
WELCOME
TO
15-213
18-213
15-513

Bits, Bytes, and Integers – Part 2

15-213: Introduction to Computer Systems
3rd Lecture, May 21, 2020

Assignment Announcements

- **Lab 0 available via Autolab.**
  - Due Fri, May 29, 11:00pm
  - No grace days
  - No late submissions
  - Just do it!
- **Lab 1 will be available via Autolab**
  - Release on evening of Fri, May 22
  - Due Fri, May 29, 11:59pm
  - Read instructions carefully: writeup, bits.c, tests.c
    - Quirky software infrastructure
  - Based on lectures 2, 3, and 4 (CS:APP Chapter 2)
  - After today's lecture you will know everything for the integer problems
  - Floating point covered Fri, May 22

Summary From Last Lecture

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
- Representations in memory, pointers, strings
- Summary

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

Two's Complement Examples (w = 5)

	-16	8	4	2	1	
10 =	0	1	0	1	0	8+2 = 10

	-16	8	4	2	1	
-10 =	1	0	1	1	0	-16+4+2 = -10

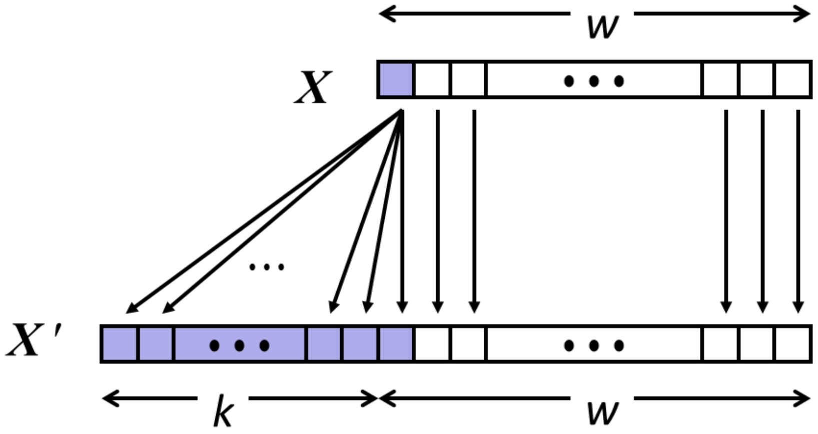
Unsigned & Signed Numeric Values

X	**B2U(*X*)**	**B2T(*X*)**
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

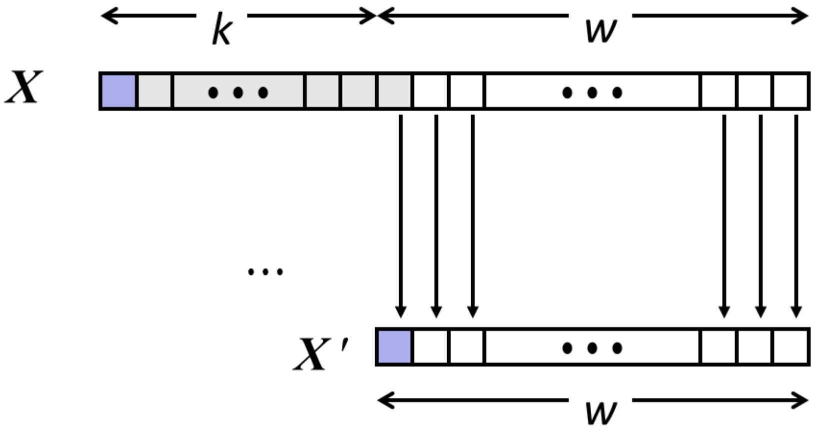
- **Equivalence**
  - Same encodings for nonnegative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- **Expression containing signed and unsigned int:** `int` is cast to `unsigned`

Sign Extension and Truncation

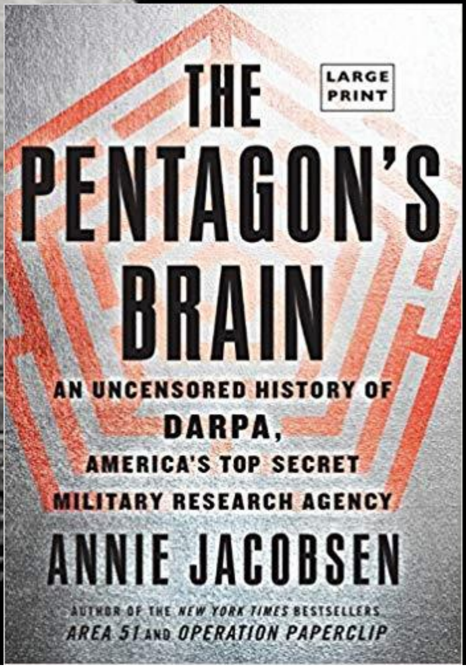
- **Sign Extension**



- **Truncation**



- **Misunderstanding integers can lead to the end of the world as we know it!**
- Thule (Qaanaaq), Greenland
- US DoD “Site J” Ballistic Missile Early Warning System (BMEWS)
- 10/5/60: world nearly ends
- Missile radar echo: 1/8s
- BMEWS reports: 75s echo(!)
- 1000s of objects reported
- NORAD alert level 5:
  - Immediate incoming nuclear attack!!!!



THE
PENTAGON'S
BRAIN

LARGE PRINT

AN UNCENSORED HISTORY OF
DARPA,
AMERICA'S TOP SECRET
MILITARY RESEARCH AGENCY

ANNIE JACOBSEN

AUTHOR OF THE NEW YORK TIMES BESTSELLERS
AREA 51 AND OPERATION PAPERCLIP

- **Kruschev was in NYC 10/5/60 (weird time to attack)**
  - someone in Qaanaaq said "why not go check outside?"
- **"Missiles" were actually THE MOON RISING OVER NORWAY**
- **Expected max distance: 3000 mi; Moon distance: .25M miles!**
- **.25M miles % sizeof(distance) = 2200mi.**
- **Overflow of distance nearly caused nuclear apocalypse!!**

Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - **Addition, negation, multiplication, shifting**
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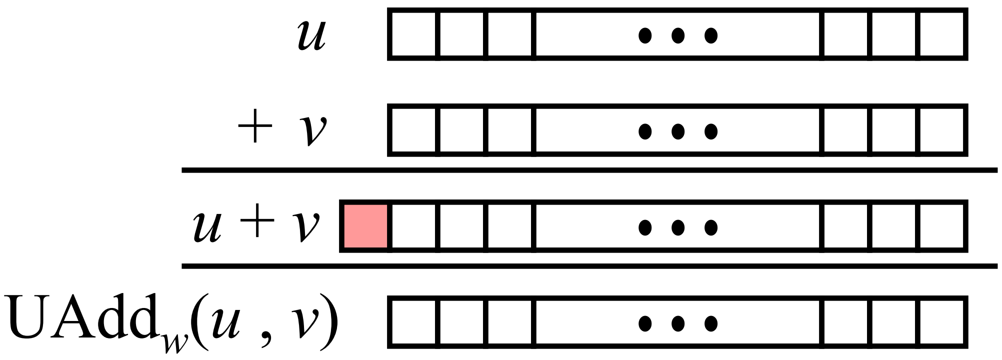
Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

$$s = \text{UAdd}_w(u , v) = u + v \mod 2^w$$

```
unsigned char     1110 1001      E9        223
                + 1101 0101    + D5      + 213
```

