21-355 Advanced Calculus I
Exam #2.

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1.1 Theorem. *All functions defined in this theorem are assumed to be infinitely differentiable real-valued functions on* $(0, \infty)$. ι *denotes the inclusion function defined by* $\iota(t) := t$ *for all* $t \in (0, \infty)$. *Then for all* $n \in \mathbb{N}^{\times}$,

$$D^n((\iota - 1)^{n-1} \log) = (n-1)! \sum_{k \in 1,\ldots,n} 1/\iota^k \tag{1}$$

Proof

Lemma. *Let* $n \in \mathbb{N}$, *then*

$$D^n(\frac{(\iota - 1)^n}{\iota}) = \frac{n!}{\iota^{n+1}} \tag{2}$$

Proof Using the binomial theorem to expand the top part of the fraction and dividing through by ι yields,

$$\frac{(\iota - 1)^n}{\iota} = \sum_{k=0}^{n} \binom{n}{k} \iota^{k-1}(-1)^{n-k} \tag{3}$$

By inspection, we observe that (3) is a polynomial in ι with degree $n-1$ plus an extra $\frac{1}{\iota}$ term. It is clear from the exponent rule for differentiation, that the nth derivative of a degree $n-1$ polynomial is 0. Therefore, the only term that matters for the derivative of the whole expression is the $\frac{1}{\iota}$ term. By the exponent rule applied n times, we get that the nth derivative of $\frac{1}{\iota} = \frac{n!}{\iota^{n+1}}$. We observe that the right-hand side is always positive by the following. If we differentiate $\frac{1}{\iota}$ by itself, we observe that the coefficient is negative when the denominator is a even power. However, when we factor in the $(-1)^n$ term, we see that this is negative if n is odd. Therefore, if the nth derivative leaves an even power in the denominator(i.e. n is odd), it is made positive by the extra -1 from the binomial expansion. Furthermore, if the nth derivative leaves an odd power in the denominator (n is even), then the $(-1)^n = 1$, therefore the sign stays positive. □

Now we prove the main theorem by induction on n.

Base Case: $n = 1$. $D(\log) = 1/\iota$. This is true by the definition of log.

Inductive Step: Applying the product rule (Rudin 5.3(b)) yields,

$$D((\iota - 1)^{n-1} \log) = (n-1)(\iota - 1)^{n-2} \log + \frac{(\iota - 1)^{n-1}}{\iota} \tag{4}$$

Now we need to take the $(n-1)th$ derivative of both sides of the addition to get the nth derivative of the original function in the theorem. We apply the Lemma to the right-hand side, and the inductive hypothesis to the left-hand side yielding,

$$D^n((\iota - 1)^{n-1} \log) = (n-1)((n-2)! \sum_{k \in 1,\ldots,n-1} 1/\iota^k) + \frac{(n-1)!}{\iota^n} \tag{5}$$

Simplifying the above expressions yields:

$$D^n((\iota - 1)^{n-1} \log) = (n-1)!(\sum_{k \in 1,\ldots,n-1} 1/\iota^k) + \frac{(n-1)!}{\iota^n} \tag{6}$$

Finally by combining terms we get:

$$D^n((\iota - 1)^{n-1} \log) = (n-1)!(\sum_{k \in 1,\dots,n} 1/\iota^k) \tag{7}$$

□

2.1 Theorem. *Let the interval $[a,b]$ and the differentiable function $f : [a,b] \to \mathbb{R}$ be given and assume that f' is Riemann-integrable. Then there exists isotone differentiable functions $g, h : [a,b] \to \mathbb{R}$ such that g' and h' are Riemann-integrable and $f = h - g$.*

Proof Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition of $[a,b]$ and define:

$$S(P,f) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \tag{8}$$

Next define a function:

$$V_a^x(f) = \sup S(P,f) \tag{9}$$

where sup is taken over all partitions of $[a,x]$.

Lemma 1. *Suppose f' is bounded on $[a,b]$. Then $V_a^b(f) \le ||f'||(b-a)$.*

Proof For all partitions of $[a,b]$, P, we write $S(P,f) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$. Now by the mean-value theorem (Rudin 5.10), we know there exists a $c_k \in [x_{k-1}, x_k]$ s.t $S(P,f) = \sum_{k=1}^{n} |f'(c_k)|\Delta x_k$. By the definition of supremum norm (Rudin 7.14) we have that $\sum_{k=1}^{n} |f'(c_k)|\Delta x_k \le ||f'|| \sum_{k=1}^{n} \Delta x_k$. However this is just equal to $||f'||(b-a)$. (Lemma and Proof influenced from *Calculus, an Introduction (Beng)* pg. 137) □

Lemma 2. *Let $V_b^a(f)$ be bounded on $[a,b]$ and let $a < x < y < b$ be given. Then $V_a^y(f) = V_a^x(f) + V_x^y(f)$.*

Proof We will first prove that $V_a^y(f) \le V_a^x(f) + V_x^y(f)$. Let P be a partition of $[a,y]$. Let $P_x = P \cup \{x\}$. Then $P_x = P' \cup P''$ where P' is a partition of $[a,x]$ and P'' is a partition of $[x,y]$ (Clearly either P' or P'' must contain x). Now we write $S(P_x, f) = S(P', f) + S(P'', f) \le V_a^x(f) + V_x^y(f)$. Taking the supremum over all paritions P of $[a,y]$ yields $V_a^y(f) \le V_a^x(f) + V_x^y(f)$.

