Auctions, Negotiations

R&N, 17.6

Player B

Player A

+5,??	-4,??
0,??	6,??

Player B

Player A	**+5,??**	**-4,??**
	0,??	**6,??**

- So far, we have assumed that both players know exactly the payoffs they get for every pair of pure strategies
- What if one of the players (Player A, e.g.) does not know the payoffs for Player B?
- Is it still possible to find an solution (equilibrium)?
- This case models a large number of important decision-making scenarios. When does this situation arise?

	Hockey	**Movie**
Hockey	**+2,+1**	**0,0**
Movie	**0,0**	**+1,+2**

- Two friends have different tastes, A likes to watch hockey games but B prefers to go see a movie. Neither likes to go to his preferred choice alone; each would rather go the other’s preferred choice rather than go alone to its own.

	Hockey	Movie
Hockey	+2,??	0,??
Movie	0,??	+1,??

	Hockey	Movie
Hockey	+2,??	0,??
Movie	0,??	+1,??

- Two friends have different tastes, A likes to watch hockey games but B prefers to go see a movie. Neither likes to go to his preferred choice alone; each would rather go the other's preferred choice rather than go alone to its own.
- *But suppose now that A is sure that he wants to share the activity with B; but he is not sure that B wants to sure the activity → A does not know B's payoff structure.*

B wishes to *meet* A with probability ½

	Hockey	**Movie**
Hockey	**+2,+1**	**0,0**
Movie	**0,0**	**+1,+2**

B wishes to *avoid* A with probability ½

	Hockey	**Movie**
Hockey	**+2,0**	**0,+2**
Movie	**0,+1**	**+1,0**

- A does not know B’s preferences, but he may know *probabilities* for each of B’s preferences → In this example, A may know how *likely* it is that B wants to meet/avoid him

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Formalization

- In these situations, each player can appear as having different *types*.
- In that example, Player B can be of two types: “wishes to meet” or “wishes to avoid”
- We going to need an additional variable: The *type* of each player denoted by t_A and t_B
- What is known by both players is the probability that the player A has type t_A, assuming the player B has type t_B, for all possible pairs of values of t_A and t_B.
- We denote that probability by:
  - $P(t_A|t_B)$ → Belief of player A’s type given player B’s type
  - $P(t_B|t_A)$ → Belief of player B’s type given player A’s type

t_B=meet

	Hockey	Movie
Hockey	**+2,+1**	**0,0**
Movie	**0,0**	**+1,+2**

t_B=avoid

	Hockey	Movie
Hockey	**+2,0**	**0,+2**
Movie	**0,+1**	**+1,0**

$P(t_A=meet \mid t_B=meet) = 1$
$P(t_A=meet \mid t_B=avoid) = 1$
$P(t_A=avoid \mid t_B=meet) = 0$
$P(t_A=avoid \mid t_B=avoid) = 0$

$P(t_B=meet \mid t_A=meet) = 1/2$
$P(t_B=meet \mid t_A=avoid) = 1/2$
$P(t_B=avoid \mid t_A=meet) = 1/2$
$P(t_B=avoid \mid t_A=avoid) = 1/2$

Payoffs

- Assume that Player A is in type t_A and considers move $s_A(t_A)$
- Assume that Player will play $s_B(t_B)$ when in type t_B
- Given that Player A does not know in which type Player B is, what is the expected payoff for Player A?
- The expected payoff for Player A for a particular value of type t_A is the sum of the payoffs he would receive for each possible type from the other player, weighted by the probability that the other player is in that type.

$$\boxed{\bar{u}_A = \sum_{\text{all possible types } t_B \text{ of Player B}} u_A\big(s_A(t_A), s_B(t_B)\big)\mathrm{P}(t_B \mid t_A)}$$

Payoff if Player A knows that Player B is of type t_B

Probability that Player B is indeed of type t_B

$$\overline{u}_A = \sum_{\substack{\text{all possible types}\\ t_B \text{ of Player B}}} u_A\big(s_A(t_A), s_B(t_B)\big)\mathrm{P}(t_B \mid t_A)$$

Since Player A does not know Player B’s type, it has to sum over all possible types to get the expected value