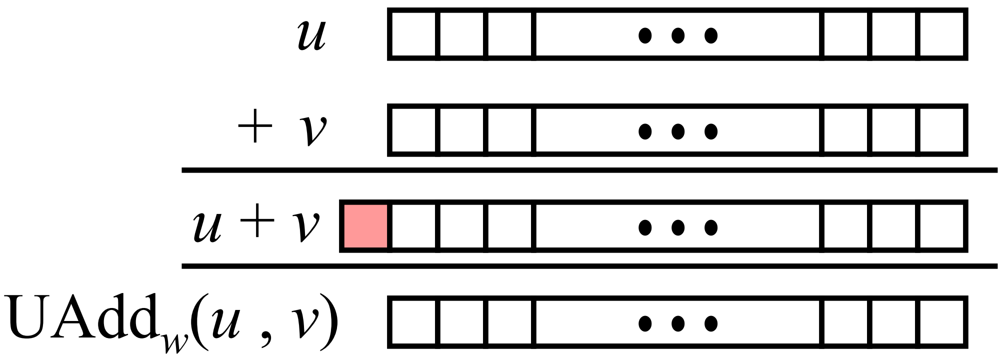
Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

$$s \quad = \quad \mathrm{UAdd}_w(u\ ,\ v) \quad = \quad u + v \ \bmod 2^w$$

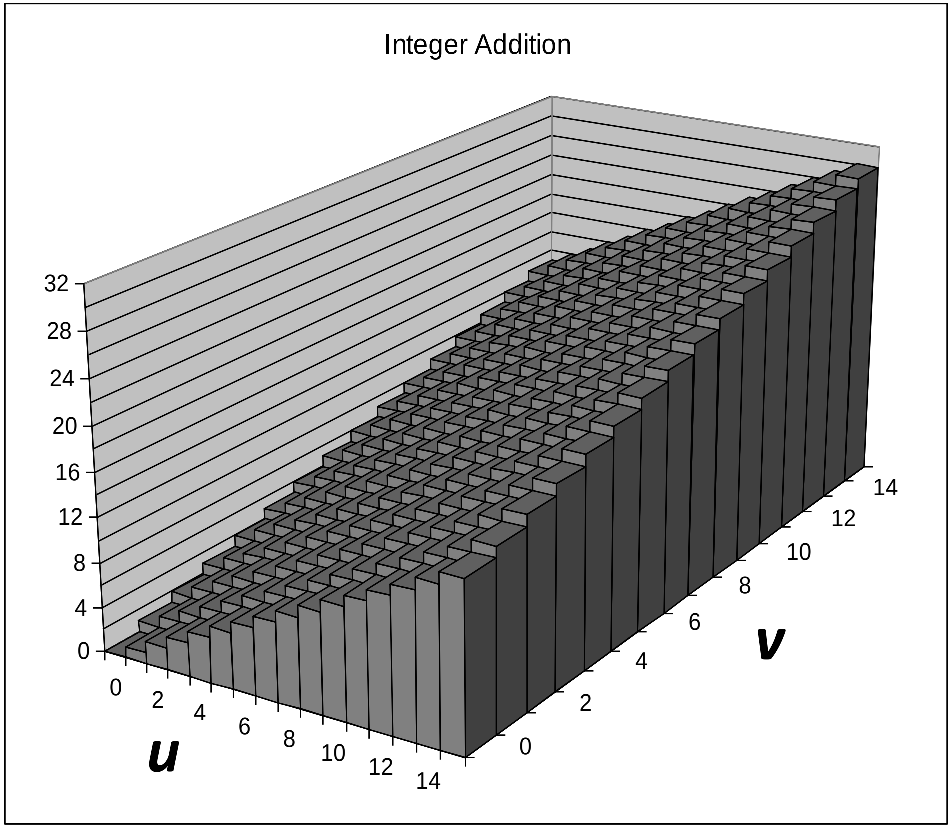
`unsigned char`	1110 1001	E9	223
	+ 1101 0101	+ D5	+ 213
	1 1011 1110	1BE	446
	1011 1110	BE	190

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualizing (Mathematical) Integer Addition

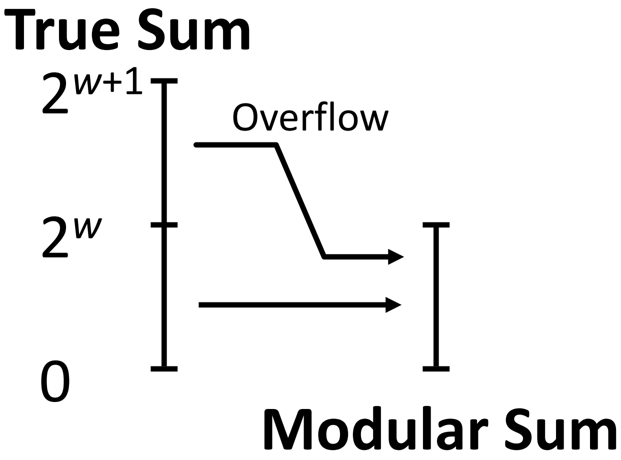
- **Integer Addition**
  - 4-bit integers *u*, *v*
  - Compute true sum $\text{Add}_4(u\ ,\ v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

$\text{Add}_4(u\ ,\ v)$



Visualizing Unsigned Addition

- **Wraps Around**
  - If true sum ≥ 2^w
  - At most once



True Sum

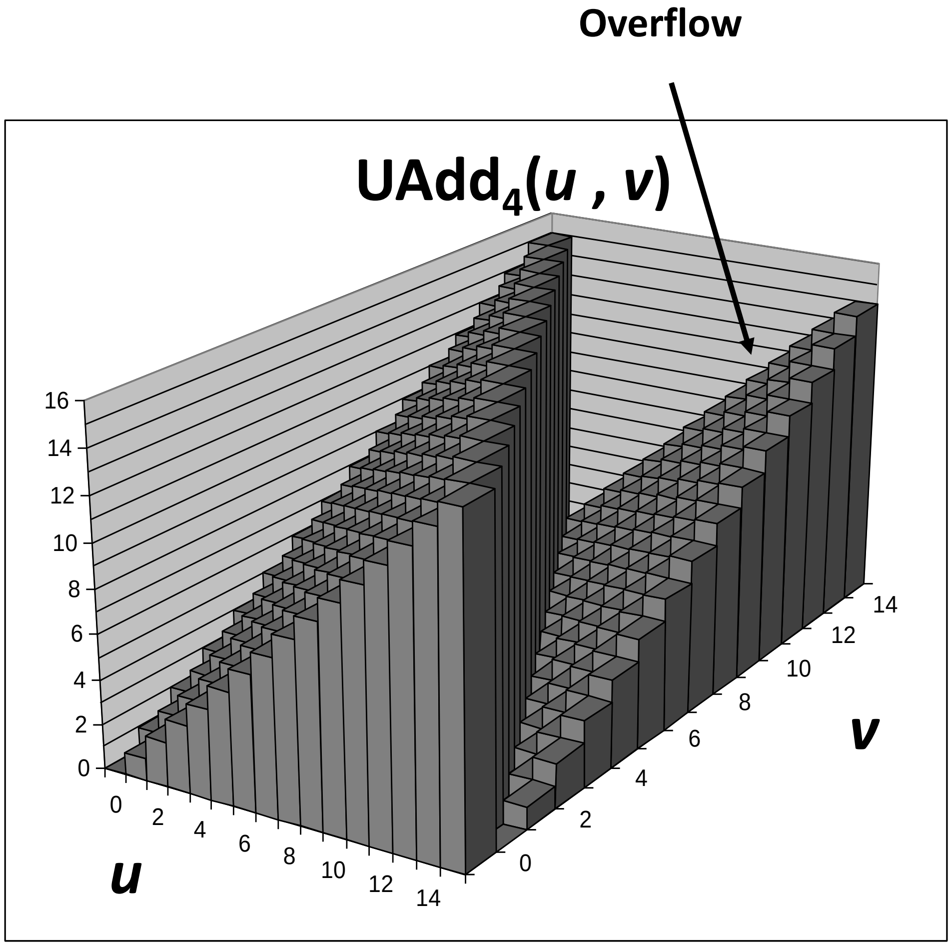
2^{w+1}

Overflow

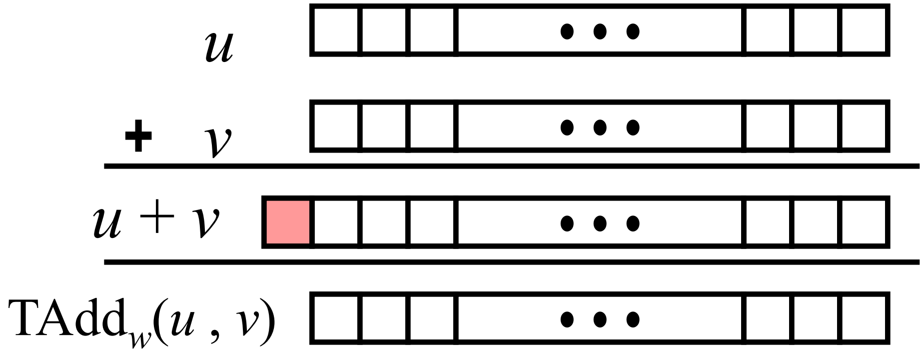
2^w

0

Modular Sum



Two’s Complement Addition



Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits

- **TAdd and UAdd have Identical Bit-Level Behavior**
  - Signed vs. unsigned addition in C:

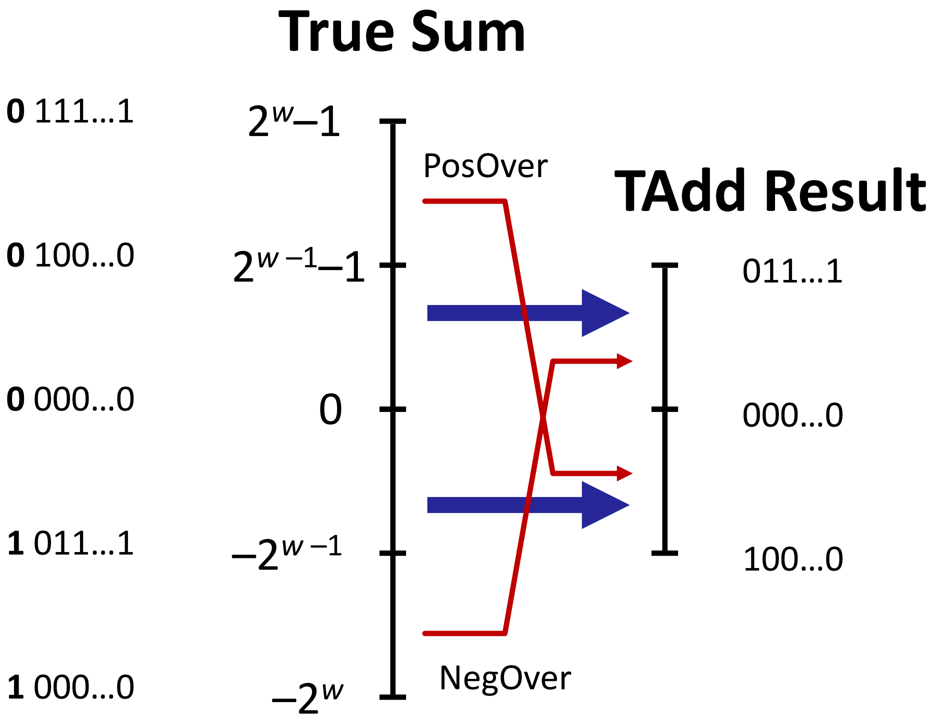
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

  - Will give `s == t`

1110 1001	E9	-23
+ 1101 0101	+ D5	+ -43
1 1011 1110	1BE	-66
1011 1110	BE	-66

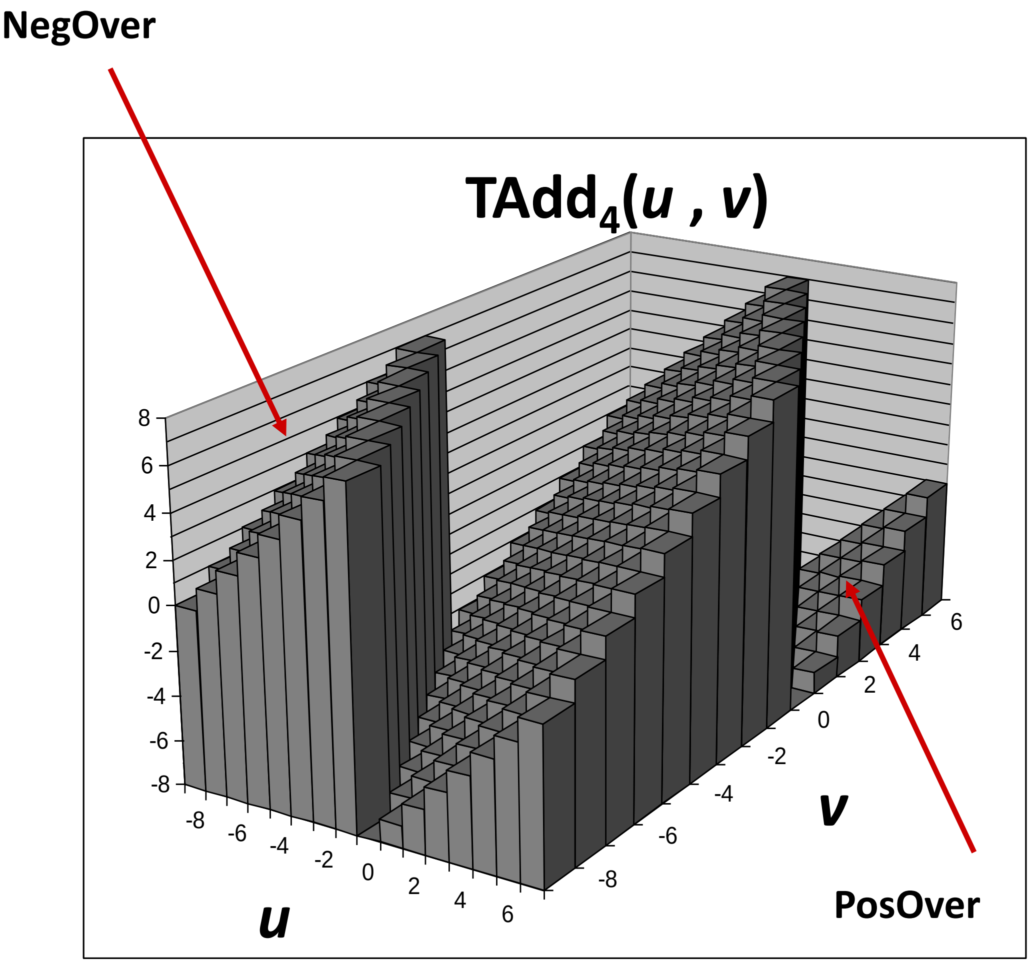
TAdd Overflow

- **Functionality**
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2’s comp. integer



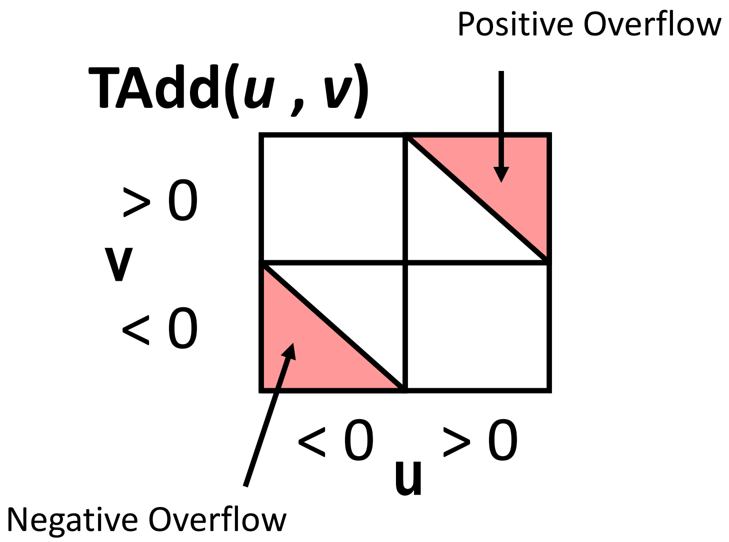
Visualizing 2's Complement Addition

- **Values**
  - 4-bit two's comp.
  - Range from -8 to +7
- **Wraps Around**
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Characterizing TAdd

