Lecture 10

Interval Analysis

I	Basic Idea
II	Algorithm
III	Optimization and Complexity
IV	Comparing interval analysis with iterative algorithms

Reference: Muchnick 7.5-7.7, 8.8

Advanced readings (optional):

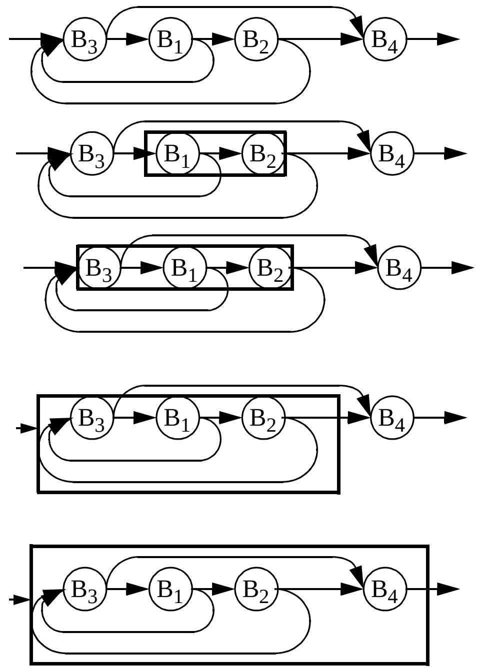
R. E. Tarjan, "A Unified Approach to Path Problems",
JACM 28 (3) July 1981, pp. 577-593.

R. E. Tarjan, "Fast Algorithms for Solving Path Problems",
JACM 28 (3) July 1981, pp. 594-614.

Motivation for Studying Interval Analysis

- **Exploit the structure of block-structured programs in data flow**
- **Tie in several concepts studied**
  - Use of structure in induction variables, loop invarient
    - motivated by nature of the problem
    - *This lecture: can we use structure for speed?*
  - Iterative algorithm for data flow
    - *This lecture: an alternative algorithm*
  - Reducibility
    - all retreating edges of DFST are back edges
    - reducible graphs converge quickly
    - *This lecture: algorithm exploits & requires reducibility*
- **Usefulness in practice**
  - Faster for "harder" analyses
  - Useful for analyses related to structure
- **Theoretically interesting - better understanding of data flow**

I. Big Picture



Basic Idea

- **In iterative analysis**
  - DEFINITION: Transfer function F_B:
    summarize effect from beginning to end of basic block B

- **In interval analysis**
  - DEFINITION: Transfer function $F_{R,B}$:
    summarize effect from beginning of R to end of basic block B
  - Recursively
    construct a larger region R from smaller regions
    construct $F_{R,B}$ from transfer functions for smaller regions
    until the program is one region
  - Let P be the region for the entire program,
    and v be initial value at entry node
    - out[B] = $F_{P,B}$ (v)
    - in [B] = $\wedge_{B'}$ out[B'], where B' is a predecessor of B

II. Algorithm

- (a) Operations on transfer functions
- (b) How to build nested regions?
- (c) How to construct transfer functions that correspond to the larger regions?

(a) Operations on Transfer Functions

- **Example: Reaching Definitions**

- $F(x) = Gen \cup (x - Kill)$
- $$\begin{aligned} F_2(F_1(x)) &= Gen_2 \cup (F_1(x) - Kill_2) \\ &= Gen_2 \cup (Gen_1 \cup (x - Kill_1)) - Kill_2) \\ &= Gen_2 \cup (Gen_1 \cup (x - Kill_1)) - Kill_2) \\ &= Gen_2 \cup (Gen_1 - Kill_2) \cup (x - (Kill_1 \cup Kill_2)) \end{aligned}$$

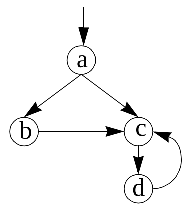
- $$\begin{aligned} F_1(x) \wedge F_2(x) &= Gen_1 \cup (x - Kill_1) \cup Gen_2 \cup (x - Kill_2) \\ &= (Gen_1 \cup Gen_2) \cup (x - (Kill_1 \cap Kill_2)) \end{aligned}$$

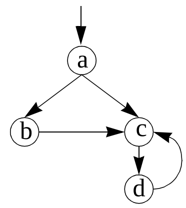
- $$\begin{aligned} F^*(x) &<= F^n(x), \forall\, n \geq 0 \\ &= x \cup F(x) \cup F(F(x)) \cup ... \\ &= x \cup (Gen \cup (x - Kill)) \cup (Gen \cup ((Gen \cup (x - Kill)) - Kill)) \cup ... \\ &= Gen \cup (x - \varnothing) \end{aligned}$$

(b) Structure of Nested Regions (An example)

- A **region** in a flow graph is a set of nodes that
  - includes a **header**, which dominates all other nodes in a region
- **T1-T2 rule (Hecht & Ullman)**
  - T1: Remove a loop
    If n is a node with a loop, i.e. an edge n->n, delete that edge
  - T2: Remove a vertex
    If there is a node n that has a unique predecessor, m,
    then m may consume n by
    deleting n and making all successors of n be successors of m.