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t_B=*meet*

	Hockey	**Movie**
Hockey	+2,+1	0,0
Movie	0,0	+1,+2

t_B=*avoid*

	Hockey	**Movie**
Hockey	+2,0	0,+2
Movie	0,+1	+1,0

$(s_B(t_B = meet), s_B(t_B = avoid))$

s_A

	(H,H)	**(H,M)**	**(M,H)**	**(M,M)**
H	**2**	**1**	**1**	**0**
M	**0**	**1/2**	**1/2**	**1**

Expected Payoffs

($s_B(t_B = meet)$, $s_B(t_B = avoid)$)

s_A	(H,H)	(H,M)	(M,H)	(M,M)
H	**2**	**1**	**1**	**0**
M	**0**	**/2**	**1/2**	**1**

Expected payoff to Player A if he chooses **H** and Player B chooses:

- **H** if it is of type meet
- **M** if it is of type avoid

Equilibrium

- The notion of equilibrium developed earlier can be (finally) extended to this case.
- It is the same definition, except that we replace payoffs by the *expected* payoff for each type of player
- A set of actions for each player $\{s^*_A(t_A), s^*_B(t_B)\}$ for all possible types t_A and t_B is an equilibrium if

$$s^*_A(t_A) = \arg\max_{s_A} \sum_{\substack{\text{all possible types}\\ t_B \text{ of Player B}}} u_A\left(s_A(t_A), s^*_B(t_B)\right) \mathrm{P}(t_B \mid t_A)$$

$$s^*_B(t_B) = \arg\max_{s_B} \sum_{\substack{\text{all possible types}\\ t_A \text{ of Player A}}} u_B\left(s^*_A(t_A), s_B(t_B)\right) \mathrm{P}(t_A \mid t_B)$$

Equilibrium

$$s^*_A(t_A) = \arg\max_{s_A} \sum_{\substack{\text{all possible types}\\ t_B \text{ of Player B}}} u_A\big(s_A(t_A), s^*_B(t_B)\big) \mathrm{P}(t_B \mid t_A)$$

Assuming Player B's uses $s^*_B(t_B)$ for all types t_B Player A cannot get higher payoff than by playing $s^*_A(t_A)$: $s^*_A(t_A)$ is the best that Player A can achieve

- Note: This is *exactly* the same definition of equilibrium as before but for the "supergame" with as many players as there are pairs *(Player, Type)* and with the definition of the expected payoffs
- Bottom line: There is an equilibrium which yields the best strategy for rational players given their beliefs about the other players' state

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t_B=*meet*

	Hockey	**Movie**
Hockey	**+2,+1**	**0,0**
Movie	**0,0**	**+1,+2**

t_B=*avoid*

	Hockey	**Movie**
Hockey	**+2,0**	**0,+2**
Movie	**0,+1**	**+1,0**

$(s_B(t_B = meet), s_B(t_B = avoid))$

s_A

	(H,H)	**(H,M)**	**(M,H)**	**(M,M)**
H	**2**	**1**	**1**	**0**
M	**0**	**1/2**	**1/2**	**1**

t_B=*meet*

	Hockey	Movie
Hockey	+2,+1	0,0
Movie	0,0	+1,+2

t_B=*avoid*

	Hockey	Movie
Hockey	+2,0	0,+2
Movie	0,+1	+1,0

$(s_B(t_B = meet),\ s_B(t_B = avoid))$

s_A

	(H,H)	(H,M)	(M,H)	(M,M)
H	2	1	1	0
M	0	1/2	1/2	1

$(s_B(t_B = meet),\ s_B(t_B = avoid))$

s_A

	(H,H)	(H,M)	(M,H)	(M,M)
H	2	1	1	0
M	0	/2	1/2	1

The strategy:

S^*_A = **H**

S^*_B=

- **H** if B is of type *meet*
- **M** if B is of type *avoid*

Is the equilibrium because:

$$\bar{u}_A(s^*_A, s^*_B) \geq \bar{u}_A(s_A, s^*_B)\ \forall s_A\ \forall t_A$$

$$\bar{u}_B(s^*_A, s^*_B) \geq \bar{u}_B(s^*_A, s_B)\ \forall s_B\ \forall t_B$$

Side Note

- Such games with beliefs over types yielding expected payoffs are termed *Bayesian Games*.
- The definitions so far were for 2 players; they extend directly to *n* players (albeit with *considerably* more painful notations)

Applications: Auctions

- One object (resource, bandwidth, job, etc.) is up for sale
- *n* available buyers
- Buyer *i* has a value for the object (which he knows, but none of the other buyers know): V_i in [0,1]
- Buyer *i* does not know V_j (for *j* neq *i*), but he assumes that the V_j's are randomly (uniformly) drawn from [0,1]
- What should be buyer *i's* strategy, assuming all the other buyers follow the best (rational) strategy?