- **Functionality**
  - True sum requires *w*+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer

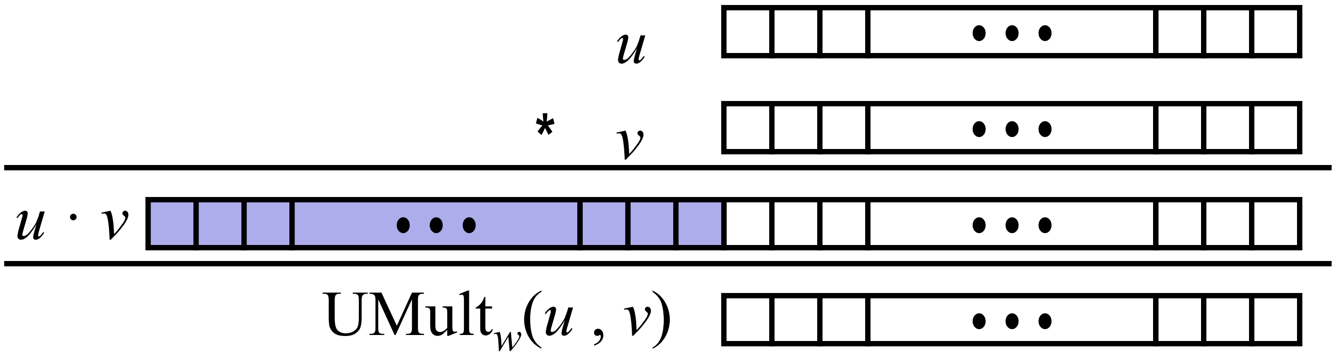


$$TAdd_w(u,v) = \begin{cases} u+v+2^w & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^w & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

Multiplication

- **Goal: Computing Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **But, exact results can be bigger than *w* bits**
  - Unsigned: up to 2*w* bits
    - Result range: $0 \leq x * y \leq (2^{w} - 1)^{2} = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \geq (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_{w})^{2}$
    - Result range: $x * y \leq (-2^{w-1})^{2} = 2^{2w-2}$
- **So, maintaining exact results…**
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C



Operands: *w* bits

True Product: 2**w* bits

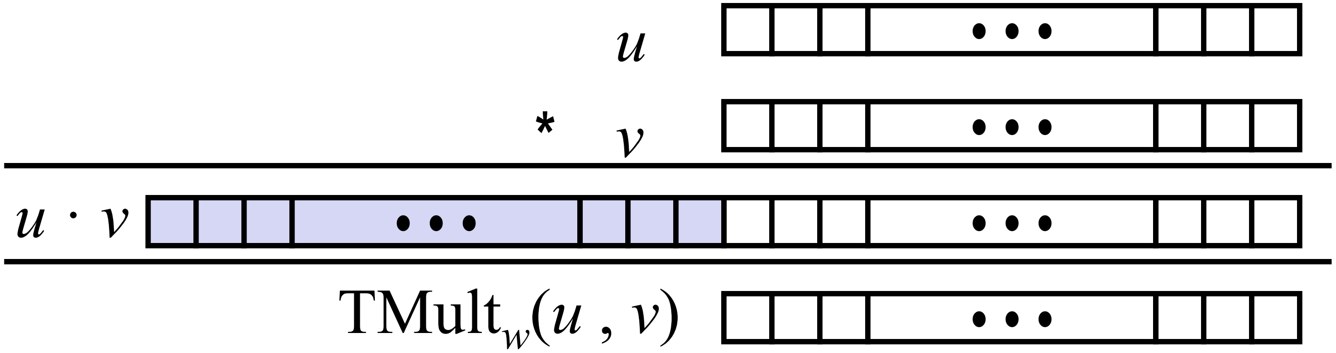
Discard *w* bits: *w* bits

- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

$\text{UMult}_w(u\ ,\ v) = u \cdot v \bmod 2^w$

```
                1110 1001      E9       223
*               1101 0101   *  D5   *   213
 1100 0001 1101 1101         C1DD     47499
                1101 1101      DD       221
```

Signed Multiplication in C



Operands: *w* bits

True Product: 2**w* bits

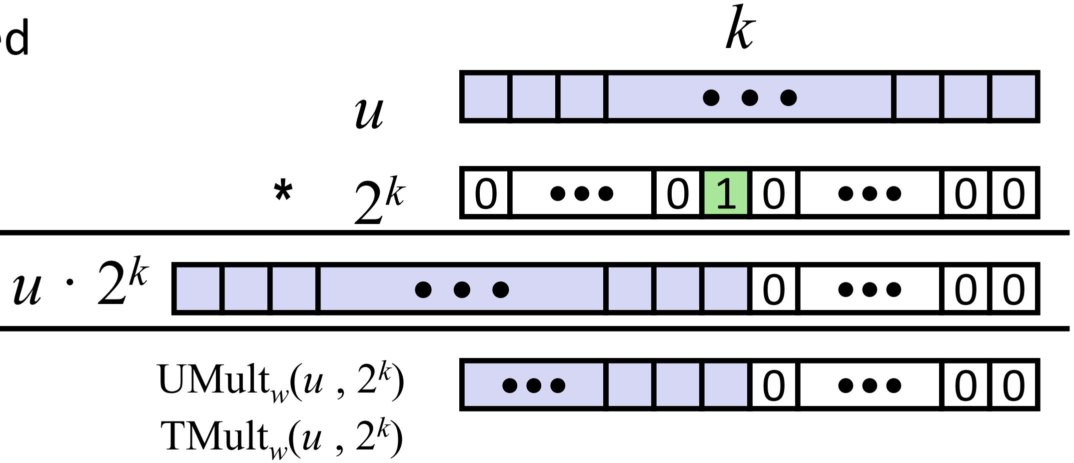
Discard *w* bits: *w* bits

- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

```
               1110 1001        E9        -23
*              1101 0101    *   D5    *   -43
---------------------------   ------  ---------
0000 0011 1101 1101           03DD        989
---------------------------   ------  ---------
               1101 1101          DD        -35
```

Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` gives `u * 2`^{k}
  - Both signed and unsigned



Operands: *w* bits

True Product: *w*+*k* bits

Discard *k* bits: *w* bits

- **Examples**
  - `u << 3 == u * 8`
  - `(u << 5) – (u << 3)== u * 24`
  - Most machines shift and add faster than multiply
    - Compiler generates this code auto

Important Lession:

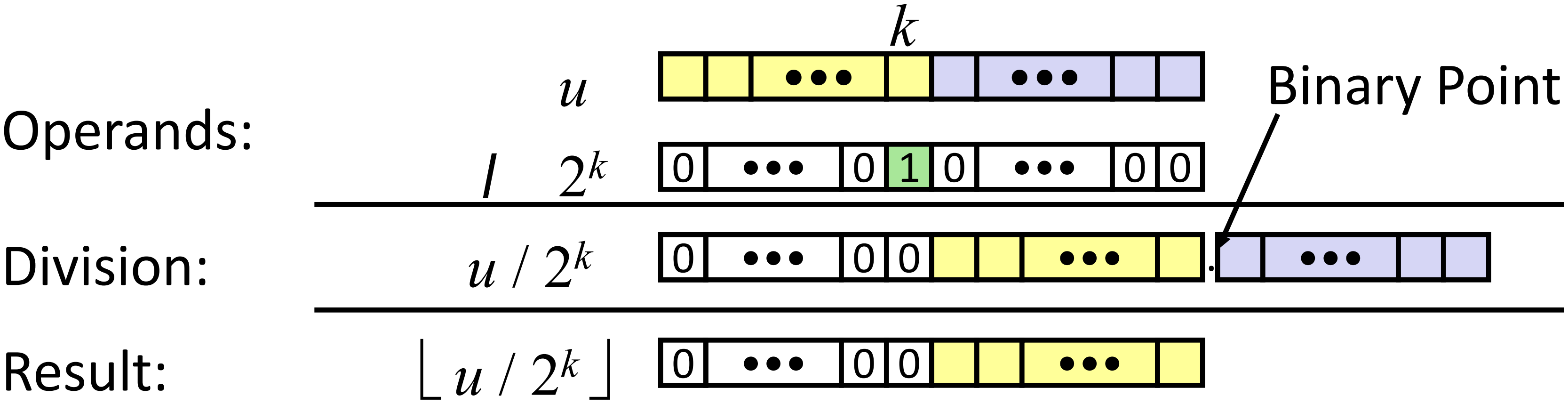
Trust Your Compiler!