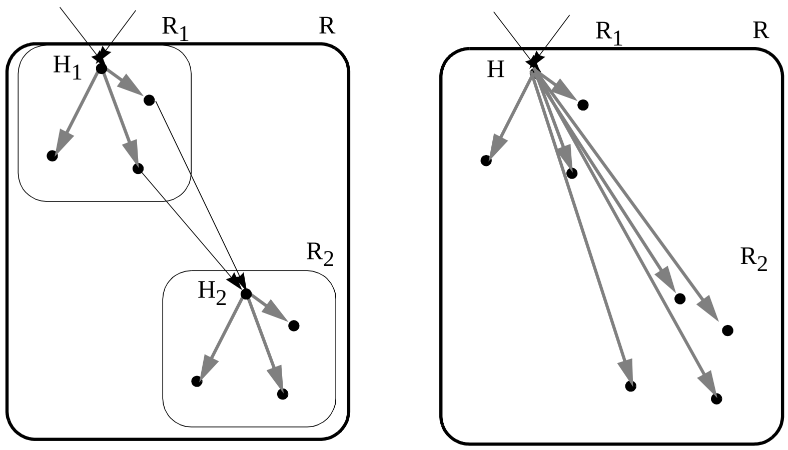
Example





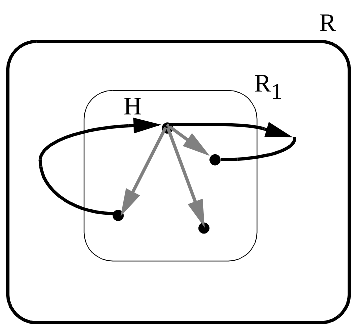
- In reduced graph:
  - each vertex represents a subgraph of original graph (a **region**).
  - each edge represents an edge in original graph
- **Limit flow graph**: result of exhaustive application of T1 and T2
  - independent of order of application.
  - if limit flow graph has a single vertex => reducible
- Can define larger regions (e.g. Allen&Cocke’s intervals)
  simple regions=>simple composition rules for transfer functions

(c) Transfer Functions for T2 Rule



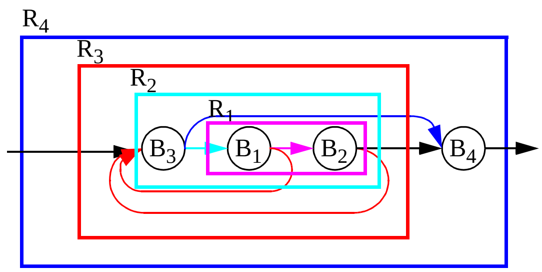
- **Transfer function**
  $F_{R,B}$**:** summarizes the effect from beginning of R to **end** of B
  $F_{R,in(H2)}$**:** summarizes the effect from beginning of R to beginning of H2
  - Unchanged for blocks B in region R_1 ($F_{R,B} = F_{R1,B}$)
  - $F_{R,in(H2)} = \wedge_{\ p} F_{R,P}$, where p is a predecessor of H_2
  - For blocks B in region R_2: $F_{R,B} = F_{R2,B} \cdot F_{R,in(H2)}$

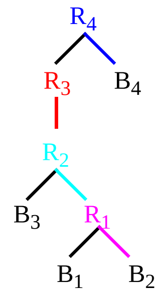
Transfer Functions for T1 Rule



- **Transfer function $F_{R,B}$**
  - $F_{R,in(H)} = (\wedge_{\ p} F_{R1,P})$ *, where p is a predecessor of H in R
  - $F_{R,B} = F_{R1,B} \cdot F_{R,in(H)}$