First-Price Sealed Auctions

- Each buyer *i* writes down a bid g_i
- Buyer i_o with the highest bid buys object at price = bid g_{io}

- Models contract bids, “descending” (“Dutch”) auctions.

First-Price Sealed Auctions

- Each buyer *i* writes down a bid g_i
- Buyer i_o with the highest bid buys object at price = bid g_{io}

- Players: *n* Buyers
- Moves: All the possible bids $g_i >= 0$ for each Player *i*
- Payoffs:
  - $V_i - g_i$ if $g_i = \max_j(g_j)$
  - 0 otherwise

- Notes: The seller is not considered here; although we avoid mentioning the problem of the ties ($g_i = g_j$ for 2 different players), tie-breaking rules must be built into the auction

First-Price Sealed Auctions

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- Players: n Buyers
- Moves: All the possible bids $g_i >= 0$ for each Player i
- Payoffs:
  - $V_i - g_i$ if $g_i = \max_j(g_j)$
  - 0 otherwise
- Previous formalism can be used:
  - Types are the different values V_i for each player i
  - Player j does not know the value V_i for player i, but it knows the belief distribution for this value (uniform in this case)

First-Price Sealed Auctions

- For every player i, the equilibrium is reached for:

$g^*_i = \text{argmax}_g$ (Expected payoff for player i)

First-Price Sealed Auctions

- For every player i, the equilibrium is reached for:

g^*_i = argmax_g (Expected payoff for player i bidding g)

- The payoff is non-zero only if i wins the auction, so:

g^*_i = argmax_g (Expected payoff for player i bidding g when i wins) x Prob(i wins)

First-Price Sealed Auctions

- Assume that for any Player j, the strategy is of the form:
- Prob(i wins) = product of Prob($g > m_jV_j$) for all the other Players j
- Prob($g > m_jV_j$) = Prob($g/m_j > V_j$) = g/m_j
- So Prob(i wins) is proportional to g^{n-1}
- The coefficient of proportionality does not depend on Player i and it is unimportant

First-Price Sealed Auctions

- For every player *i*, the equilibrium is reached for:

$$g^*_i = \text{argmax}_g \text{ (Expected payoff for player } i \text{ bidding } g\text{)}$$

- The payoff is non-zero only if *i* wins the auction, so:

$$g^*_i = \text{argmax}_g \text{ (Expected payoff for player } i \text{ bidding } g \text{ when } i \text{ wins x Prob(} i \text{ wins))}$$

$= (V_i\text{-}g)$

= Prob($g >$ all of the other n-1 bids) proportional to g^{n-1}

First-Price Sealed Auctions

- For every player *i*, the equilibrium is reached for:

$$g^*_i = \text{argmax}_g \text{ (Expected payoff for player } i \text{ bidding } g \text{ when } i \text{ wins x Prob(} i \text{ wins))}$$

$$g^*_i = \text{argmax}_g \,(V_i\text{-}g)\, g^{n-1}$$

- The maximum is reached when the derivative is zero:

$$(n\text{-}1)(V_i - g^*)\, g^{*n-2} - g^{*n-1} = 0$$

$$(n\text{-}1)(V_i - g^*) - g^* = 0$$

$$\boxed{g^* = (1\text{-}1/n)\ V_i}$$

First-Price Sealed Auctions

- For every player i, the equilibrium is reached for: $\boxed{g_i^* = (1-1/n)\ V_i}$
- Meaning: If all other players follow the this strategy, i can do no better than using this value of g_i
- Note: As the number of buyers (n) increases, the bids have to come closer to the value assigned by each player.