Multiplication

- **Goal: Computing Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **But, exact results can be bigger than *w* bits**
  - Unsigned: up to 2*w* bits
    - Result range: $0 \le x * y \le (2^{w} - 1)^{2} = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_{w})^{2}$
    - Result range: $x * y \le (-2^{w-1})^{2} = 2^{2w-2}$
- **So, maintaining exact results…**
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Power-of-2 Divide with Shift

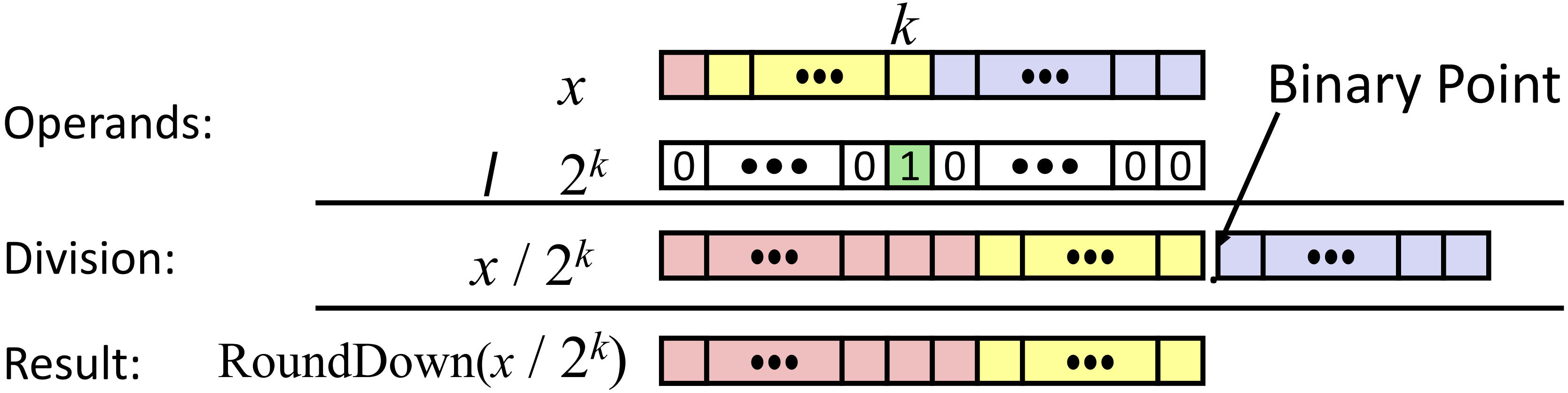
- **Quotient of Unsigned by Power of 2**
  - `u >> k` gives $\lfloor u / 2^k \rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Signed Power-of-2 Divide with Shift

- **Quotient of Signed by Power of 2**
  - `x >> k` gives $\lfloor x \ / \ 2^k \rfloor$
  - Uses arithmetic shift
  - Rounds wrong direction when `u < 0`

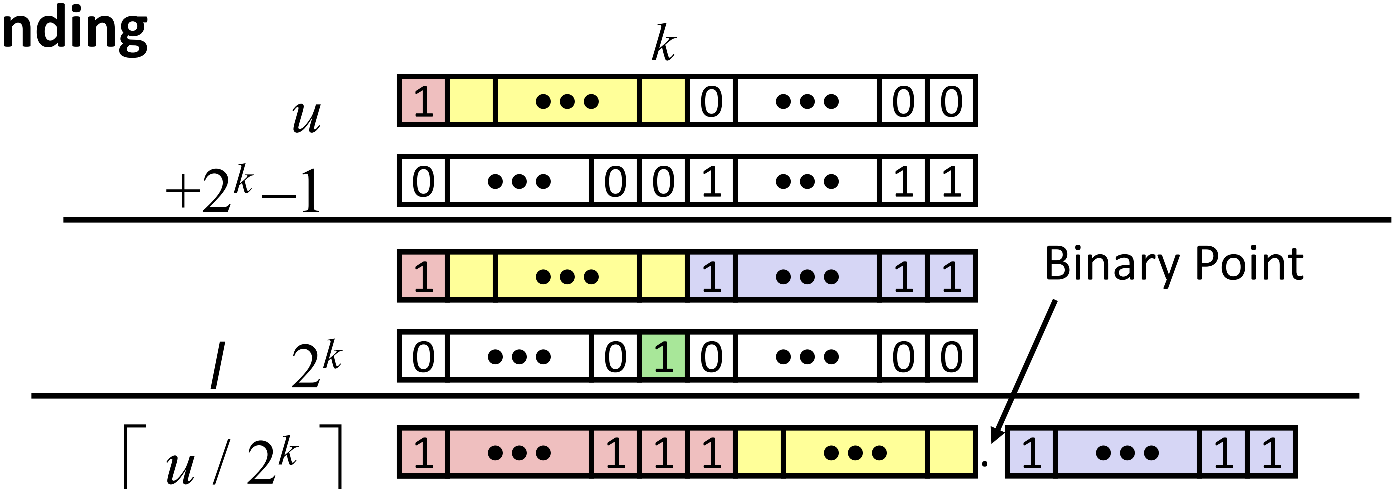


	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Correct Power-of-2 Divide

- **Quotient of Negative Number by Power of 2**
  - Want $\lceil \mathtt{x} \ / \ 2^k \rceil$ (Round Toward 0)
  - Compute as $\lfloor (\mathtt{x}+2^k-\mathtt{1}) / \ 2^k \rfloor$
    - In C: `(x + (1<<k)-1) >> k`
    - Biases dividend toward 0

Case 1: No rounding



Dividend:

k

u 1 ••• 0 ••• 0 0

$+2^k-1$ 0 ••• 0 0 1 ••• 1 1

1 ••• 1 ••• 1 1

Divisor:

