Bits, Bytes, and Integers – Part 2

15-213/14-513/15-513: Introduction to Computer Systems 3rd Lecture, September 6, 2022

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Waitlist questions

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- 14-513: INI Enrollment (ini-academic@andrew.cmu.edu)
- Please don't contact the instructors with waitlist questions.

Reminder about labs 0 and 1

■ Lab 0 is due today September 6 at 11:59pm

- No late days, no grace days
- Email instructors if you need an extension
- It's supposed to be easy—if it takes you more than a couple hours' effort, you may not be prepared for this course

■ Lab 1 (data lab) is out

- Due Thursday September 15 at 11:59pm ET
- Start early!
- Lab 2 (bomb lab) comes out on Thursday September 8

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- Integers
 - Representation: unsigned and signed; negation and addition
 - Conversion, casting, extension, truncation
 - Multiplication, division, shifting
- Byte order in memory, pointers, strings

Activity: https://canvas.cmu.edu/courses/30386/assignments/525231

Use the "People" section of Canvas to assign yourself to a group.

Do models 0 and 1, then stop.

Encoding "Integers"

Unsigned

Given a bit w bits long...

Given a bit vector
$$x$$
, $B2U(x) = \sum_{i=0}^{w-1} x_i \cdot 2^i$ w bits long...

Signed (twos complement)

B2T(x) =
$$-x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$
Sign Bit

Examples (w = 5)

±16	8	4	2	1
0	1	0	1	0

$$0 + 8 + 0 + 2 + 0 = 10$$

$$16 + 8 + 0 + 2 + 0 = 26$$

$$-16 + 8 + 0 + 2 + 0 = -10$$

Negation: Complement & Increment

Negate through complement and increase

$$\sim x + 1 == -x$$

■ Why?

$$-x + x == 0$$
 (by definition)

$$-x + x + 1 == 0$$

$$(\sim x+1) + x == 0$$

x 1001110

Example: x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
У	-15213	C4 93	11000100 10010011

Complement & Increment Examples

$$x = 0$$

	Decimal	Hex	Binary
0	0	00 00	0000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	0000000 00000000

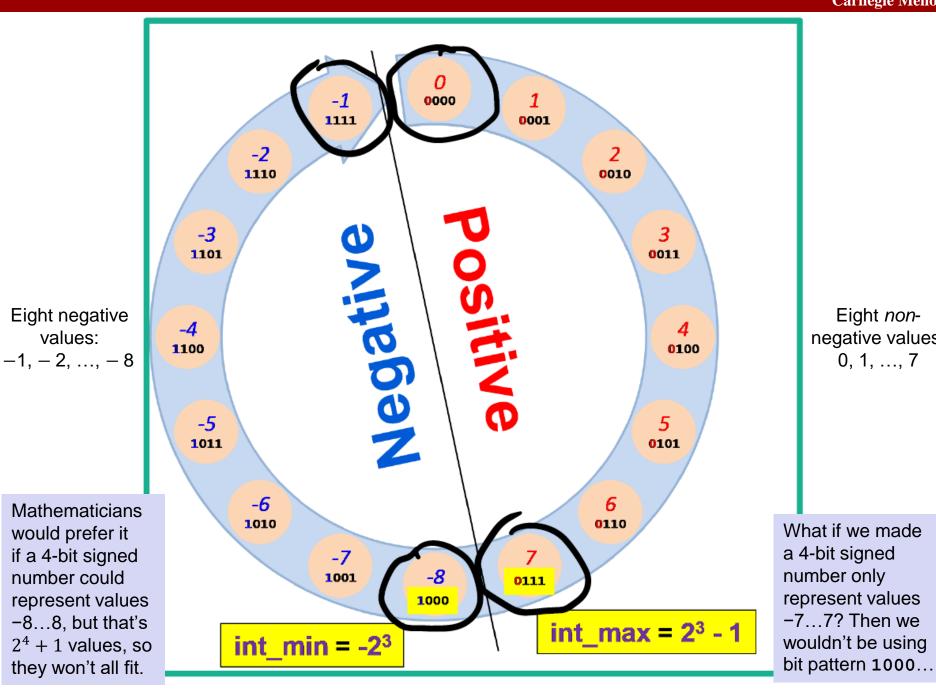
$$x = T_{\min}$$

	Decimal	Hex	Binary
x	-32768	80 00	10000000 00000000
~x	32767	7F FF	01111111 11111111
~x+1	-32768	80 00	10000000 00000000



