# 21-355 Advanced Calculus I
# Exam #1.

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Let the non-empty metric spaces $(X, d)$ and $(X', d')$ and the mapping $f : X \to X'$ be given.

**1.1 Theorem.** *The function* $d^* : X \times X' \to [0, \infty[$ *defined by* $d^*((x, x'), (y, y')) := \max\{d(x, y), d'(x', y')\}$ *for all* $(x, x'), (y, y') \in X \times X'$ *is a metric on* $X \times X'$.

**Proof** In order to establish that $d^*$ is a metric, we need to show that is adheres to the three properties of a metric.

1. $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$ : Because $d^*$ relies solely on $d$ and $d'$, we know that since they are both metrics, they evaluate to positive values only. Therefore, the max of them must be positive. For the case when $p = q$, we know both $d$ and $d'$ will be zero, and thus the max will be zero as well, satisfying the condition.
2. $d(p, q) = d(q, p)$ : Again since $d^*$'s behaviour is completely determined by the established metrics $d$ and $d'$ we know that this will be satisfied. The max value between $d$ and $d'$ will be the same for $(p, q)$ as for $(q, p)$.
3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$ : Without loss of generality, say that $d$ is used as the max for $d^*(p, q)$. We know then from the fact that $d$ is a metric that:

$$d(p, q) \leq d(p, r) + d(r, q) \tag{1}$$

Applying $d$ to $p$ in this case means just take the $d$ distance instead of the max between the $d$ and $d'$ distance. Now supposing that for $(p, r)$ it turns out that $d'$ is the max. The right-hand side of (1) increases if this is the case and the relation still holds. The same argument applies if $(r, q)$ uses the $d'$ distance. The right-hand side goes up again, and the relationship is still intact. Note that this argument is completely symmetric, we could have picked the $(p, q)$ distance to be the distance obtained from $d'$ and the argument flows exactly the same way. So we have $d(p, q) \leq d^*(p, r) + d^*(r, q)$ and $d'(x) \leq d^*(p, r) + d^*(r, q)$ which implies $d^*(x) \leq d^*(p, r) + d^*(r, q)$.

Now since $d^*$ satisfies all three properties, we can say that it is a metric on $X \times X'$. □

**1.2 Theorem.** *The mapping* $p : X \times X' \to X$ *defined by*

$$p((x, x')) := x \quad \forall (x, x') \in X \times X' \tag{2}$$

*is uniformly continuous. Furthermore,* $f$ *is continuous if and only if the mapping* $g : X \to X \times X'$ *defined by*

$$g(x) := (x, f(x)) \quad \forall x \in X \tag{3}$$

*is continuous.*

**Proof** We first prove the first statement. In order to show uniform continuity, we need to show that for any $\epsilon > 0$,

$$d(p(s), p(t)) < \epsilon \tag{4}$$

Whenever $d^*(s,t) < \delta$ for all $s = (x, x'), t = (y, y') \in X \times X'$. Let $\epsilon > 0$ be given. Letting $\delta = \epsilon/2$ satisfies condition (4). This is seen to be true by noting that for any $s$ and $t$ s.t $d^*(s,t) < \delta$, $d(x,y) < \delta = \epsilon/2 < \epsilon$. Note that the $d$ distance is all that matters because of the nature of $p$. $p$ evaluates to the element from $X$, so if the distance between these two elements, $s$, $t$, is less than $\epsilon$, then $d(p(s), p(t))$ must be less than $\epsilon$ as well. □

To prove the next statement we look first at the "if" part, and then prove the converse.

In order to show $g(x)$ is continuous at a point $p$, we need to prove the following: For all $\epsilon > 0$, there exists a $\delta > 0$ s.t:

$$d^*(g(p), g(x)) < \epsilon \tag{5}$$

For all $x \in X$ s.t. $d(p,x) < \delta$. Now since $f$ is continuous we know there exists a $\delta_f$ s.t. for all $\epsilon > 0$, $d'(f(p), f(x)) < \epsilon$ for all $x \in X$ s.t. $d(p,x) < \delta_f$. Using this, we introduce $\delta_g = \min\{\epsilon/2, \delta_f\}$. The claim now is that $\delta_g$ satisfies (5). To see that this $\delta_g$ works, we note that $d^*(g(p), g(x))$ uses either $d$ or $d'$ as the metric depending on which is larger. If it uses $d$, then since $d(p,x) < \delta_g \leq \epsilon/2$, then $d^*(g(x), g(p)) < \epsilon$. If we use $d'$ instead for $d^*$, then since $d(p,x) < \delta_g \leq \delta_f$, $d^*(g(p), g(x)) < \epsilon$ by the continuity of $f$. Therefore $g(x)$ is continuous at all points $p \in X$.