$/ \quad 2^k$ 0 ••• 0 1 0 ••• 0 0

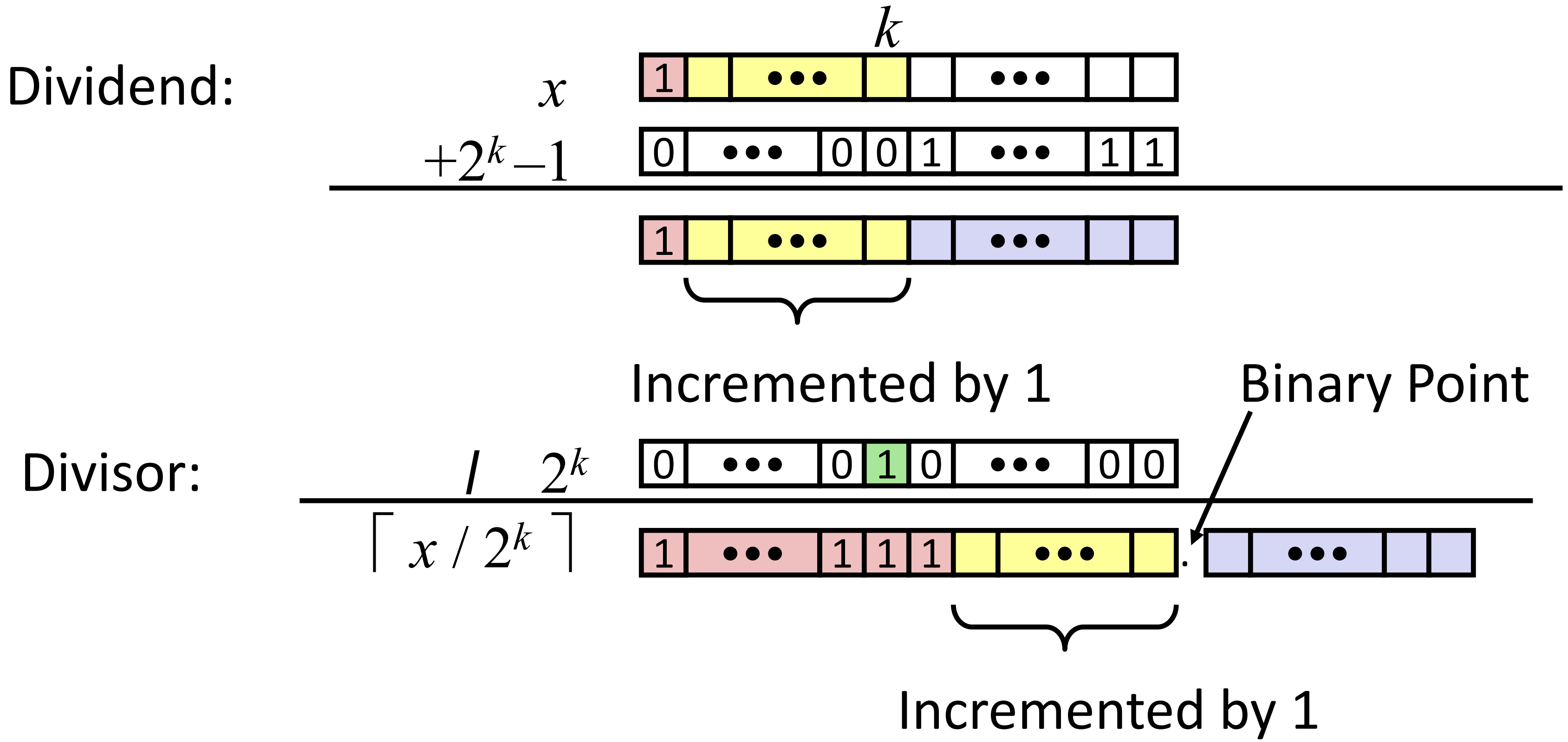
$\lceil u / 2^k \rceil$ 1 ••• 1 1 1 ••• . 1 ••• 1 1

Binary Point

Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Negation: Complement & Increment

- **Negate through complement and increase**

  `~x + 1 == -x`

- **Example**
  - Observation: `~x + x == 1111…111 == -1`

$$
\begin{array}{rr|c|c|c|c|c|c|c|c|}
 & x & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
+ & \sim x & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

x = 15213

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`~x`	-15214	`C4 92`	`11000100 10010010`
`~x+1`	-15213	`C4 93`	`11000100 10010011`
`y`	-15213	`C4 93`	`11000100 10010011`

Complement & Increment Examples

x = 0

	Decimal	Hex	Binary
0	0	`00 00`	`00000000 00000000`
~0	-1	`FF FF`	`11111111 11111111`
~0+1	0	`00 00`	`00000000 00000000`

x = TMin

	Decimal	Hex	Binary
`x`	-32768	`80 00`	`10000000 00000000`
`~x`	32767	`7F FF`	`01111111 11111111`
`~x+1`	-32768	`80 00`	`10000000 00000000`

Canonical counter example

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- Representing information as bits
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- **Integers**
  - Representation: unsigned and signed
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  - Addition, negation, multiplication, shifting
  - **Summary**
- Representations in memory, pointers, strings

Arithmetic: Basic Rules

- **Addition:**
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2^w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2^w

- **Multiplication:**
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

- ***Don't*** **use without understanding implications**
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

Counting Down with Unsigned

- **Proper way to use unsigned as loop index**

```
unsigned i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

- **See Robert Seacord, *Secure Coding in C and C++***
  - C Standard guarantees that unsigned addition will behave like modular arithmetic
    - 0 – 1 → *UMax*
- **Even better**

```
size_t i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

  - Data type **`size_t`** defined as unsigned value with length = word size

Why Should I Use Unsigned? (cont.)

- ***Do* Use When Performing Modular Arithmetic**
  - Multiprecision arithmetic
- ***Do* Use When Using Bits to Represent Sets**
  - Logical right shift, no sign extension
- ***Do* Use In System Programming**
  - Bit masks, device commands,…

Quiz Time!

Check out:

https://canvas.cmu.edu/courses/16836

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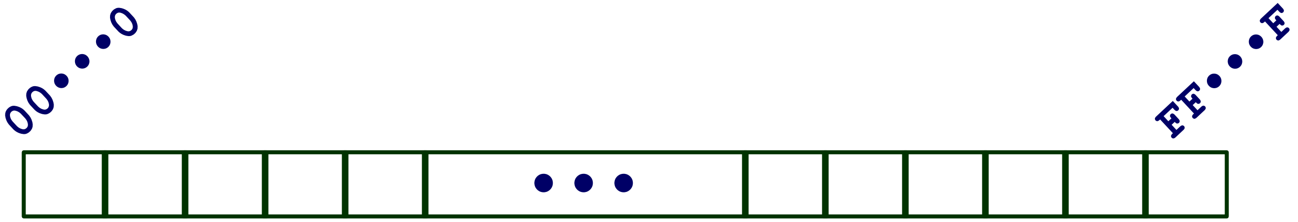
- **Representations in memory, pointers, strings**

Byte-Oriented Memory Organization



- **Programs refer to data by address**
  - Conceptually, envision it as a very large array of bytes
    - In reality, it’s not, but can think of it that way
  - An address is like an index into that array
    - and, a pointer variable stores an address

- **Note: system provides private address spaces to each “process”**
  - Think of a process as a program being executed
  - So, a program can clobber its own data, but not that of others

Machine Words

- **Any given computer has a “Word Size”**
  - Nominal size of integer-valued data
    - and of addresses

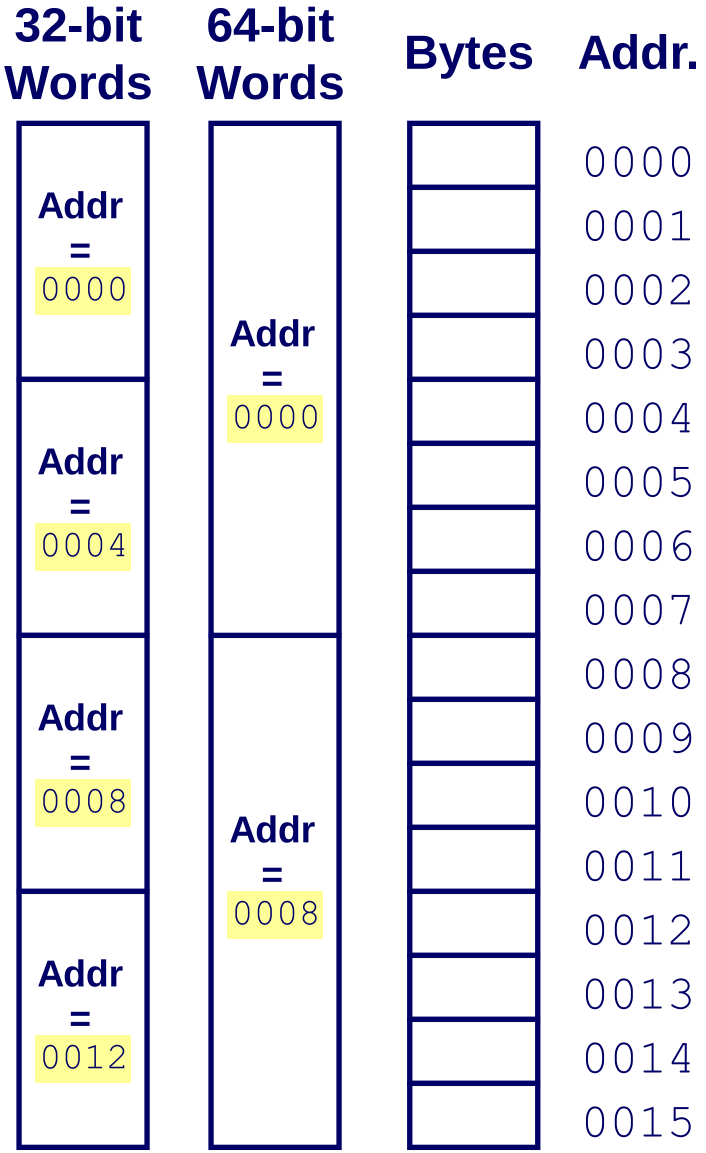
  - Until recently, most machines used 32 bits (4 bytes) as word size
    - Limits addresses to 4GB (2^{32} bytes)

