# Floating Point

15-213/18-243: Introduction to Computer Systems
4th Lecture, 26 May 2011

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# Last Time: Integers

- **Representation: unsigned and signed**
- **Conversion, casting**
  - Bit representation maintained but reinterpreted
- **Expanding, truncating**
  - Truncating = mod
- **Addition, negation, multiplication, shifting**
  - Operations are mod $2^w$
  - Pay attention to division of negative numbers

- **"Ring" properties hold**
  - Associative, commutative, distributive, additive 0 and inverse
- **Ordering properties do not hold**
  - $u > 0$ does not mean $u + v > v$
  - $u, v > 0$ does not mean $u \cdot v > 0$

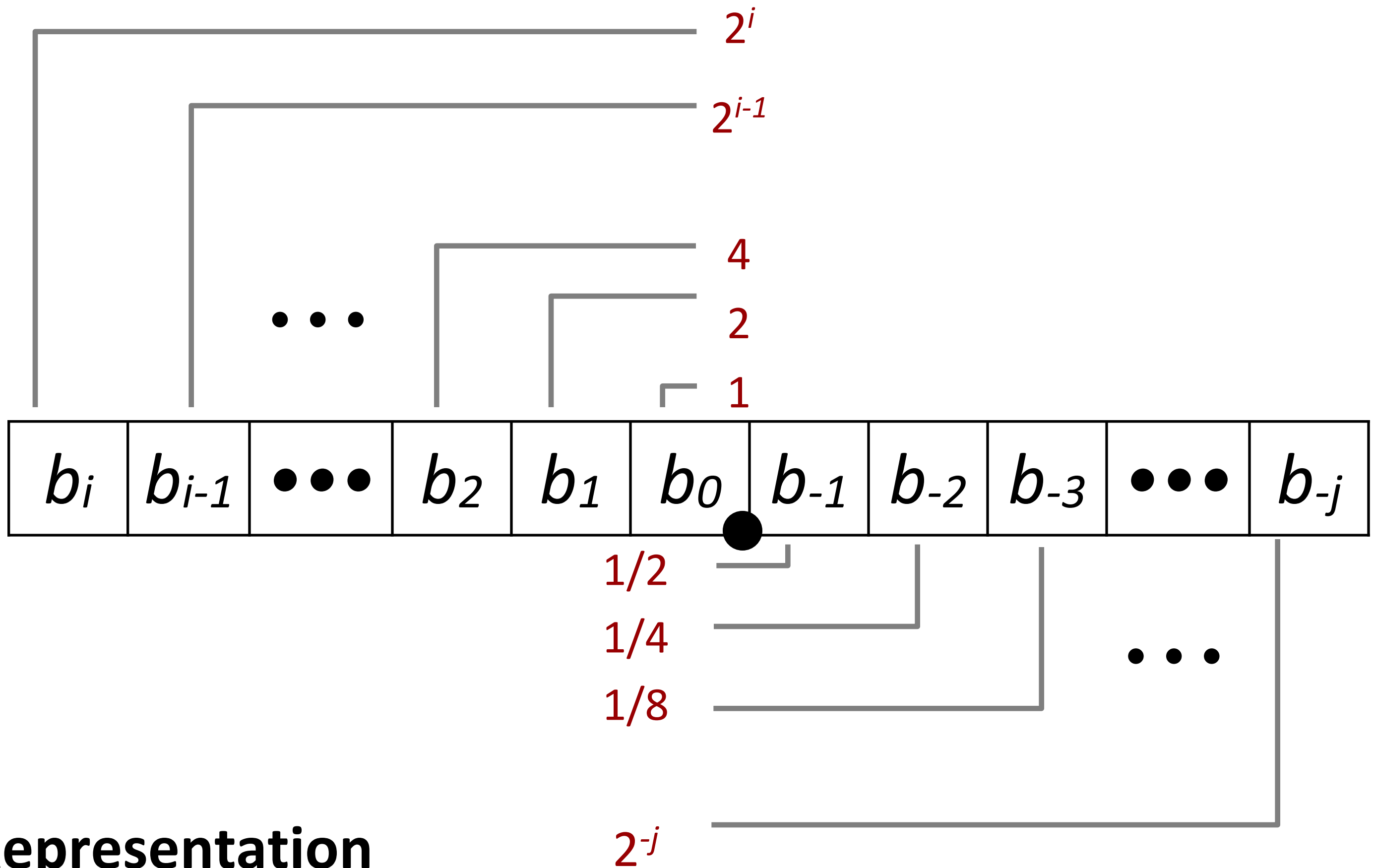
# Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

