Lecture 9

Digit-based sorting and data structures

9.1 Overview

In this lecture we examine digit-based sorting and digit-based data structures. Our previous data structures treated keys as abstract objects that could only be examined via comparisons. The methods discussed in this lecture will instead treat them as a sequence of digits or a sequence of characters. Material in this lecture includes:

- Radix sort: a method for sorting strings or integers.
- Tries: a data structure that can be viewed as an on-the-fly version of radix sort.

9.2 Introduction

So far, we have looked at sorting and storing items whose keys can be anything at all, so long as we have a notion of "less than". Today we will look at methods for the case that keys are natural objects like strings or integers.

To start, say we have n objects to sort whose keys are all integers in a small range: $1, \ldots, r$ (e.g., say I have a bunch of homeworks to sort by section). In this case, what would be a fast way to sort? One easy method is bucket-sort:

Bucket-sort:

- Make an array A of size r, of "buckets" (perhaps implemented as linked lists).
- Make one pass through the n objects. Insert object with key k into the kth bucket $A[k]$.
- Finally, make a pass through the array of buckets, concatenating as you go.

The first step of bucket-sort takes time $O(r)$, the second step takes time $O(n)$, and the last step takes time $O(r)$ or $O(r+n)$ depending on how the buckets are implemented. In any case, the total time is $O(n+r)$.

So, bucket-sort is linear time if $r = O(n)$. Notice one thing interesting about it is that by using the magic of indirect addressing, we are making an r-way decision at each step (something you can't do in the comparison model). Unfortunately, bucket-sort is only good if the range of keys is small. This leads us to our next method, radix-sort.

9.3 Radix Sort

Suppose our keys are numbers, or strings, that can be viewed as a *sequence* of digits (or characters) each in a range of size r. In that case, there is a natural sorting algorithm called Radix Sort we can use to sort them. ("radix" is the base r). Actually there are two versions: one that's conceptually easier called Most-Significant-First radix sort, where we go top-down digit by digit (or byte by byte), and another that's trickier to think of but easy to code called Least-Significant-First radix sort where we go in the other direction.

9.3.1 Most-significant-first (MSF) radix sort

Most-significant-first radix sort begins by sorting keys into buckets according to their most significant character or digit. For instance, if we are sorting strings, we would get a bucket for 'a', a bucket for 'b', a bucket for 'c' and so on. So, now the strings are roughly sorted in that any two strings that begin with different letters are in the correct order.[1] Now we just recursively sort each bucket that has more than one element using the same procedure (sorting into buckets by the next most significant character and so on) and then concatenate the buckets.

If we ignore time spent scanning empty buckets, then to sort n strings we just spend $O(n)$ time at each level. So, if strings are length L, then the total time is $O(nL)$. The time spent scanning empty buckets could be a problem if r is large, but if we view r as a constant, then just goes into the $O()$.

9.3.2 Least-significant-first (LSF) radix sort

Here is another idea that is easier to implement but trickier to see why it works. Let's switch from strings to numbers since that will make the method a little cleaner.

In this algorithm, we first perform a bucketsort using only the *least* significant digit. That is, we place the keys into buckets according to the ones digit and then concatenate the buckets. Next we bucketsort by the tens digit, then the hundreds and so on. This sounds weird, but the claim is that if we perform each bucketsort in a *stable* manner that doesn't rearrange the order of items that go into the same bucket (a sorting method is called **stable** if it doesn't rearrange equal keys) then this will end up correctly sorting the items. Let's see what happens on an example:

Example: Suppose our input is `[28, 15, 24, 17, 13, 22]`.

- We first sort by the ones digit, producing: `[22, 13, 24, 15, 17, 28]`.
- Now we sort by the tens digit using a stable bucketsort (so the relative order of items with the same tens digit remains unchanged), producing: `[13, 15, 17, 22, 24, 28]`.

[1] If we are sorting numbers, we need to pad to the left with zeros to have the correct semantics.

Why does the algorithm work? Let's prove by induction that after the ith pass, the items are correctly sorted according to the least i digits. This is clearly true for the base case $i = 1$. Now, let's do the general case. We know that after the ith pass, the items that differ in the ith digit will be in the desired order with respect to each other (because we just sorted them by that digit!) but what about the items that are equal in this digit? Well, by induction, these were in the desired order *before* we began the ith pass, and since our bucketsort is *stable*, they remain in the correct order afterwards. So we are done.[2]

If numbers have L digits, then running time is $O(L(r + n)) = O(Ln)$ if $r = O(n)$.

Advantages: this method is easy to implement since there is no need to keep buckets separate or even to do recursion: We just have a loop that for `j = L` down to `1` calls `bucketsort(A,j)` which does a bucketsort using the jth character of each string for sorting.

Relation to bounds on comparison-based sorting: If we have n different numbers, then their length is at least $\log_r n$. In this case, the running time is $O(n \log_r n)$. The reason we get this instead of $n \log_2 n$ is we are using indirect-addressing to make an r-way decision in 1 time step. On the negative side, if some keys are much longer than others, then L could be a lot bigger than $\log_r n$. On the positive side, each operation is just an operation on a single digit.

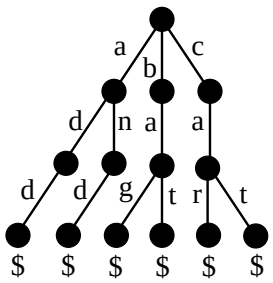
9.4 Tries

The word *trie* comes from re*trie*val. These are also called *digital search trees.* Tries are to MSF radix sort like binary search trees are to quicksort.

In a trie, you put letters on the edges and you walk down the tree reading off the word. In particular, each node will have an array of size r (e.g., $r = 26$ or 128 or 256) of child pointers. To store a string, you just follow down the associated child for each letter in the string from first to last. For instance, say we wanted to store the words:

{`and, bat, add, bag, cat, car`}

When doing an insert, we end each string with "$" (a special end character) in case some strings are substrings of others. To do a lookup, we just walk down the tree letter-by-letter and then see if the node we get to at the end has a "$" in it. (If we ever get to a null pointer, the we know the key is not there — e.g., if we look up "apple" then we will notice in the 2nd step that it can't possibly be in the trie). For instance, in the above example, we would have:



[2]If keys are strings, and they have different lengths, then to match the usual notion of what we mean by "sorted in alphabetical order", we should pad them to the right with blanks that are defined to be less than 'a'. E.g., {car, card} should be viewed {car␣, card}. This is the flip-side of the previous footnote.

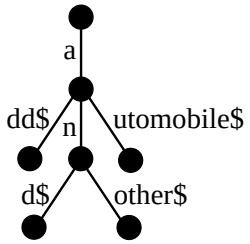
(In terms of implementation, each node has an array of child pointers, so when the picture shows an edge having a letter, like 'c', this really means that the child is the "cth" child.)

The time to do a search is only O(length of key). The same is true for doing an insert, if we view r as a constant. (If r is really big then we should worry about the time spent allocating the arrays of child pointers). So, this is really nice. Also, prefix searching is especially easy (e.g., if you wanted to make a text editor that did word-completion). The main drawback is that the overhead is high because you have so much pointer-following to do. E.g., what if we added "automobile" to the above trie? Also, you have a factor of r extra space since you need to have an array of size r in each node.

Since the design is so nice conceptually, people have thought about ways of making it more practical. In particular, some things you can do are:

- Compress paths that don't branch into a single edge.
- Only split when needed.

So, for instance, the set {add, automobile, and, another} would look like:



This is called a "Patricia tree" (practical algorithm to retrieve information coded in alphanumeric).