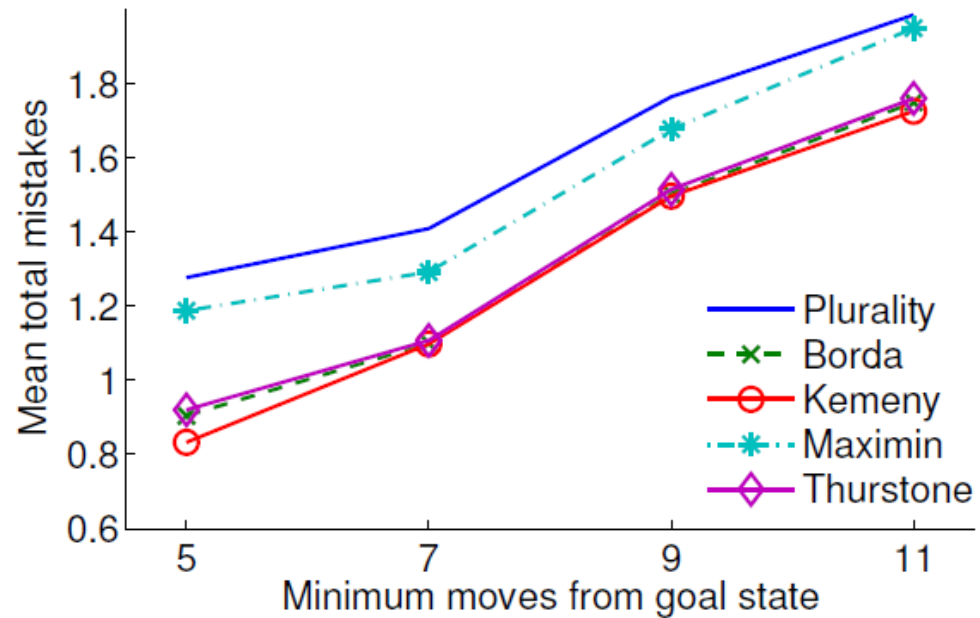
Drop here! Continue to rearrange the order by dragging and dropping until you're satisfied.

2	4	3
7	5	
8	1	6

D

Closest to solution  
(Fewest moves)

Furthest from solution  
(Most moves)



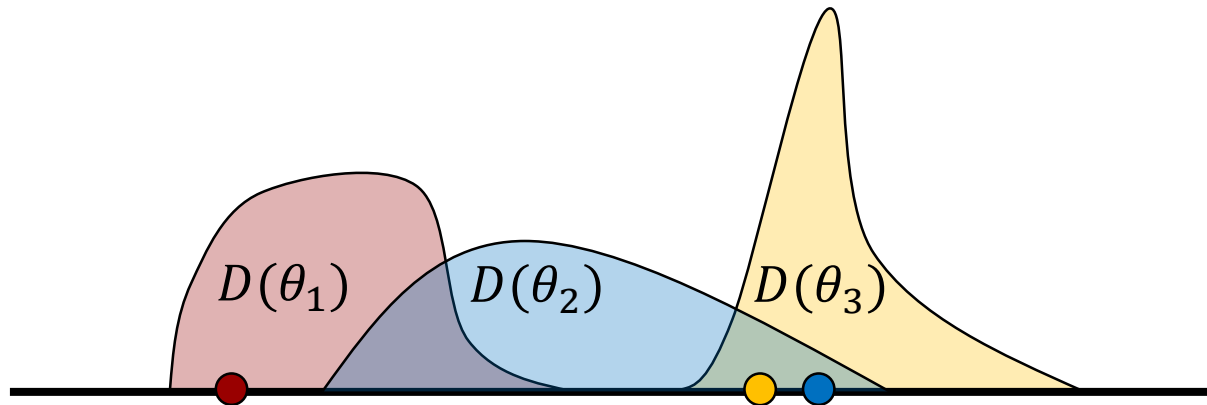
[Mao et al. 2013]

# RANDOM UTILITY MODELS

- Parameters  $\theta = (\theta_1, \dots, \theta_m)$ 
  - $m$  = number of alternatives
  - Each alternative  $x_j$  modeled by **utility distribution**  $D(\theta_j)$
- A voter's **utility**  $U_j$  for alternative  $x_j$  is drawn independently from  $D(\theta_j)$
- Voters rank alternatives by  $U_1, \dots, U_m$ :

$$\Pr[x_2 \succ x_1 \succ x_3 \mid \theta_1, \theta_2, \theta_3] = \Pr_{U_j \sim D(\theta_j)} [U_2 > U_1 > U_3]$$