Eight nonnegative values:

0, 1, ..., 7



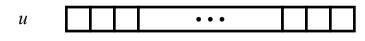
values:

Unsigned Addition

Operands: w bits

True Sum: w+1 bits

Discard Carry: w bits



+ <i>v</i>		• • •		

1		I	Ī
u + v		•••	

$UAdd_{w}(u, v)$

Standard Addition Function

- Ignores carry output
- **■** Implements Modular Arithmetic

$$s = UAdd_w(u, v) = u + v \mod 2^w$$

unsigned char	+	1110 1101	 E9 + D5	233 + 213
	_		 	

Hex Deciman

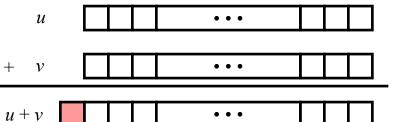
Kr.	V	Δ.
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
В	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Unsigned Addition

Operands: w bits

True Sum: w+1 bits

Discard Carry: w bits



 $UAdd_{w}(u, v)$

Standard Addition Function

- Ignores carry output
- **■** Implements Modular Arithmetic

$$s = UAdd_w(u, v) = u + v \mod 2^w$$

unsigned char	+	1110 1101		E9 + D5	233 + 213
	1	1011	1110	1BE	446
		1011	1110	BE	190

Hex Decimal Binary

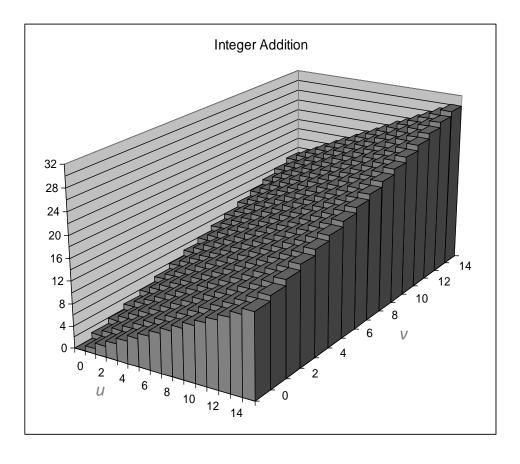
Kr.	V	Δ.
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
В	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualizing (Mathematical) Integer Addition

Integer Addition

- 4-bit integers u, v
- Compute true sum $Add_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface

 $Add_4(u, v)$

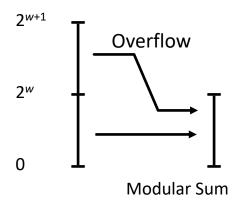


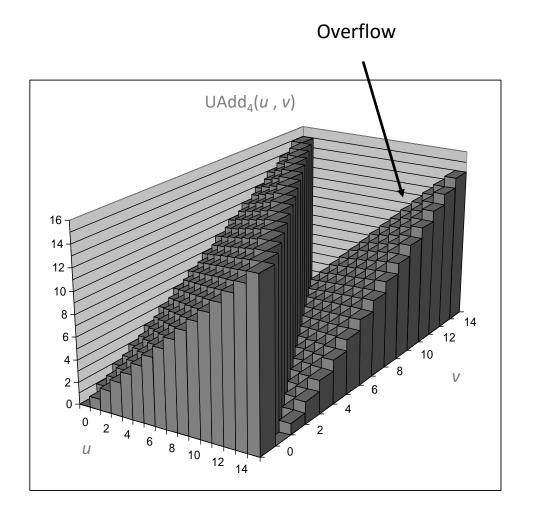
Visualizing Unsigned Addition

Wraps Around

- If true sum $\geq 2^w$
- At most once

True Sum





Two's Complement Addition

■ TAdd and UAdd have Identical Bit-Level Behavior

Signed vs. unsigned addition in C:

BE

-23

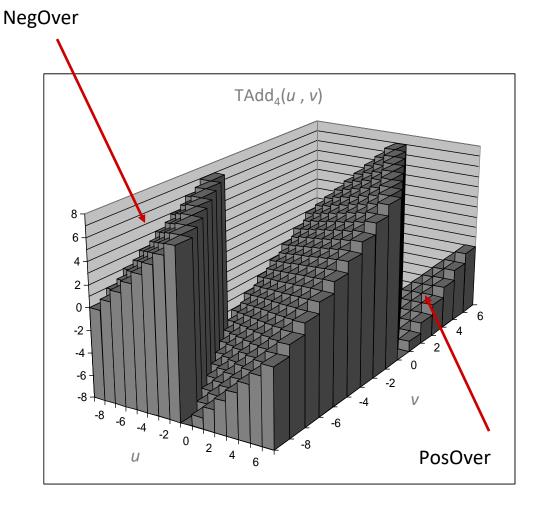
Visualizing 2's Complement Addition

Values

- 4-bit two's comp.
- Range from -8 to +7

Wraps Around

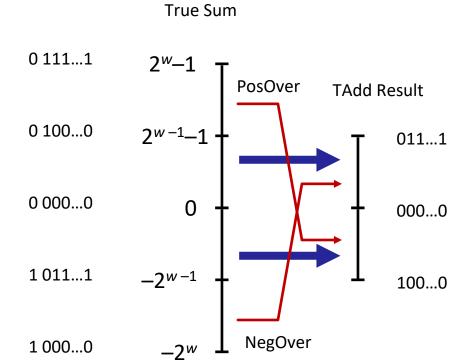
- If sum $\geq 2^{w-1}$
 - Becomes negative
 - At most once
- If sum $< -2^{w-1}$
 - Becomes positive
 - At most once



TAdd Overflow

Functionality

- True sum requires w+1 bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer



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Boolean Algebra

Developed by George Boole in 19th Century

- Algebraic representation of logic
 - Encode "True" as 1 and "False" as 0

And

■ A&B = 1 when both A=1 and B=1

&	0	1
0	0	0
1	0	1

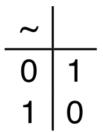
Or

■ A | B = 1 when either A=1 or B=1

ı	0	1
0	0	1
1	1	1

Not

~A = 1 when A=0



Exclusive-Or (Xor)

■ A^B = 1 when either A=1 or B=1, but not both

٨	0	1
0	0	1
1	1	0

Activity: https://canvas.cmu.edu/courses/30386/assignments/525231

Do model 2, then stop.

General Boolean Algebras

- Operate on Bit Vectors
 - Operations applied bitwise

```
01101001 01101001 01101001

& 01010101 | 01010101 ^ 01010101 ~ 01010101

01000001 01111101 00111100 1010101
```

All of the Properties of Boolean Algebra Apply

Example: Representing & Manipulating Sets

Representation

- Width w bit vector represents subsets of {0, ..., w−1}
- $a_j = 1 \text{ if } j \in A$
 - 01101001 { 0, 3, 5, 6 }
 - **76543210**
 - 01010101 { 0, 2, 4, 6 }
 - 76543210

Operations

	&	Intersection	01000001	{ 0, 6 }
•		Union	01111101	{ 0, 2, 3, 4, 5, 6 }
•	٨	Symmetric difference	00111100	{ 2, 3, 4, 5 }
	~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

■ Operations &, |, ~, ^ Available in C

- Apply to any "integral" data type
 - long, int, short, char, unsigned
- View arguments as bit vectors
- Arguments applied bit-wise

Examples (Char data type)

- $\sim 0x41 \rightarrow$
- ~0x00 →
- $0x69 \& 0x55 \rightarrow$
- $0x69 \mid 0x55 \rightarrow$

Hex Decimanary

0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
В	11	1011
С	12	1100
D	13	1101
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Bit-Level Operations in C

■ Operations &, |, ~, ^ Available in C

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- Arguments applied bit-wise

Examples (Char data type)