Now we show the converse. Since $g(x)$ is continuous at any point $p$, we know of the existence of a $\delta_g$ s.t for all $\epsilon > 0$, $d^*(g(p), g(x)) < \epsilon$ for all $x \in X$ s.t $d(p,x) < \delta_g$. The key observation now is

$$d^*(g(p), g(x)) \geq d'(f(p), f(x)) \tag{6}$$

This is easily seen to be true by noting that $d^*$ takes the max of the two metrics $d$ and $d'$, therefore the value of $d^*$ must be greater than or equal to the $d'$ distance. (6) allows us to say that for any $\epsilon > 0$, $d'(f(p), f(x)) < \epsilon$ for all $x \in X$ s.t $d(p,x) < \delta_g$. Therefore $f$ is continuous at all points $p \in X$. □

**1.3 Theorem.** *The subset $G := \{(x, f(x)) \mid x \in X\}$ of $X \times X'$ is called the* **graph of** *$f$. The following statements are equivalent:*
*(1): $G$ is a compact subset of $X \times X'$.*
*(2): $X$ is compact and $f$ is continuous.*

**Proof**

**Lemma.** *The mapping $h : X \to G$ where $h(x) := (x, f(x))$ is bijective.*

**Proof** $h$ is clearly 1-to-1. The first element in the pair $(x, f(x))$ guarantees this. $h$ is also "onto" since the range of $h$ is $G$ by definition. So since $h$ is 1-to-1 and "onto" it is a bijection. □

**(1⇒2)**

Since $h$ is bijective, there exists a function $q(x) := h^{-1}(x) : G \to X$. The exact definition of $q$ is $q(x) := x$. By Theorem 1.2 we know that $q$ is uniformly continuous and therefore continuous. Since $h$ is bijective, we know that $X$ and $G$ are the same cardinality. Therefore, $q(G) = X$. By Rudin 4.14, since $G$ is compact, $q(G) = X$ is compact. Since $q$ is continuous and a 1-1 mapping (since in $G$ there is at most 1 pair with $x$ in it) from a compact space $G$ to $X$, by Rudin 4.17 we know that the inverse of $q$, which is $h$, is continuous. Now by Theorem 1.2, since $h$ is continuous, so is $f$, and the implication is proven.

**(2⇒1)**

This implication follows directly from Rudin 4.14. By Theorem 1.2 we know that if $f$ is continuous, then $h(x) := (x, f(x))$ is continuous as well. Again note that our $h$ maps all of $X$ to all of $G$, so $h(X) = G$ since $h$ is a bijection. So since $h$ is continuous and $X$ is compact, $h(X) = G$ implies that $G$ is compact by Rudin 4.14.

Since (1) ⇒ (2) and (2) ⇒ (1) then, (1) ⇔ (2) and we are done. □

**2.1 Theorem.** *Let the intervals $D$ and $D'$ of $\mathbb{R}$ and the bijection $f : D \to D'$ be given. If $f$ is continuous, then the inverse mapping $g = f^{-1} : D' \to D$ is continuous.*

**Proof**

**Lemma 1.** *Given an invertible mapping $f$, then $f$ is either isotone or antitone.*

**Proof** If $f$ is neither isotone nor antitone, that means that there exists at least two points $r$ and $s$ and a point $p$ such that $f(r) = p$ and $f(s) = p$. Because of this, the inverse of $f$, $g = f^{-1}$, is not a function since $g(p) = r$ and $g(p) = s$. Therefore the inverse does not exist, contradicting the fact that $f$ is invertible. So we conclude that $f$ is either isotone or antitone. □

**Lemma 2.** *Given an isotone, invertible mapping $f : D \to D'$, $g = f^{-1} : D' \to D$ is isotone as well.*

**Proof** Pick two points $x_1'$,$x_2' \in D'$ s.t $x_1' < x_2'$. Let $x_1 = g(x_1')$, $x_2 = g(x_2')$. Clearly, $x_1' = f(x_1)$ and $x_2' = f(x_2)$. Since $f$ is isotone, $x_1' < x_2' \Rightarrow x_1 < x_2$. Furthermore, this relation implies that $g$ is isotone. □

**Lemma 3.** *Given an isotone, invertible mapping $f : D \to D'$, $g = f^{-1} : D' \to D$, if $f$ is continuous, then $g$ is continuous.*