# Fractional binary numbers

- What is $1011.101_2$?

# Fractional Binary Numbers




- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

# Fractional Binary Numbers: Examples

- **Value** **Representation**

| Value | Representation |
|---|---|
| 5 3/4 | $101.11_2$ |
| 2 7/8 | $10.111_2$ |
| 63/64 | $1.0111_2$ |

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

# Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

- **Value** **Representation**
  - 1/3 $0.0101010101[01]\ldots_2$
  - 1/5 $0.001100110011[0011]\ldots_2$
  - 1/10 $0.0001100110011[0011]\ldots_2$

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# IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

# Floating Point Representation

- **Numerical Form:**

$$(-1)^s \, M \; 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two

- **Encoding**
  - MSB s is sign bit ***s***
  - exp field encodes ***E*** (but is not equal to E)
  - frac field encodes ***M*** (but is not equal to M)

# Precisions

- **Single precision: 32 bits**

| s | exp | frac |
| --- | --- | --- |
| 1 | 8-bits | 23-bits |

- **Double precision: 64 bits**

| s | exp | frac |
| --- | --- | --- |
| 1 | 11-bits | 52-bits |

- **Extended precision: 80 bits (Intel only)**

| s | exp | frac |
| --- | --- | --- |
| 1 | 15-bits | 63 or 64-bits |

# Normalized Values

- **Condition: exp ≠ 000…0 and exp ≠ 111…1**

- **Exponent coded as *biased* value: *E* = *Exp* – *Bias***
  - *Exp*: unsigned value exp
  - *Bias* = $2^{k-1}$ - 1, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- **Significand coded with implied leading 1: *M* =** $1.\text{xxx}\ldots\text{x}_2$
  - xxx…x: bits of frac
  - Minimum when 000…0 (*M* = 1.0)
  - Maximum when 111…1 (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

# Normalized Encoding Example

- **Value:** `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  $M = 1.\underline{1101101101101}_2$

  $\texttt{frac} = \underline{1101101101101}0000000000_2$

- **Exponent**

  $E = 13$

  $Bias = 127$

  $Exp = 140 = 10001100_2$

- **Result:**

| s | exp | frac |
|---|---|---|
| 0 | 10001100 | 11011011011010000000000 |

# Denormalized Values

- **Condition:** exp = 000…0

- **Exponent value: $E = 1 - Bias$ (instead of $E = -Bias$)**
- **Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$**
  - **`xxx…x`**: bits of **`frac`**
- **Cases**
  - **`exp`** = 000…0, **`frac`** = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - **`exp`** = 000…0, **`frac`** ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