- $\sim 0x41 \rightarrow 0xBE$
 - $\sim 0100\ 0001_2 \rightarrow 1011\ 1110_2$
- $\sim 0x00 \rightarrow 0xFF$
 - $\sim 0000\ 0000_2 \rightarrow 1111\ 1111_2$
- $0x69 \& 0x55 \rightarrow 0x41$
 - $0110\ 1001_2\ \&\ 0101\ 0101_2\ \to\ 0100\ 0001_2$
- $0x69 \mid 0x55 \rightarrow 0x7D$
 - $0110\ 1001_2\ |\ 0101\ 0101_2 \to 0111\ 1101_2$

Hex Decimany

0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111
	1 2 3 4 5 6 7 8 9 10 11 12 13 14

Contrast: Logic Operations in C

- Contrast to Bit-Level Operators
 - Logic Operations
 - View 0 as "Fals
 - Anything nonze
 - Alway
 - . Early Watch out for && vs. & (and || vs. |)...
- Example one of the more common oopsies in
 - !0x41 → C programming
 - $!0x00 \rightarrow$
 - $!!0x41 \rightarrow 0x01$
 - 0x69 && 0x55 → 0x01
 - $0x69 \parallel 0x55 \rightarrow 0x01$
 - p && *p (avoids null pointer access)

Activity: https://canvas.cmu.edu/courses/30386/assignments/525231

Do model 3, then stop.

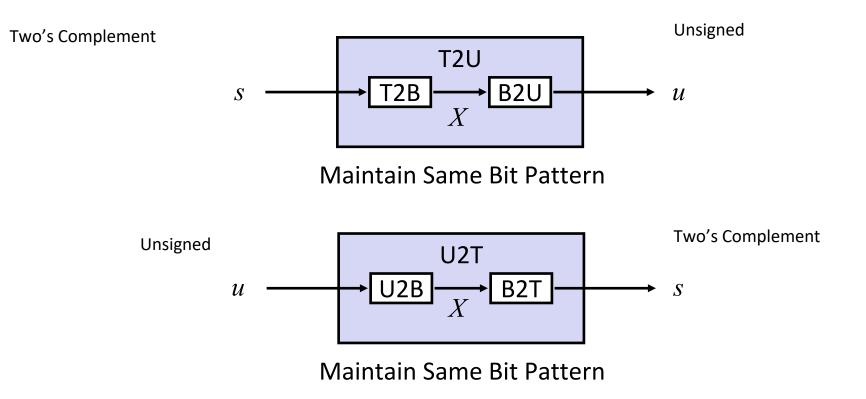
Logical versus Bitwise

X	!X	!!X	!!X == X	X	~X	~~X	~~X == X
-1	0	1	No	-1	0	-1	Yes
0	1	0	Yes	0	-1	0	Yes
1	0	1	Yes	1	-2	1	Yes
2	0	1	No	2	-3	2	Yes

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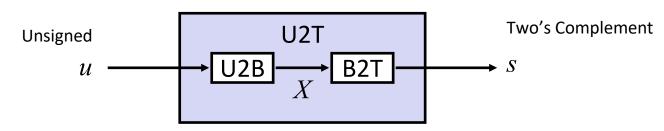
Mapping Between Signed & Unsigned



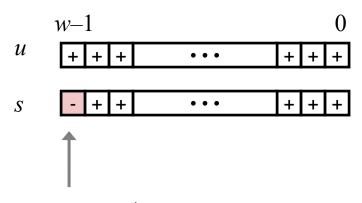
Mappings between unsigned and two's complement numbers:

Keep bit representations and reinterpret

Relation between Signed & Unsigned



Maintain Same Bit Pattern



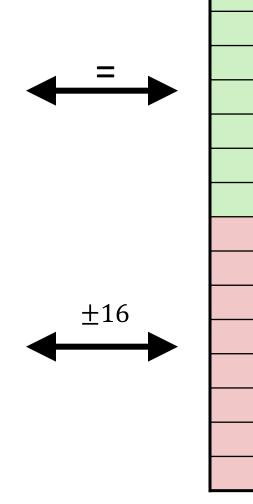
Large positive weight becomes

Large negative weight

Mapping Signed ↔ Unsigned

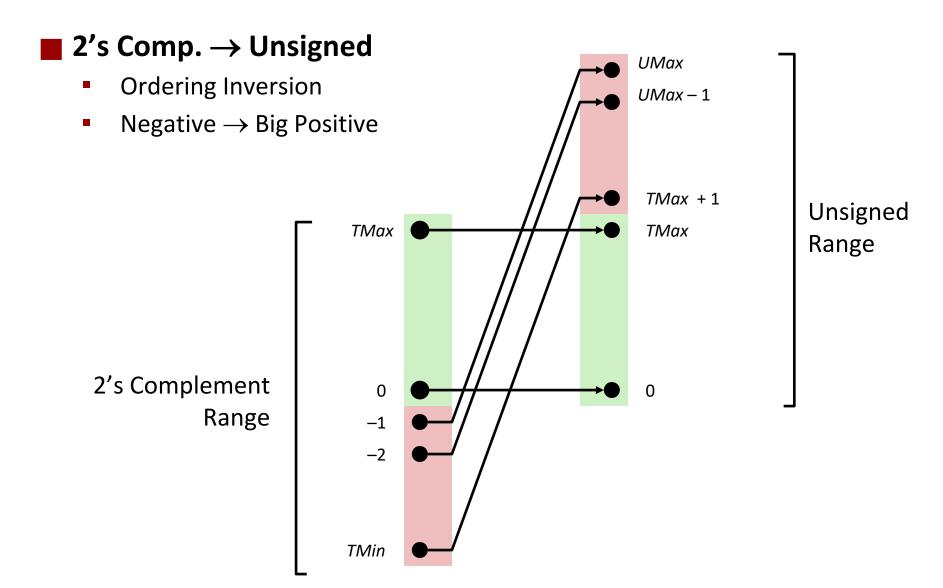
Bits
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

Signed	
0	
1	
2	
3	
4	
5	
6	
7	
-8	
-7	
-6	
-5	
-4	
-3	
-2	
-1	



Unsigned
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

Conversion Visualized



Signed vs. Unsigned in C

Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffixOU, 4294967259U

Casting

Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

Implicit casting also occurs via assignments and procedure calls

Casting Surprises

Expression Evaluation

- If there is a mix of unsigned and signed in single expression,
 signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples:

Constant 1	Constant 2	Relation	Evaluation
0	0υ	==	Unsigned
-1	0	<	Signed
-1	0υ	>	Unsigned
INT_MAX	INT_MIN	>	Signed
(unsigned) INT_MAX	INT_MIN	<	Unsigned
-1	-2	>	Signed
(unsigned)-1	-2	>	Unsigned
INT_MAX	((unsigned)INT_MAX) + 1	<	Unsigned
INT_MAX	(int)(((unsigned)INT_MAX) + 1)	>	Signed

Summary Casting Signed ↔ Unsigned: Basic Rules

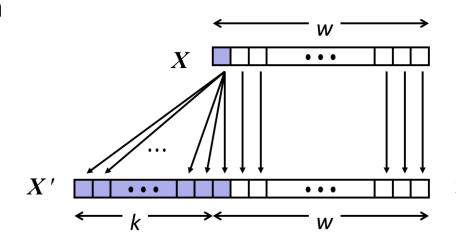
- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- Expression containing signed and unsigned int
 - int is cast to unsigned!!

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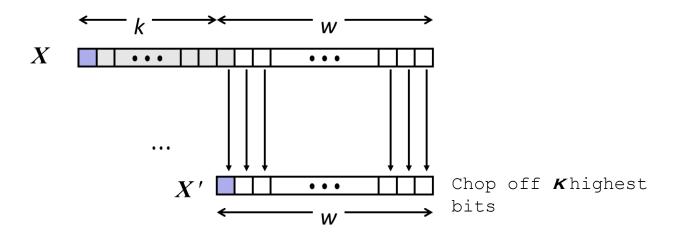
Sign Extension and Truncation