To show equality we now exhibit that $V_a^y(f) \ge V_a^x(f) + V_x^y(f)$. Let P' be a partition of $[a,x]$ and P'' be a partition of $[x,y]$. Now write $S(P',f) + S(P'',f) \le V_a^y(f)$ for any partitions P' and P''. Again, we take the supremum over all partitions of $[a,y]$ and get $V_a^x(f) + V_x^y(f) \le V_a^y(f)$ which is what we wanted. (Lemma and Proof influenced from *Calculus, an Introduction (Beng)* pg. 137 (Although the proof is all my own, the equality tipped me off for the main theorem (selection of h))) □

Since f' is Riemann-integrable, we know that f' is bounded, and by Lemma 1 can say that $V_a^b(f)$ is bounded on $[a,b]$. Let $h(x) = V_a^x(f)$ and $g(x) = h(x) - f(x)$. Lemma 2 guarentees that $h(x)$ is isotone. In order to show that f can be fully broken down, we still need to show that g is isotone. Let $x, y \in [a,b]$ s.t. $x < y$. In order for g to be isotone, it must be the case that

$$V_a^x(f) - f(x) \le V_a^y(f) - f(y) \tag{10}$$

since this is how we defined g. Rearranging terms yields:

$$f(y) - f(x) \le V_a^y(f) - V_a^x(f) \tag{11}$$

Applying Lemma 2 again to the right-hand side yields: $f(y) - f(x) \le V_x^y(f)$. Partitioning $[x,y]$ via the parition $P = \{x,y\}$ yields the following. $S(P,f) = |f(y) - f(x)|$ which is in turn less than or equal to $V_x^y(f)$ since V is defined to be the sup over all partitions. Therefore, $f(y) - f(x) \le V_x^y(f)$ which implies that g is isotone. After all this, we have deduced two Riemann-integrable isotone functions such that $f = h - g$. (This proof using the so-called bounded variation V was inspired through the book I cite. The proofs are done by myself however) □

3.1 Theorem. *Let the three-times differentiable function* $f : [-1, 1] \to \mathbb{R}$ *be given and assume that* $f(-1) = f(0) = f'(0) = 0$ *and* $f(1) = 1$. *Then there exists* $t \in (-1, 1)$ *such that* $f'''(t) \geq 3$.

Proof The proof of this theorem will flow through Taylor's theorem (Rudin 5.15). Let $\alpha = 0$, $n = 3$, and $\beta = -1$ in the Taylor formulation. Substituting these values into the theorem yields:

$$f(-1) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!}(-1)^k - \frac{f^{(3)}(c)}{6} \tag{12}$$

Now since $f(-1) = f(0) = f'(0) = 0$ we can write:

$$0 = \frac{f''(0)}{2} - \frac{f^{(3)}(c)}{6} \tag{13}$$

Rearranging terms and simplifying yields:

$$f^{(3)}(c) = 3f''(0) \tag{14}$$

Therefore we have established that there exists a $c \in (-1, 0)$ such that the above holds. Now we do the same procedure again, except this time we let $\beta = 1$.

$$f(1) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} + \frac{f^{(3)}(t)}{6} \tag{15}$$

Substituting and rearranging terms yields:

$$f^{(3)}(t) = 6 - 3f''(0) \tag{16}$$

Therefore we have established that there exists a $t \in (0, 1)$ such that the above holds. Now in an attempt to show that either $f^{(3)}(t)$ or $f^{(3)}(c)$ is greater than 3, we introduce this equation:

$$f^{(3)}(t) + f^{(3)}(c) = 6 \tag{17}$$

Substituting yields:

$$3f''(0) + 6 - 3f''(0) = 6 \tag{18}$$

Since the above is clearly true, we have established that either $f^{(3)}(t) \geq 3$ or $f^{(3)}(c) \geq 3$ (otherwise their sum could not add to 6). Since we know we can get a $c \in (-1, 0)$ and a $t \in (0, 1)$, then there exists an $s \in (-1, 1)$ that makes $f^{(3)}(s) \geq 3$. □

4.1 Theorem. *Let the continuous function* $f : [0, 1] \to \mathbb{R}$ *be given and assume that* $f(0) = 0$. *Then, for every* $\epsilon > 0$ *there exists a polynomial function* $p : \mathbb{R} \to \mathbb{R}$ *such that* $p(0) = 0$ *and* $|f(t) - p(t)| \leq \epsilon$ *for all* $t \in [0, 1]$.

Proof In the proof of Rudin 7.26, the claim is made (and proved of course) that $|P_n(x) - f(x)| < \epsilon$ for sufficiently large values of n. Furthermore, the assumptions that the proof makes are identical to those assumptions made about f and p in the statement of this theorem. Therefore if we let $p(t) = \lim_{n \to \infty} P_n(t)$ this implies that $|p(t) - f(t)| < \epsilon$. This clearly implies that $|f(t) - p(t)| < \epsilon$. □