First Example

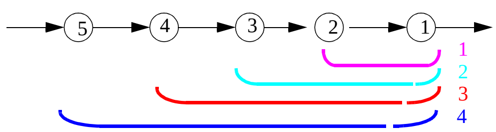




R	T_1/T_2	R'	$F_{R,in(R')}$	$F_{R,B1}$	$F_{R,B2}$	$F_{R,B3}$	$F_{R,B4}$
R_1	T_2	B_2	F_{B1}	F_{B1}	$F_{B2}\cdot F_{R1,in(B2)}$		
R_2	T_2	R_1	F_{B3}	$F_{R1,B1}\cdot F_{R2,in(R1)}$	$F_{R1,B2}\cdot F_{R2,in(R1)}$	F_{B3}	
R_3	T_1	R_2	$(F_{R2B1}\wedge F_{R2B2})^*$	$F_{R2,B1}\cdot F_{R3,in(R2)}$	$F_{R2,B2}\cdot F_{R3,in(R2)}$	$F_{R2,B3}\cdot F_{R3,in(R2)}$	
R4	T_2	B_4	$F_{R3B3}\wedge F_{R3B2}$	$F_{R3,B1}$	$F_{R3,B2}$	$F_{R3,B3}$	$F_{B4}\cdot F_{R4,in(B4)}$

- R: region name
- R': region whose header will be subsumed

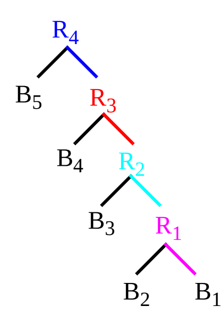
III. Complexity of Algorithm



R	T_1/T_2	R'	$F_{R,in(R')}$	$F_{R,B1}$	$F_{R,B2}$	$F_{R,B3}$	$F_{R,B4}$	$F_{R,B5}$
R_1	T_2	B_1	F_{B2}	$F_{B1}\cdot F_{B2}$	F_{B2}			
R_2	T_2	R_1	F_{B3}	$F_{R1,B1}\cdot F_{B3}$	$F_{R1,B2}\cdot F_{B3}$	F_{B3}		
R_3	T_2	R_2	F_{B4}	$F_{R2,B1}\cdot F_{B4}$	$F_{R2,B2}\cdot F_{B4}$	$F_{R2,B3}\cdot F_{B4}$	F_{B4}	
R_4	T_2	R_3	F_{B5}	$F_{R3,B1}\cdot F_{B5}$	$F_{R3,B2}\cdot F_{B5}$	$F_{R3,B3}\cdot F_{B5}$	$F_{B4}\cdot F_{B5}$	F_{B5}

R	$F_{R4,in(R)}$
R_4	I
R_3	$F_{B5}\cdot F_{R4,in(R4)}$
R_2	$F_{B4}\cdot F_{R4,in(R3)}$
R_1	$F_{B3}\cdot F_{R4,in(R2)}$
B_1	$F_{B2}\cdot F_{R4,in(R1)}$

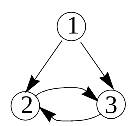
B	$F_{R4,B}$
B_5	$F_{B5}\cdot I$
B_4	$F_{B4}\cdot F_{R4,in(R3)}$
B_3	$F_{B3}\cdot F_{R4,in(R2)}$
B_2	$F_{B2}\cdot F_{R4,in(R1)}$
B_1	$F_{B1}\cdot F_{R4,in(B1)}$



Optimization

- **Let m = number of edges, n = number of nodes**
- **Ideas for optimization**
  - If we compute $F_{R,B}$ for every region B is in, then it is very expensive
  - We are ultimately only interested in the entire region (E); we need to compute only $F_{E,B}$ for every B.
    - There are many common subexpressions between $F_{E,B1}$, $F_{E,B2}$, ...
    - Number of $F_{E,B}$ calculated = m
  - Also, we need to compute $F_{R,in(R')}$, where R' represents the region whose header is subsumed.
    - Number of $F_{R,B}$ calculated, where R is not final = n
- Total number of $F_{R,B}$ calculated: (m + n)
  - Data structure keeps "header" relationship
    - Practical algorithm: O(m log n)
    - Complexity: $O(m\alpha(m,n))$, α is inverse Ackermann function

Reducibility



- If no T1, T2 is applicable before graph is reduced to single node split node and continue
- Worst case: exponential
- Most graphs (including GOTO programs) are reducible

IV. Comparison with Iterative Data Flow

- **Applicability**
  - Definitions of F* can make technique
    more powerful than iterative algorithms
  - Backward flow -- reverse graph is not typically reducible.
    Requires more effort to adapt to backward flow than iterative alg.
  - More important for interprocedural optimization
- **Speed**
  - Irreducible graphs
    - Iterative algorithm can process irreducible parts uniformly
    - Serious "irreducibility" can be slow with elimination
  - Reducible graph & Cycles do not add information (common)
    - Iterative: (depth + 2) passes
      depth is 2.75 average, independent of code length
    - Elimination: Theoretically almost linear, typically O(m log n)
  - Reducible & Cycles add information
    - Iterative takes longer to converge
    - Elimination remains the same