Sign Extension



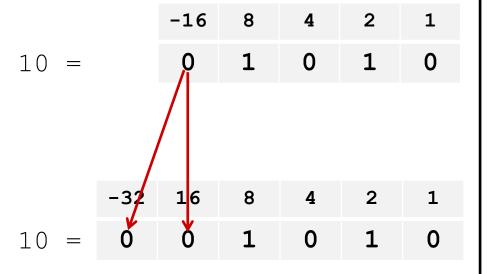
Make K copies of sign bit

Truncation

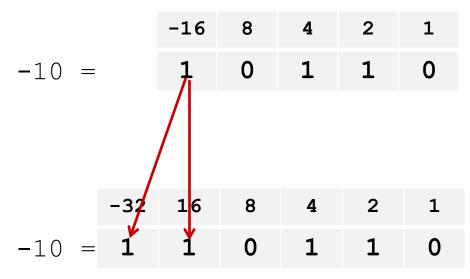


Sign Extension: Simple Example

Positive number



Negative number



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Truncation: Simple Example

No sign change

$$-16$$
 8 4 2 1 -6 = **1 1 0 1 0**

$$-8$$
 4 2 1 -6 = 1 0 1 0

 $-6 \mod 16 = 26U \mod 16 = 10U = -6$

Sign change

$$10 = \begin{bmatrix} -16 & 8 & 4 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$-8$$
 4 2 1 -6 = 1 0 1 0

 $10 \mod 16 = 10U \mod 16 = 10U = -6$

$$-16$$
 8 4 2 1 -10 = 1 0 1 1 0

 $-10 \mod 16 = 22U \mod 16 = 6U = 6$

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- Integers
 - Representation: unsigned and signed; negation
 - Conversion, casting
 - Extension, truncation, shifting
 - Addition, multiplication
- Representations in memory, pointers, strings

Shifting

Left Shift: x << y

- Shift bit-vector x left y positions
- Throw away extra bits on left
- Fill with 0's on right
- Equivalent to multiplying by 2^{y}

Right Shift: x >> y

- Shift bit-vector x right y positions
- Throw away extra bits on right
- Two kinds:
 - "Logical": Fill with 0's on left
 - "Arithmetic": Replicate most significant bit on left
- Almost equivalent to dividing by 2^{y}

Undefined Behavior (in C)

Shift amount < 0 or ≥ word size</p>

Argument x	01100010
<< 3	00010000
Logical >> 2	00 <mark>011000</mark>
Arithmetic >> 2	00 <mark>011000</mark>

Argument x	101 <mark>000</mark> 10
<< 3	00010
Logical >> 2	00 <mark>101000</mark>
Arithmetic >> 2	11 <mark>101000</mark>

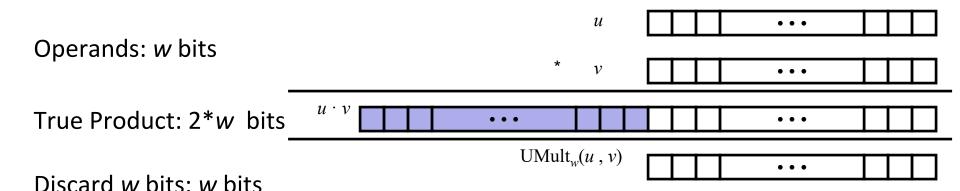
Multiplication

- **Goal:** Computing Product of w-bit numbers x, y
 - Either signed or unsigned
- But, exact results can be bigger than w bits
 - Unsigned: up to 2w bits
 - Result range: $0 \le x * y \le (2^w 1)^2 = 2^{2w} 2^{w+1} + 1$
 - Two's complement min (negative): Up to 2w-1 bits
 - Result range: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
 - Two's complement max (positive): Up to 2w bits, but only for $(TMin_w)^2$
 - Result range: $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
- So, maintaining exact results...
 - would need to keep expanding word size with each product computed
 - is done in software, if needed
 - e.g., by "arbitrary precision" arithmetic packages

Activity: https://canvas.cmu.edu/courses/30386/assignments/525231

Do model 4, then stop.