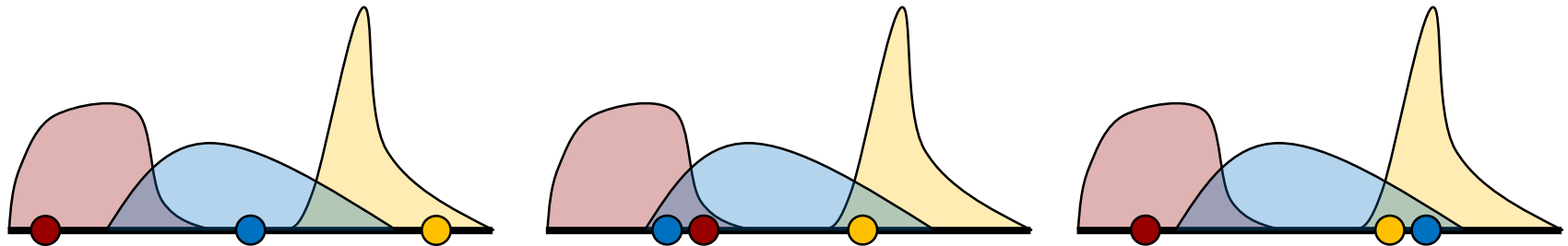
# RANDOM UTILITY MODELS



Generating a single vote

$$x_2 \succ x_3 \succ x_1$$

# RANDOM UTILITY MODELS



Voter 1

$$x_3 \succ x_2 \succ x_1$$

Voter 2

$$x_3 \succ x_1 \succ x_2$$

Voter 3

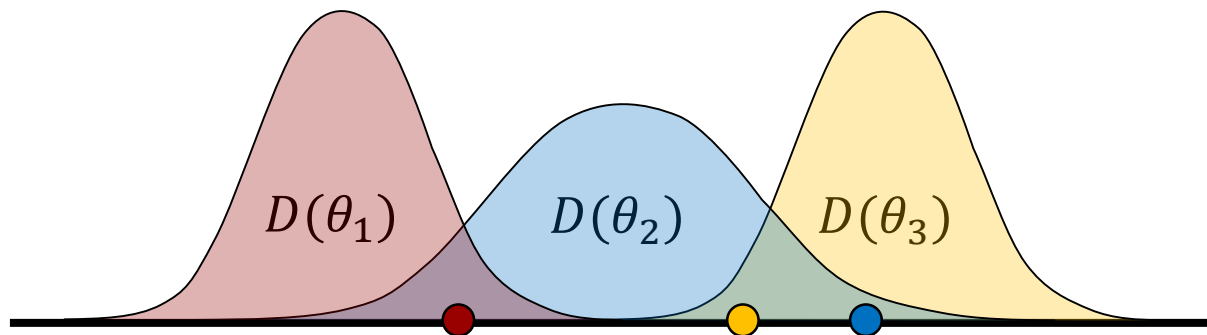
$$x_2 \succ x_3 \succ x_1$$

Generating a preference profile

$$\Pr[\succ_1, \dots, \succ_n \mid \boldsymbol{\theta}] = \prod_{i \in N} \Pr[\succ_i \mid \boldsymbol{\theta}]$$

# THE THURSTONE MODEL

- Defined by a normal distribution
  - For each  $x_j$ ,  $\theta_j = (\mu_j, \sigma_j)$
  - $D(\theta_j) = \mathcal{N}(\mu_j, \sigma_j^2)$
- Computing  $\Pr[\succ | \theta]$  believed to be hard



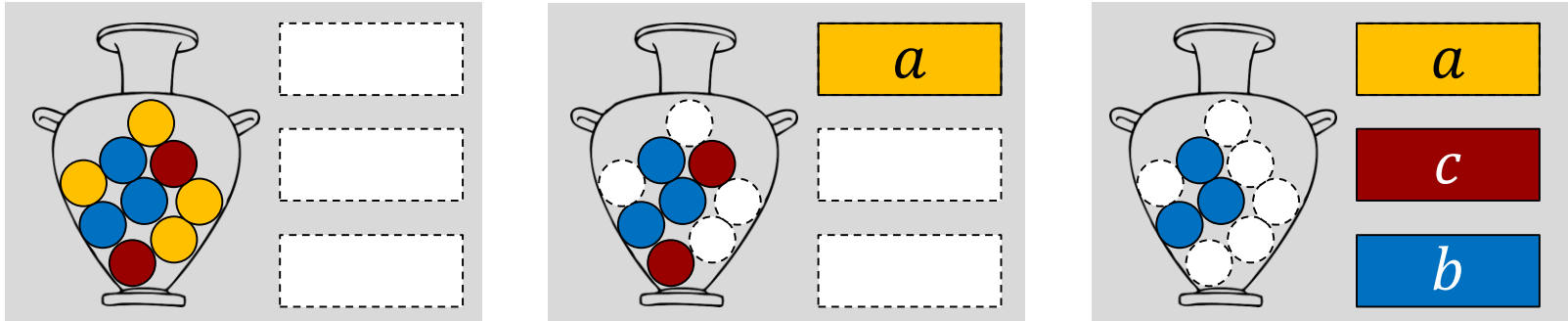
# THE PLACKETT-LUCE MODEL

- Defined by a Gumbel distribution
  - For each  $x_j$ ,  $\theta_j = (\mu_j, \beta_j)$
  - $D(\theta_j) = \mathcal{G}(\mu_j, \beta_j)$
- Equivalently, there exist weights  $w_1, \dots, w_m$  such that  $\Pr[x_{j_1} > x_{j_2} \cdots > x_{j_m} \mid \mathbf{w}]$  is given by

$$\frac{w_{j_1}}{w_{j_1} + \cdots + w_{j_m}} \cdot \frac{w_{j_2}}{w_{j_2} + \cdots + w_{j_m}} \cdots \frac{w_{j_{m-1}}}{w_{j_{m-1}} + w_{j_m}}$$



# THE PLACKETT-LUCE MODEL



Urn interpretation

$$\Pr[a \succ c \succ d \mid (4,3,2)] = \frac{4}{9} \cdot \frac{2}{5}$$

# BEYOND SOCIAL CHOICE

- We previously interpreted pairwise comparisons as voters comparing alternatives
- But these comparisons can be the results of competitions between players
- In these situations, we typically want to update our estimates of player ratings **online**
- The famous **Elo system** originally used the Thurstone model







TrueSkill™ system used to rank Halo players  
Also based on the Thurstone model  
[Herbrich et al. 2006]



# SUMMARY

- Terminology:
  - Models: Condorcet, Mallows, random insertion, Thurstone, Plackett-Luce
  - Kendall tau distance
  - The Kemeny rule
- Algorithms:
  - IP for Kemeny
- Big ideas:
  - Voting as search for truth