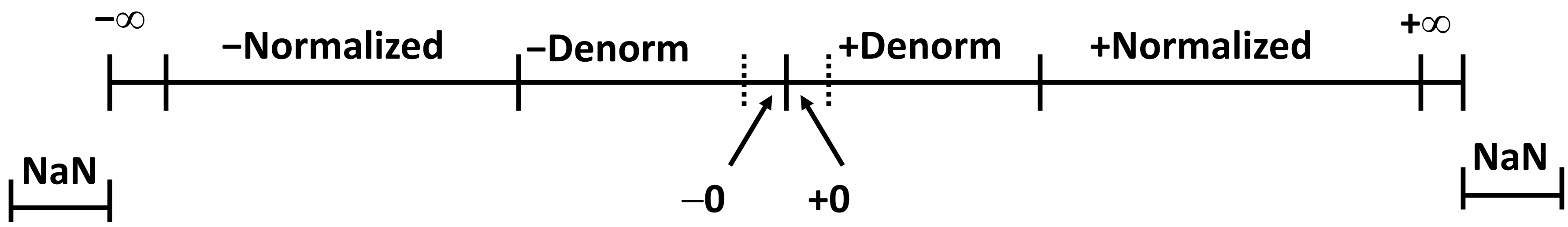
# Special Values

- **Condition: `exp`** = 111…1

- **Case: `exp`** = 111…1, **`frac`** = 000…0
  - Represents value $\infty$ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$

- **Case: `exp`** = 111…1, **`frac`** $\neq$ 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty \times 0$

# Visualization: Floating Point Encodings



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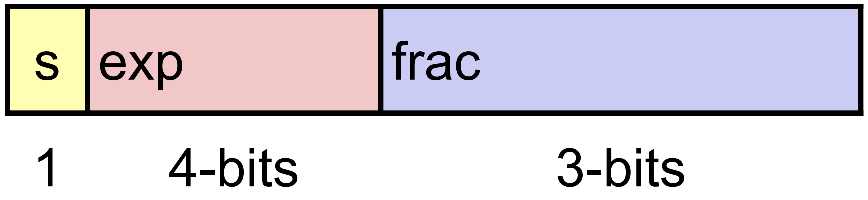
# Tiny Floating Point Example




- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the **`frac`**

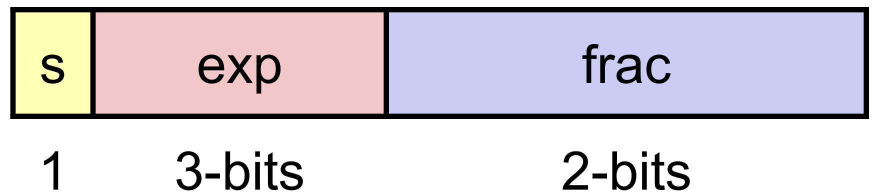
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

# Dynamic Range (Positive Only)

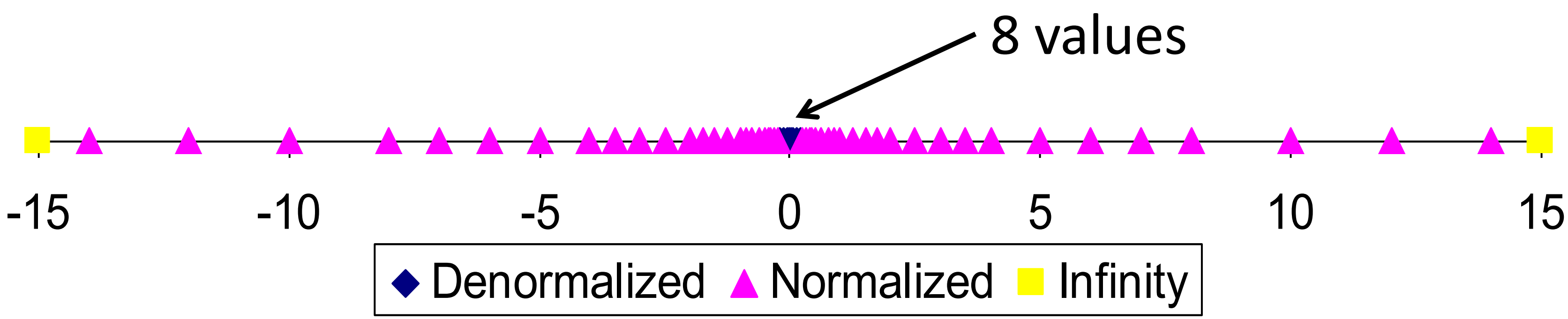
| | s | exp | frac | E | Value | |
|---|---|---|---|---|---|---|
| | 0 | 0000 | 000 | -6 | 0 | |
| | 0 | 0000 | 001 | -6 | 1/8*1/64 = 1/512 | closest to zero |
| Denormalized numbers | 0 | 0000 | 010 | -6 | 2/8*1/64 = 2/512 | |
| | … | | | | | |
| | 0 | 0000 | 110 | -6 | 6/8*1/64 = 6/512 | |
| | 0 | 0000 | 111 | -6 | 7/8*1/64 = 7/512 | largest denorm |
| | 0 | 0001 | 000 | -6 | 8/8*1/64 = 8/512 | smallest norm |
| | 0 | 0001 | 001 | -6 | 9/8*1/64 = 9/512 | |
| | … | | | | | |
| | 0 | 0110 | 110 | -1 | 14/8*1/2 = 14/16 | |
| | 0 | 0110 | 111 | -1 | 15/8*1/2 = 15/16 | closest to 1 below |
| Normalized numbers | 0 | 0111 | 000 | 0 | 8/8*1 = 1 | |
| | 0 | 0111 | 001 | 0 | 9/8*1 = 9/8 | closest to 1 above |
| | 0 | 0111 | 010 | 0 | 10/8*1 = 10/8 | |
| | … | | | | | |
| | 0 | 1110 | 110 | 7 | 14/8*128 = 224 | |
| | 0 | 1110 | 111 | 7 | 15/8*128 = 240 | largest norm |
| | 0 | 1111 | 000 | n/a | inf | |

# Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3

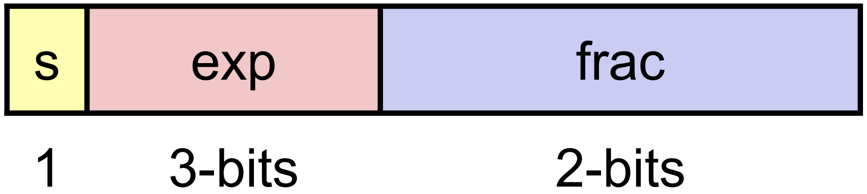


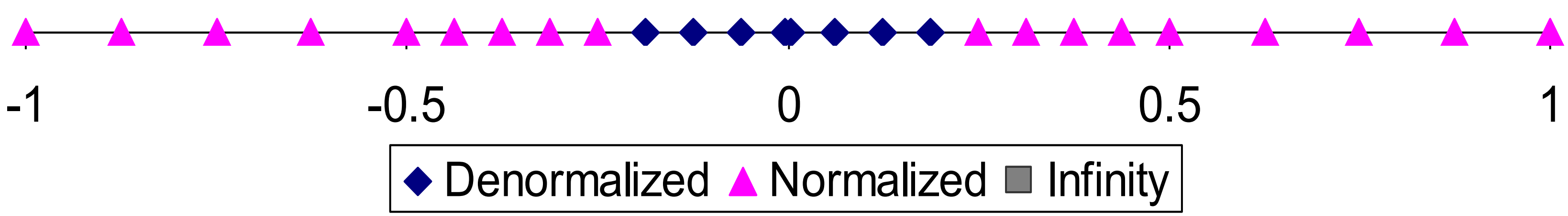

- **Notice how the distribution gets denser toward zero.**



# Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3



# Interesting Numbers `{single,double}`

| *Description* | *exp* | *frac* | *Numeric Value* |
|---|---|---|---|
| ■ **Zero** | **00…00** | **00…00** | **0.0** |
| ■ **Smallest Pos. Denorm.** | **00…00** | **00…01** | $2^{-\{23,52\}} \times 2^{-\{126,1022\}}$ |
| ▪ Single ≈ $1.4 \times 10^{-45}$ | | | |
| ▪ Double ≈ $4.9 \times 10^{-324}$ | | | |
| ■ **Largest Denormalized** | **00…00** | **11…11** | $(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$ |
| ▪ Single ≈ $1.18 \times 10^{-38}$ | | | |
| ▪ Double ≈ $2.2 \times 10^{-308}$ | | | |
| ■ **Smallest Pos. Normalized** | **00…01** | **00…00** | $1.0 \times 2^{-\{126,1022\}}$ |
| ▪ Just larger than largest denormalized | | | |
| ■ **One** | **01…11** | **00…00** | **1.0** |
| ■ **Largest Normalized** | **11…10** | **11…11** | $(2.0 - \varepsilon) \times 2^{\{127,1023\}}$ |
| ▪ Single ≈ $3.4 \times 10^{38}$ | | | |
| ▪ Double ≈ $1.8 \times 10^{308}$ | | | |