Unsigned Multiplication in C

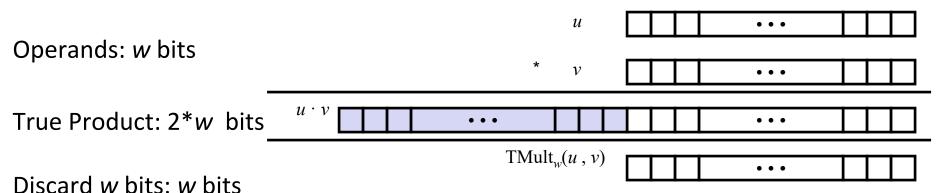


Standard Multiplication Function

Ignores high order w bits

■ Implements Modular Arithmetic

Signed Multiplication in C



Standard Multiplication Function

- Ignores high order w bits
- Some of which are different for signed vs. unsigned multiplication
- Lower bits are the same

		1110	1001		E9		-23
*		1101	0101	*	D5	*	-43
0000	0011	1101	1101	C	03DD		989
		1101	1101		DD		-35

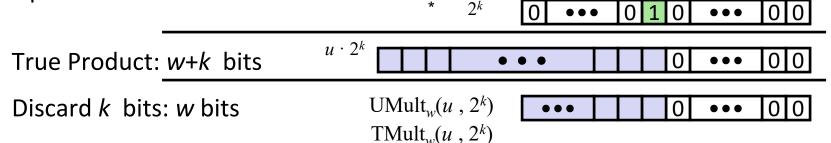
k

Power-of-2 Multiply with Shift

Operation

- $\mathbf{u} \ll \mathbf{k}$ gives $\mathbf{u} * \mathbf{2}^k$
- Both signed and unsigned

Operands: w bits



u

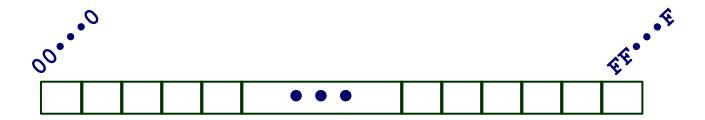
Examples

- u << 3 == u * 8
- u << 5 u << 3 == u * 24
- Most machines shift and add faster than multiply
 - Compiler generates this code automatically

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Byte-Oriented Memory Organization



Programs refer to data by address

- Imagine all of RAM as an enormous array of bytes
- An address is an index into that array
 - A pointer variable stores an address

System provides a private address space to each "process"

- A process is an instance of a program, being executed
- An address space is one of those enormous arrays of bytes
- Each program can see only its own code and data within its enormous array
- We'll come back to this later ("virtual memory" classes)

Machine Words

Any given computer has a "Word Size"

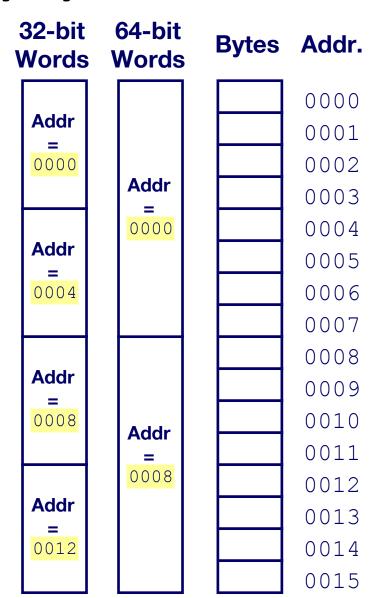
- Nominal size of integer-valued data
 - and of addresses
- Until recently, most machines used 32 bits (4 bytes) as word size
 - Limits addresses to 4GB (2³² bytes)
- Increasingly, machines have 64-bit word size
 - Potentially, could have 16 EB (exabytes) of addressable memory
 - That's 18.4×10^{18} bytes
 - Machines still support multiple data formats
 - Fractions or multiples of word size
 - Always integral number of bytes

Yes, both of these numbers are correct.

This discrepancy is known as the Great Storage Industry Marketing Lie. Ask me about it after class if you really want to know.