  - Increasingly, machines have 64-bit word size
    - Potentially, could have 18 EB (exabytes) of addressable memory
    - That’s 18.4 X 10^{18}

  - Machines still support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Word-Oriented Memory Organization

- **Addresses Specify Byte Locations**
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



32-bit Words

64-bit Words

Bytes

Addr.

Addr = 0000

Addr = 0004

Addr = 0008

Addr = 0012

Addr = 0000

Addr = 0008

0000
0001
0002
0003
0004
0005
0006
0007
0008
0009
0010
0011
0012
0013
0014
0015

Example Data Representations

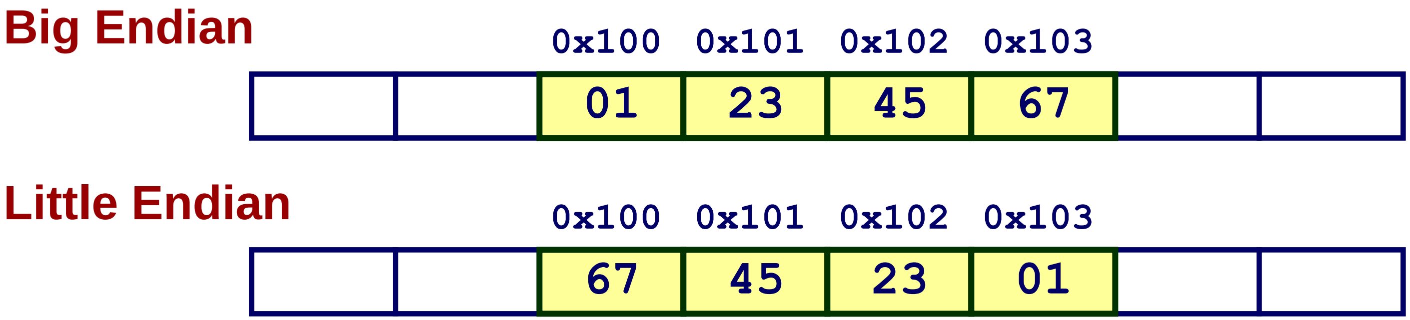
C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
pointer	4	8	8

Byte Ordering

- **So, how are the bytes within a multi-byte word ordered in memory?**
- **Conventions**
  - Big Endian: Sun (Oracle SPARC), PPC Mac, *Internet*
    - Least significant byte has highest address
  - Little Endian: *x86*, ARM processors running Android, iOS, and Linux
    - Least significant byte has lowest address

Byte Ordering Example

- **Example**
  - Variable x has 4-byte value of 0x01234567
  - Address given by &x is 0x100



Big Endian

		0x100	0x101	0x102	0x103		
		01	23	45	67		

Little Endian

		0x100	0x101	0x102	0x103		
		67	45	23	01		

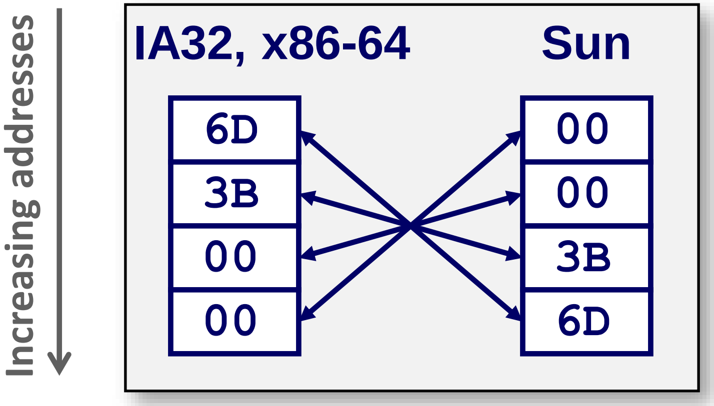
Representing Integers

Decimal: 15213

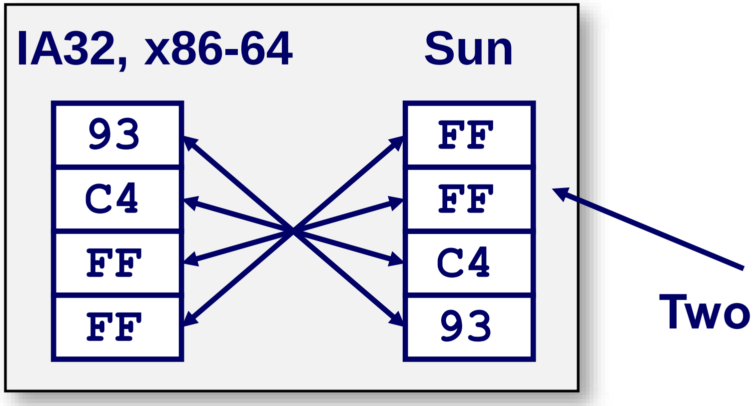
Binary: 0011 1011 0110 1101

Hex: 3 B 6 D

`int A = 15213;`

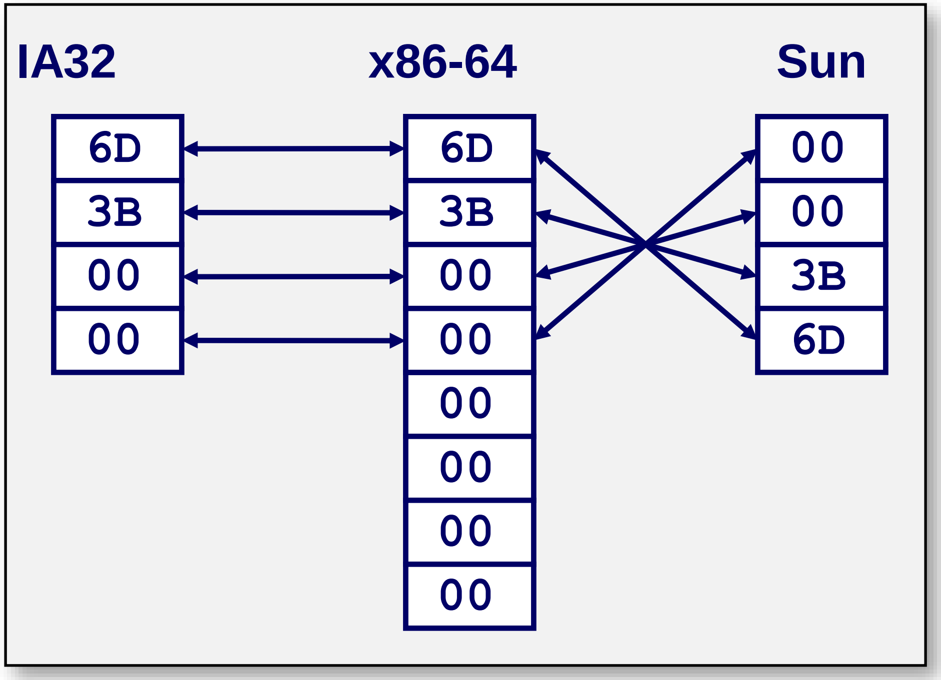


`int B = -15213;`



Two's complement representation

`long int C = 15213;`



Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
  size_t i;
  for (i = 0; i < len; i++)
    printf(”%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:
%p: Print pointer
%x: Print Hexadecimal

