15-859NN Spectral Graph Theory, Spring 2021

Homework 0 **Due: Friday February 19 in class**

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Instructions. Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

This homework is just a list of fundamantal linear algebra facts you should know. You should have an idea how the proofs also go.

(25) 1. **Orthogonal Projection**

Suppose that $Ax = b$ is an over constrained linear system and we would like to find an x to minimize $|Ax - b|_2^2$, the L_2^2 distance, where the columns of A are independent.

(a) Show that the answer to our minimization problem is:
1) The system $A^T A\bar{x} = A^T b$ always has a solution.
2) The $\bar{x}$ is solution to our problem.

(b) The projection of $b \in \mathbb{R}^m$ onto the column space of an m by n matrix A is the linear matrix $A(A^T A)^{-1} A^T b$.

(25) 2. **Spectral Theorem**

Suppose that A is a symmetric n by n real matrix. Show that A has the following properties:

(a) The eigenvalues are all real.

(b) A has a complete set of eigenvalues and eigenvectors, i.e., Its eigenvectors span a space of dimension n.

(c) $A = U^T \Lambda U$ where the rows of U are an orthonormal set of eigenvectors for A and Λ is a diagonal matrix of eigenvalues for A.

(d) If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $x_1, \ldots, x_n$ are the respective orthonormal eigenvectors as column vectors then

$$A = \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T$$

(25) 3. **Matrix Exponential**

Assuming that A is real symmetric, use the Spectral Theorem show that e^A is well defined and give a simple expression for it. That is, how do the eigenvalues and eigenvectors of A relate to those of e^A?