First-Price Sealed Auctions

- Useful fact to know: The expected value of the max of n numbers randomly and uniformly drawn from [0,1] is $n/(n+1)$

➔ For the equilibrium $g_i^* = (1-1/n)\ V_i$, the expected highest bid is

$$(1-1/n)(n/(n+1)) = 1-2/(n+1)$$

Second-Price Sealed Auctions

- Each buyer *i* writes down a bid g_i
- Buyer i_o with the highest bid buys object at price = *second highest bid*

- It models "ascending" ("English") auctions.

Second-Price Sealed Bid

- Players: *n* Buyers
- Moves: All the possible bids $g_i >= 0$ for each Player *i*
- Payoffs:
  - $V_i - g^o$ if $g_i = \max_j(g_j)$ $g^o = \max_{j \text{ neq } i}(g_j)$
  - 0 otherwise

- Notes: The seller is not considered here; although we avoid mentioning the problem of the ties ($g_i = g_j$ for 2 different players), tie-breaking rules must be built into the auction

Second-Price Sealed Bid

- Let g^o be the second highest bid
- For Player *i*, payoff is:
  - $V_i - g^o$ if $g_i > g^o$
  - 0 otherwise

- If $V_i > g^o$ then any bid that wins the auction is optimal (maximum payoff). In particular, $g_i = V_i$ wins.
- If $V_i < g^o$ then any bid that loses the auction is optimal (maximum payoff is 0). In particular, $g_i = V_i$ loses.

- Therefore $g_i = V_i$ yields the highest payoff for Player i, irrespective of the other players' bids
- Therefore it is a dominant strategy and the equilibrium is:

$$\boxed{g^*_i = V_i}$$

Application: Negotiation

- Seller (S) and Buyer (B)
  - S assigns a value V_S to the object
  - B assigns a value V_B to the object
  - Neither player knows the other player's assigned value, but they both know a distribution over the values (for example, drawn randomly and uniformly from [0 1])
- S writes down a bid g_S
- B writes down a bid g_B

- If $g_B <= g_S$ no trade occurs: Payoffs = 0
- If $g_B > g_S$ then B pays $(g_S + g_B)/2$ to S and receives the object
  - S Payoff = $(g_S + g_B)/2 - V_S$
  - B Payoff = $V_B - (g_S + g_B)/2$

Negotiation (Double Auction)

- This scenario can be modeled again using the same formalism as before:
  - Players: S and B
  - Types: The values V_S and V_B in [0 1]
  - Actions: The bids g_S and g_B in [0 1]
  - Beliefs: Uniform distribution on [0 1] for V_S and V_B
- Payoffs:

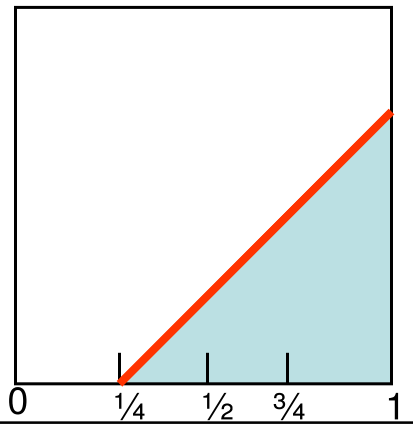
u_S = Expected payoff to S

= (Expected payoff to S if trade occurs) x Prob(trade occurs)

= (Expected payoff to S if trade occurs) x Prob($g_B > g_S$)

Negotiation (Double Auction)

- Equilibrium:
  - $g^*_B(V_B) = 1/12 + 2/3\ V_B$
  - $g^*_S(V_S) = 1/4 + 2/3\ V_S$
- Trade occurs if: $V_B > V_S + 1/4$



Notes

- Tools developed for general games earlier can be used to analyze auctions
- Second-price auction is often preferred
- Auction design is obviously important in economics (e.g., auction of radio spectrum…); it is becoming increasingly important for the design of autonomous agents
- Many other topics:
  - Cooperative auctions
  - Many goods
  - Other mechanisms (time limits, multiple bids, etc.)
  - Risk adverse buyers
  - Non-uniform beliefs

Summary

- Extension of “games” formalism to cases in which the payoffs are uncertain
- Extension of the notion of “equilibrium” to these cases by introducing the notion of “player type” and by replacing playoffs by “expected payoffs” computed using beliefs over player types (e.g., probability that Player B wants to meet Player A)
- Useful model for auctions and negotiations
- Examples:
  - First-price auctions
  - Second-price auctions
  - Double auction negotiation