`show_bytes` Execution Example

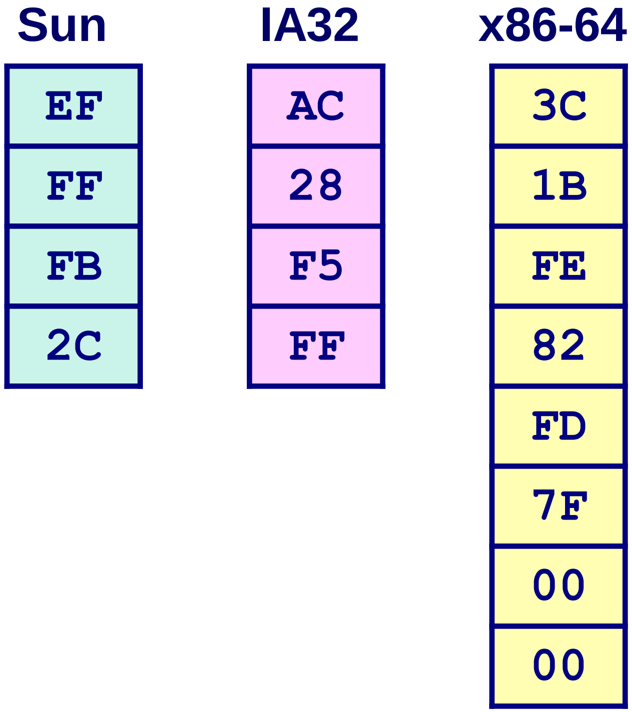
```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;
0x7fffb7f71dbc    6d
0x7fffb7f71dbd    3b
0x7fffb7f71dbe    00
0x7fffb7f71dbf    00
```

Representing Pointers

```
int B = -15213;
int *P = &B;
```



Sun	IA32	x86-64
EF	AC	3C
FF	28	1B
FB	F5	FE
2C	FF	82
		FD
		7F
		00
		00

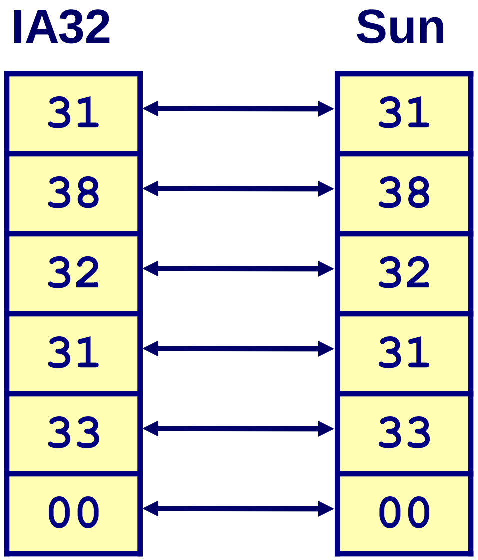
Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

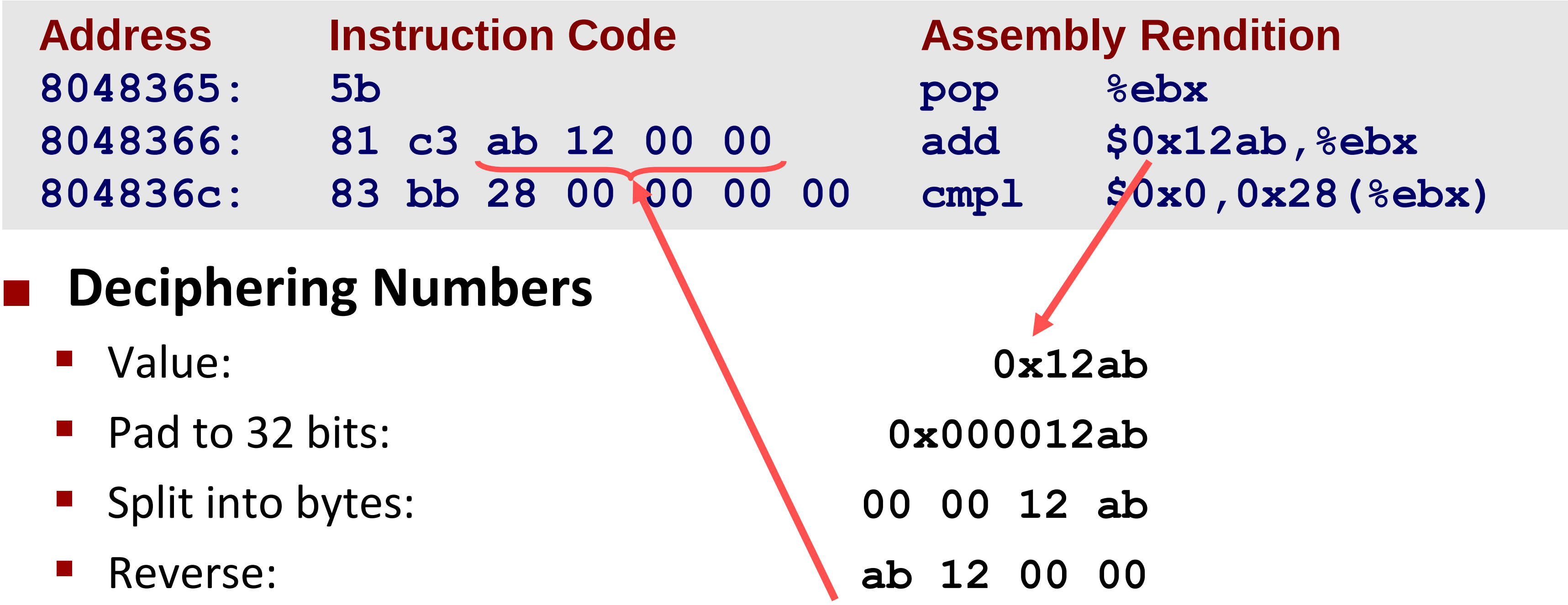
`char S[6] = "18213";`

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character “0” has code 0x30
      - Digit *i* has code 0x30+*I*
    - ***man ascii*** *for code table*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue



Reading Byte-Reversed Listings

- **Disassembly**
  - Text representation of binary machine code
  - Generated by program that reads the machine code
- **Example Fragment**



```
Address        Instruction Code                 Assembly Rendition
8048365:       5b                               pop    %ebx
8048366:       81 c3 ab 12 00 00                add    $0x12ab,%ebx
804836c:       83 bb 28 00 00 00 00             cmpl   $0x0,0x28(%ebx)
```

- **Deciphering Numbers**
  - Value: `0x12ab`
  - Pad to 32 bits: `0x000012ab`
  - Split into bytes: `00 00 12 ab`
  - Reverse: `ab 12 00 00`

Summary

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
  - **Representation: unsigned and signed**
  - **Conversion, casting**
  - **Expanding, truncating**
  - **Addition, negation, multiplication, shifting**
- **Representations in memory, pointers, strings**
- **Summary**

Integer C Puzzles

x < 0	⇒	((x*2) < 0)	✗
ux >= 0			✓
x & 7 == 7	⇒	(x<<30) < 0	✓
ux > -1			✗
x > y	⇒	-x < -y	✗
x * x >= 0			✗
x > 0 && y > 0	⇒	x + y > 0	✗
x >= 0	⇒	-x <= 0	✓
x <= 0	⇒	-x >= 0	✗
(x|-x)>>31 == -1			✗
ux >> 3 == ux/8			✓
x >> 3 == x/8			✗
x & (x-1) != 0			✗

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```