# Special Properties of Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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- **Rounding, addition, multiplication**
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# Floating Point Operations: Basic Idea

- $x +_f y = \mathrm{Round}(x + y)$

- $x \times_f y = \mathrm{Round}(x \times y)$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

# Rounding

- **Rounding Modes (illustrate with $ rounding)**

| | **$1.40** | **$1.60** | **$1.50** | **$2.50** | **–$1.50** |
|---|---|---|---|---|---|
| Towards zero | $1 | $1 | $1 | $2 | –$1 |
| Round down (−∞) | $1 | $1 | $1 | $2 | –$2 |
| Round up (+∞) | $2 | $2 | $2 | $3 | –$1 |
| Nearest Even (default) | $1 | $2 | $2 | $2 | –$2 |

- **What are the advantages of the modes?**

# Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

| | | |
|---|---|---|
| 1.2349999 | 1.23 | (Less than half way) |
| 1.2350001 | 1.24 | (Greater than half way) |
| 1.2350000 | 1.24 | (Half way—round up) |
| 1.2450000 | 1.24 | (Half way—round down) |

# Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

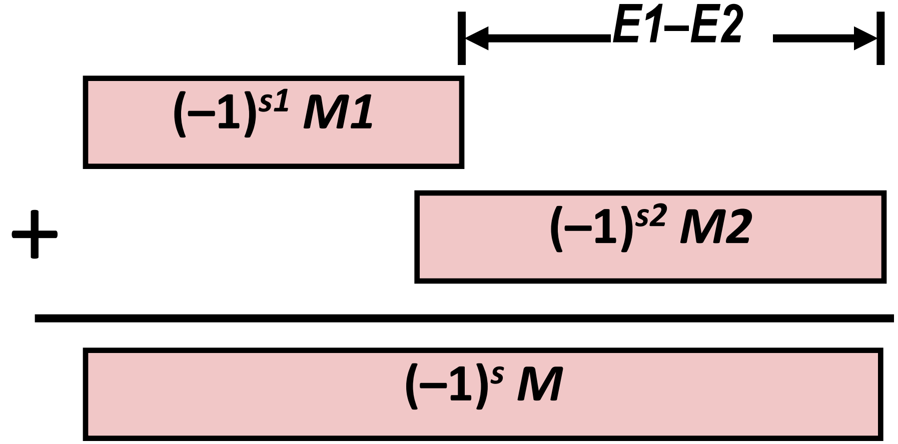
| Value | Binary | Rounded | Action | Rounded Value |
|---|---|---|---|---|
| 2 3/32 | $10.00011_2$ | $10.00_2$ | (<1/2—down) | 2 |
| 2 3/16 | $10.00110_2$ | $10.01_2$ | (>1/2—up) | 2 1/4 |
| 2 7/8 | $10.11100_2$ | $11.00_2$ | ( 1/2—up) | 3 |
| 2 5/8 | $10.10100_2$ | $10.10_2$ | ( 1/2—down) | 2 1/2 |

# FP Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \mathbf{x}\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign $s$: $s1$ ^ $s2$
  - Significand $M$: $M1$ x $M2$
  - Exponent $E$: $E1 + E2$
- **Fixing**
  - If $M \geq 2$, shift $M$ right, increment $E$
  - If $E$ out of range, overflow
  - Round $M$ to fit `frac` precision
- **Implementation**
  - Biggest chore is multiplying significands

# Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume $E1 > E2$




- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign $s$, significand $M$:
    - Result of signed align & add
  - Exponent $E$: $E1$