Addresses Always Specify Byte Locations

- Address of a word is address of the first byte in the word
- Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	8	8
float	4	4	4
double	8	8	8
pointer	4	8	8

Byte Ordering

- So, how are the bytes within a multi-byte word ordered in memory?
- Conventions
 - Big Endian: Sun, PPC Mac, network packet headers
 - Least significant byte has highest address
 - Little Endian: x86, ARM processors running Android, iOS, and Windows
 - Least significant byte has lowest address

Byte Ordering Example

Example

- Variable x has 4-byte value of 0x01234567
- Address given by &x is 0x100

Big Endian			0x100	0x101	0x102	0x103	
			01	23	45	67	
Little Endian		0x100	0x101	0x102	0x103		
			67	45	23	01	

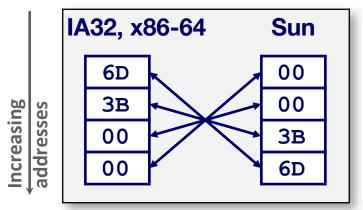
Representing Integers

Decimal: 15213

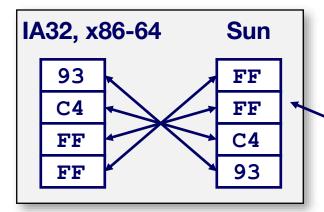
Binary: 0011 1011 0110 1101

Hex: 3 B 6 D

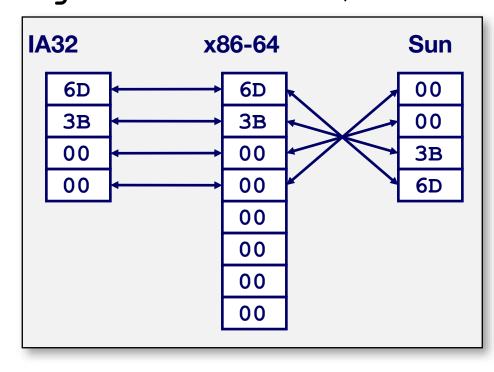




int B = -15213;



long int C = 15213;



Two's complement representation

Examining Data Representations

Code to Print Byte Representation of Data

Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len) {
    size_t i;
    for (i = 0; i < len; i++)
        printf("%p\t0x%.2x\n",start+i, start[i]);
    printf("\n");
}</pre>
```

Printf directives:

%p: Print pointer

%x: Print Hexadecimal

show bytes Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

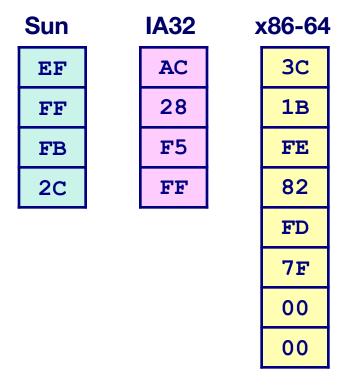
Result (Linux x86-64):

```
int a = 15213;
0x7fffb7f71dbc 6d
0x7fffb7f71dbd 3b
0x7fffb7f71dbe 00
0x7fffb7f71dbf 00
```

Representing Pointers

int
$$B = -15213;$$

int *P = &B



Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

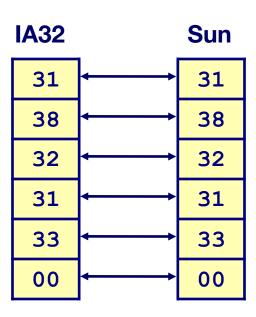
char S[6] = "18213";

Strings in C

- Represented by array of characters
- Each character encoded in ASCII format
 - Standard 7-bit encoding of character set
 - Character "0" has code 0x30
 - Digit i has code 0x30+i
- String should be null-terminated
 - Final character = 0

Compatibility

Byte ordering not an issue



Representing x86 machine code

x86 machine code is a sequence of *bytes*

- Grouped into variable-length instructions, which look like strings...
- But they contain embedded little-endian numbers...

Example Fragment

Address	Instruction Code	Assembly Rendition
8048365:	5b	pop %ebx
8048366:	81 c3 ab 12 00 00	add \$0x12ab, %ebx
804836c:	83 bb 28 00 00 00 00	cmpl \$0x0,0x28(%ebx)

Deciphering Numbers

- Value:
- Pad to 32 bits:
- Split into bytes:
- Reverse:

0x12ab

0x000012ab

00 00 12 ab

ab 12 00 00