**Proof** Select a point $p \in D'$. The condition we need to satisfy to establish the continuity of $g$ is the following: For every $\epsilon > 0$ there exists a $\delta$ s.t

$$|g(p) - g(x')| < \epsilon \tag{7}$$

For all $x' \in D'$ s.t. $|p - x'| < \delta$. Let $x_p = g(p)$ s.t $f(x_p) = p$ and let $\epsilon > 0$ be given. Then set $\delta = \min\{f(x_p) - f(x_p - \epsilon), f(x_p + \epsilon) - f(x_p)\}$. We know that $\delta$ will be a positive value since $f$ is isotone. To see that this $\delta$ satisfies (7) we make the following observations. Let $Gi = [p-\delta, p+\delta]$, $Gf = [g(p)-\epsilon, g(p)+\epsilon]$, $Fi = [x_p - \epsilon, x_p + \epsilon]$, $Ff = [f(x_p - \epsilon), f(x_p + \epsilon)]$. It should be noted that these intervals are well-defined since by Lemma 2 $g$ is isotone so both $Gi$ and $Gf$ and non-empty. The same argument applies to $Fi$ and $Ff$. Since $f$ is continuous and by our selection of $\delta$, we know that $Gi \subset Ff$. No value can be in $Gi$ that is not in $Ff$ since $f$ is continuous. Applying $g$ to $Ff$ yields $Fi$, therefore applying $g$ to $Gi$ yields a subset of $Fi$. Furthermore, by the definition of $f$ and $g$, $Gf = Fi$. This implies that $g(Gi) \subset Gf$ which shows that (7) is satisfied given our chosen $\delta$.

To show continuity at the endpoints of $D' = [c, d]$ we just need to modify the above proof slightly. To show $g$ is continuous at $c$, we just to need to modify the definition of $Gi, Gf, Fi$ and $Ff$ to exclude elements that would not be in $D'$. Specifically, those elements would be $p - \delta$ in $Gi$. Losing these "points" does not affect the rest of the argument. The same rationale applies at $d$ as well. Therefore $g$ is continuous over all of $D'$. □

By Lemma 1, we know that $f$ is either isotone or antitone. If $f$ is isotone we apply Lemma 3 and we are done. If $f$ is antitone, then we apply Lemma 3 to $-f$ (which is isotone and continuous by Rudin Theorem 4.9) and we are done. This is seen to be true since the inverse of $-f$ is the same as for $f$ except for the sign switch. We know from Rudin 4.9 that sign-switching does not affect the continuity. Therefore the theorem holds for all bijections $f$. □

**3.1 Theorem.** *The mapping $q : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $q(t) := 1/t$ for all $t \in \mathbb{R} \setminus \{0\}$ is continuous*

**Proof** We start out by writing $q$ in terms of two new functions $f$ and $g$ defined as such, $f(x) = 1$, $g(t) = t$. Clearly $f(x)$ is continuous, and $g(x)$ is continuous by Theorem 4.6 from Rudin since $\lim_{x\to p} x = p$. Now by Theorem 4.9 in Rudin we know that $q(t) := f(t)/g(t)$ is continuous if $f$ and $g$ are continuous. Since we established the continuity of $f$ and $g$ above, then we have that $q(t) := 1/t$ is continuous. □

**3.2 Theorem.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be given, and let $f$ and $g$ be continuous. Let $D := \{t \in \mathbb{R} \mid g(t) \neq 0\}$ and define $h : D \to \mathbb{R}$ by $h(t) := f(t)/g(t)$ for all $t \in D$. Then $h$ is continuous.*

**Proof** We consider only limit points of $D$ since there is nothing to prove for isolated points. Since $f$ and $g$ are continuous, we know that $\forall\, p \in D$ $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x\to p} g(x) = g(p)$ by Rudin Theorem 4.6. Now by Rudin Theorem 4.4 we know that $\lim_{x\to p} h(p) = \lim_{x\to p} \frac{f(x)}{g(x)} = \frac{f(p)}{g(p)} = h(p)$. Applying Rudin Theorem 4.6 again, we know the function $h(x) := \frac{f(x)}{g(x)}$ is continuous because of the statement about the limit.

A few examples of uniformly continuous functions of the form $h(x) := \frac{f(x)}{g(x)}$ are:

1. $h(x) := \frac{\sin x}{x}$ - The derivative of this is bounded which means it is uniformly continuous. The limit as $x$ gets large is 1 which further strengthens the argument. Obviously we are not considerering the case when $x = 0$.

2. $h(x) := \frac{x}{1}$ - A trivial example in case my first one is wrong. This reduces to $h(x) := x$ which is clearly uniformly continuous.

3. $h(x) := \frac{1}{1}$. - To cover my back I throw this in. $h(x) := 1$ is trivially uniform continuous and technically is usuable from the problem statement.

Some nonuniformly continuous functions:

1. $h(x) := \frac{1}{x}$. Let $\epsilon > 0$ and $\delta > 0$ be given and select a point $p \in X$ s.t. $|p| < \delta$. Now since around 0 $h$ takes off to infinity, taking an $x$ close to zero we can make $|h(x) - h(p)|$ larger than $\epsilon$. We can do this however while still maintaining $|x - p| < \delta$. The key to this being true is that $h$ is unbounded around 0, which allows me to adjust the difference between $h(x)$ and $h(p)$ without adjusting the distance between $x$ and $p$.

2. $h(x) := \frac{x^2}{1}$. In class it was mentioned that $x^2$ is not uniformly continuous, so I put it in here too.

$\square$