- **Fixing**
  - If $M \geq 2$, shift $M$ right, increment $E$
  - if $M < 1$, shift $M$ left $k$ positions, decrement $E$ by $k$
  - Overflow if $E$ out of range
  - Round $M$ to fit `frac` precision

# Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? ***Yes***
    - But may generate infinity or NaN
  - Commutative? ***Yes***
  - Associative? ***No***
    - Overflow and inexactness of rounding
  - 0 is additive identity? ***Yes***
  - Every element has additive inverse ***Almost***
    - Except for infinities & NaNs
- **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
    - Except for infinities & NaNs

# Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? ***Yes***
    - But may generate infinity or NaN
  - Multiplication Commutative? ***Yes***
  - Multiplication is Associative? ***No***
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? ***Yes***
  - Multiplication distributes over addition? ***No***
    - Possibility of overflow, inexactness of rounding

- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? ***Almost***
    - Except for infinities & NaNs

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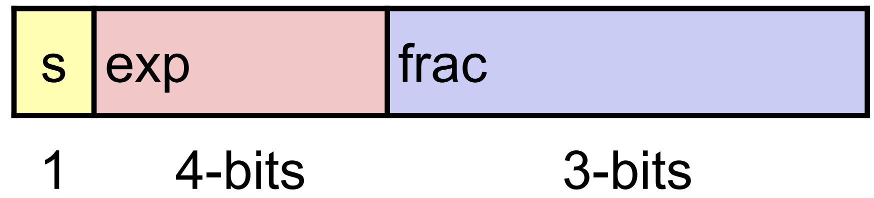
# Floating Point in C

- **C Guarantees Two Levels**
  - `float` single precision
  - `double` double precision
- **Conversions/Casting**
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

# Creating Floating Point Number




- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

| | |
|---|---|
| 128 | 10000000 |
| 13 | 00001101 |
| 17 | 00010001 |
| 19 | 00010011 |
| 138 | 10001010 |
| 63 | 00111111 |

# Normalize

| s | exp | frac |
| --- | --- | --- |
| 1 | 4-bits | 3-bits |

- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

| *Value* | *Binary* | *Fraction* | *Exponent* |
| --- | --- | --- | --- |
| 128 | 10000000 | 1.0000000 | 7 |
| 13 | 00001101 | 1.1010000 | 3 |
| 17 | 00010001 | 1.0001000 | 4 |
| 19 | 00010011 | 1.0011000 | 4 |
| 138 | 10001010 | 1.0001010 | 7 |
| 63 | 00111111 | 1.1111100 | 5 |

# Rounding

- **Binary Fractional Numbers**
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$

| *Value* | *Fraction* | *Incr?* | *Rounded* |
|---|---|---|---|
| 128 | 1.0000000 | N | 1.000 |
| 15 | 1.1010000 | N | 1.101 |
| 17 | 1.0001000 | N | 1.000 |
| 19 | 1.0011000 | Y | 1.010 |
| 138 | 1.0001010 | Y | 1.001 |
| 63 | 1.1111100 | Y | 10.000 |

# Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

| *Value* | *Rounded* | *Exp* | *Adjusted* | *Result* | *float_8* |
|---|---|---|---|---|---|
| 128 | 1.000 | 7 | | 128 | 01110000 |
| 13 | 1.101 | 3 | | 13 | 01010101 |
| 17 | 1.000 | 4 | | 16 | 01011000 |
| 19 | 1.010 | 4 | | 20 | 01011010 |
| 138 | 1.001 | 7 | | 144 | 01110001 |
| 63 | 10.000 | 5 | 1.000/6 | 64 | 01101000 |

Remember that E = e – bias
Bias = 7

# Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **`d`** nor **`f`** is NaN

- x == (int)(float) x
- x == (int)(double) x
- f == (float)(double) f
- d == (float) d
- f == -(-f);
- 2/3 == 2/3.0
- d < 0.0 ⇒ ((d*2) < 0.0)
- d > f ⇒ -f > -d
- d * d >= 0.0
- (d+f)-d == f

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# Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x $2^E$**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers